NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 1 Term 1 2009

Name:________________________________ **Mathematics Class:**_________

Time Allowed: **60 minutes + 2 minutes reading time**

Available Marks: 52

Instructions:

- Questions are *not* of equal value.
- Start each question on a new page.
- Put your name on the top of each page.
- Attempt all four questions.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Write on one side of the page only. Do not work in columns.
- Each question will be collected separately.
- If you do not attempt a question, submit a blank page with your name and the question number clearly displayed.

Question	1	2	3	4	Total	
E2				/6	/6	
E6		/10	/5		/15	
E3	/13		/8		/21	
E9		/4		/6	/10	
					/52	%

Question 1: (13 marks) **Marks**

(a) Simplify, expressing your answer in the form $a + ib$ where a and b are real numbers.

(i) $(6-2i)^2$ **1**

(ii) $\dfrac{\overline{5+i}}{1+i}$ **2**

(b) Solve $x^2 - 2ix + 3 = 0$ over the complex field. **2**

(c) Write $-\sqrt{3}+i$ in modulus-argument form and hence evaluate $\left(-\sqrt{3}+i\right)^{12}$. **2**

(d) Find the real values of a and b if $(a+ib)^2 = 40-42i$ and $a > 0$. **3**

(e) Let z be the complex number for which $|z| = 4$ and arg $z = \dfrac{2\pi}{3}$.

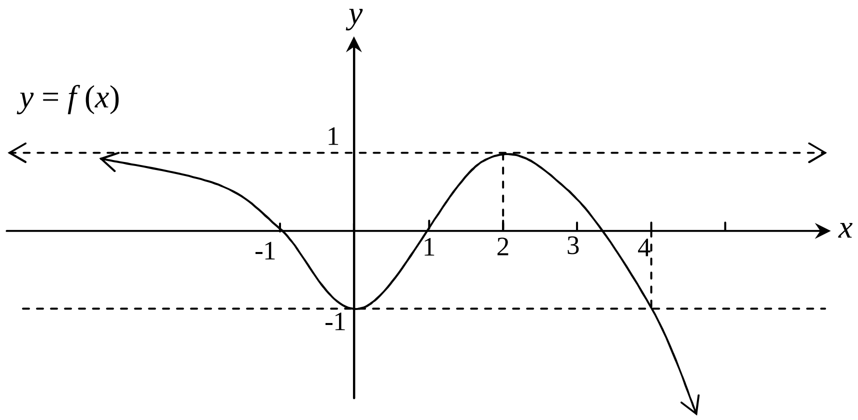
(i) On an Argand diagram, plot the point P that represents z. **1**

(ii) Without calculation and on the same diagram, plot the points X and Y which represent the complex numbers $\sqrt{z}$ and $\bar{z}$ respectively, showing the relationships between the points. **2**

Question 2: (14 marks) **Answer on the answer sheet provided.** **Marks**

(a) The graph of $y = f(x)$ is illustrated below. There is a horizontal asymptote at $y = 1$, a minimum turning point at $(0,-1)$ and a maximum turning point at $(2,1)$.

The curve passes through $(4,-1)$



On the separate number planes provided, sketch the graphs of each of the following, clearly labelling any turning points, intercepts and the equations of any asymptotes.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \left[f(x) \right]^3$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $|y| = f(x)$ **2**

(v) $y = 2^{f(x)}$ **2**

(b) If $g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le 4 \\ f(8-x) & \text{if } 4 < x \le 8 \end{cases}$ where $f(x)$ is defined as above.

(i) Evaluate $g(6)$. **1**

(ii) Neatly sketch $y = g(x)$. **2**

(iii) Solve $g(x) > -1$. **1**

Question 3: (13 marks) Start a new page **Marks**

(a) Consider $x^3 + y^3 = 3xy + 3$.

(i) Find the value of $\frac{dy}{dx}$ at the point (2, 1). **2**

(ii) Find any points at which the derivative is undefined. **3**

(b) (i) Simplify $(-4i)^3$. **1**

(ii) Hence write in modulus-argument form all the complex numbers z such that $z^3 = 64i$. **2**

(iii) If α and β are the two non-real cube roots of $64i$ with positive arguments prove that $\alpha^{12n} + \beta^{12n} = 2^{24n+1}$ where n is an integer. **2**

(c) The points A and B represent the complex numbers $z = 2 + i$ and $w = 2iz$ respectively. O is the origin.

(i) Show that $\angle AOB$ is a right angle. **1**

(ii) If $OACB$ is a rectangle, find the complex number represented by C, **2**

Question 4: (12 marks) **Start a new page** **Marks**

(a) (i) On the Argand diagram, sketch the locus of P which represents z where **3**

$$z + \bar{z} = |z|^2 .$$

(ii) Find the values of k for which there is at least one solution to the simultaneous equations **3**

$$\arg(z+1) = k \text{ and } z + \bar{z} = |z|^2 .$$

(b) (i) Sketch the graphs of $y = x(x-3)$ and $y = 3 + 2|x| - x^2$ on the same number plane given that $x = -1$ is a point of intersection. **4**

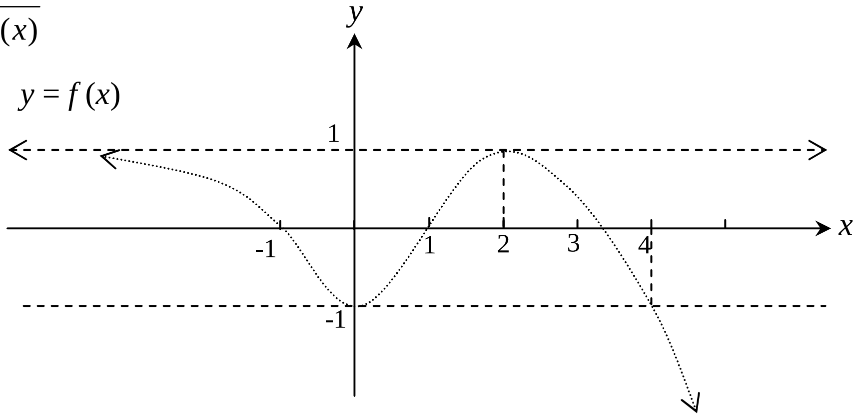
(ii) Use your graphs to solve the inequality $\dfrac{3+2|x|-x^2}{x(x-3)} \le 1$. **2**

End of Paper

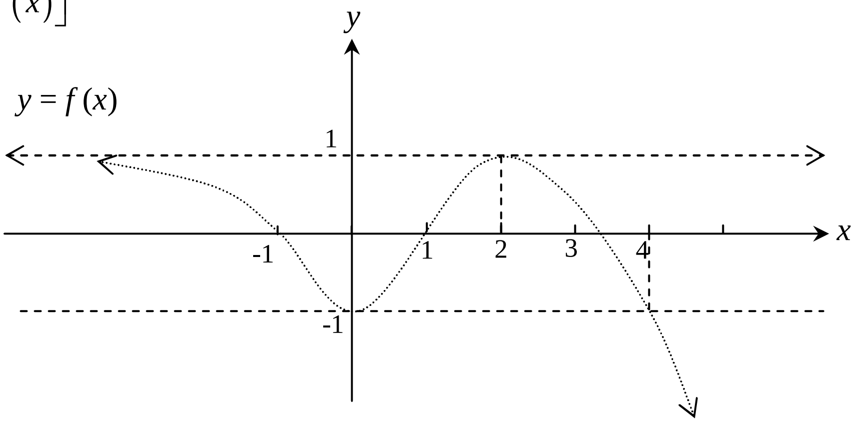
Name:____ **Class:_______**

<u>Answers for Question 2</u>

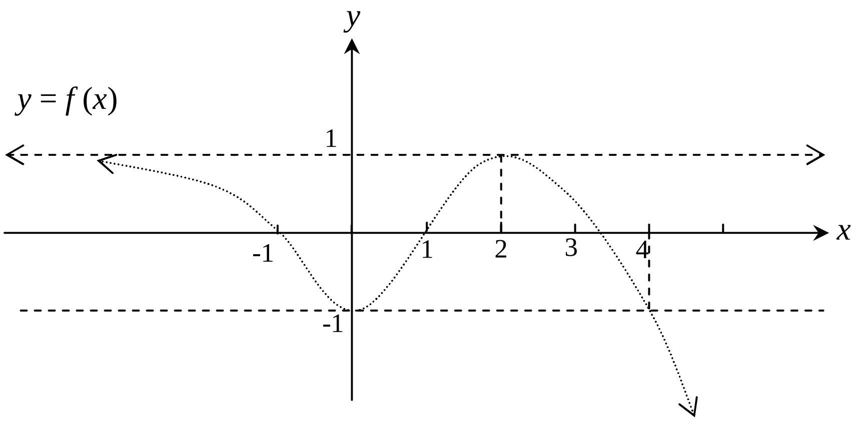
a) (i) $y = \dfrac{1}{f(x)}$



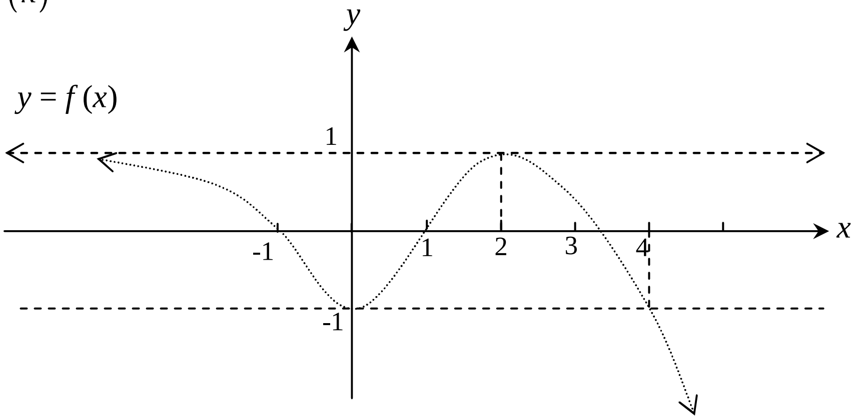
a) (ii) $y = \left[f\left(x\right) \right]^3$



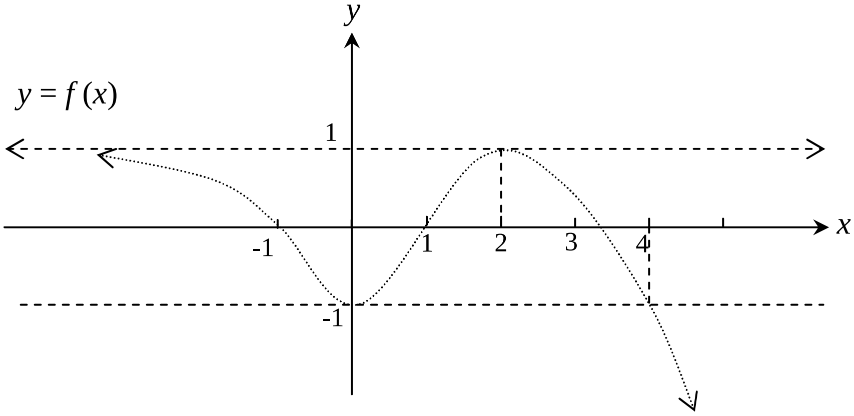
a) (iii) $y^2 = f\left(x\right)$



a) (iv) $|y| = f(x)$



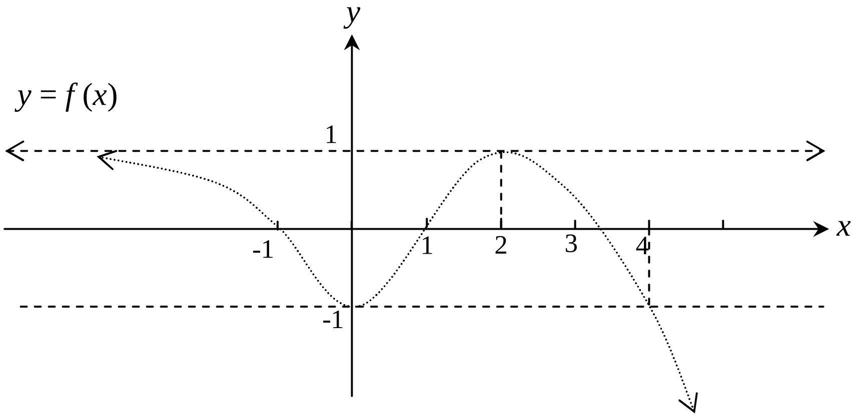
a) (v) $y = 2^{f(x)}$



b) $g(x) = \begin{cases} f(x) & \text{if } 0 \le x \le 4 \\ f(8-x) & \text{if } 4 < x \le 8 \end{cases}$

(i) Evaluate $g(6)$ ____________________________________

(ii) Neatly sketch $y = g(x)$.



b) (ii) Solve $g(x) > -1$

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Question 1:

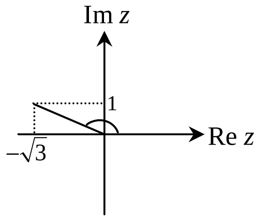
(a) (i)
$$\begin{aligned}(6-2i)^2 &= 36-24i+4i^2\\ &= 36-24i-4\\ &= 32-24i\end{aligned}$$

(ii)
$$\begin{aligned}\frac{\overline{5+i}}{1+i} &= \frac{5-i}{1+i}\times\frac{1-i}{1-i}\\ &= \frac{5-6i-1}{1+1}\\ &= \frac{4-6i}{2}\\ &= 2-3i\end{aligned}$$

(b) $x^2-2ix+3=0$

$$\begin{aligned}x &= \frac{2i\pm\sqrt{(-2i)^2-4(1)(3)}}{2}\\ &= \frac{2i\pm\sqrt{-4-12}}{2}\\ &= \frac{2i\pm 4i}{2}\\ &= -i,\ 3i\end{aligned}$$

(c) $\left|-\sqrt{3}+i\right| = \sqrt{3+1} = 2$ and $\arg\left(-\sqrt{3}+i\right) = \pi - \frac{\pi}{6}$
$= \frac{5\pi}{6}$



$\therefore\ \left(-\sqrt{3}+i\right) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Now
$$\begin{aligned}\left(-\sqrt{3}+i\right)^{12} &= \left[2\left(\cos\tfrac{5\pi}{6}+i\sin\tfrac{5\pi}{6}\right)\right]^{12}\\ &= 2^{12}\left(\cos\left(\tfrac{12\times 5\pi}{6}\right)+i\sin\left(\tfrac{12\times 5\pi}{6}\right)\right) \quad \text{by de Moivre's theorem}\\ &= 2^{12}\left(\cos 10\pi + i\sin 10\pi\right)\\ &= 2^{12}(1+0i)\\ &= 2^{12}\end{aligned}$$

(d) $(a+ib)^2 = 40-42i$ and $a>0$.

$a^2+2ab-b^2 = 40-42i$

Equating real and imaginary parts

$a^2-b^2 = 40$......................(1)

$ab = -21$...........................(2)

But $\left|(a+ib)^2\right| = |40-42i|$

$\therefore\ a^2+b^2 = \sqrt{40^2+42^2}$
$= 58$.......................(3)

Adding (1) and (3): $2a^2 = 98$

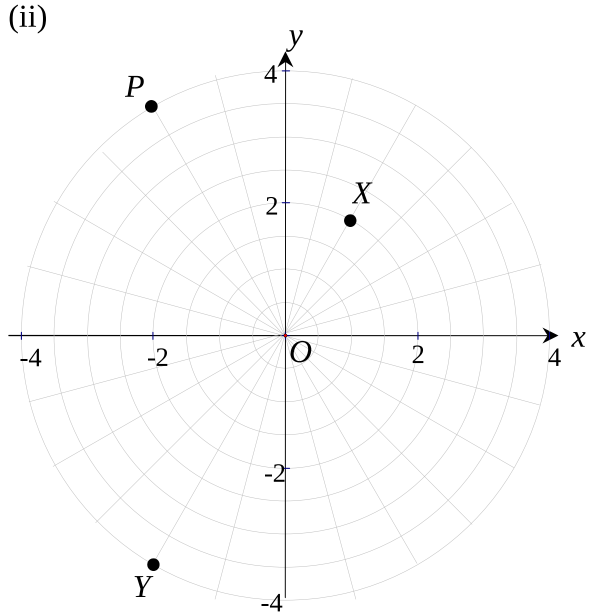
$a^2 = 49$ but $a>0$

$a = 7$

Sub into (2): $7b = -21$

$b = -3$ $\therefore\ a=7,\ b=-3$

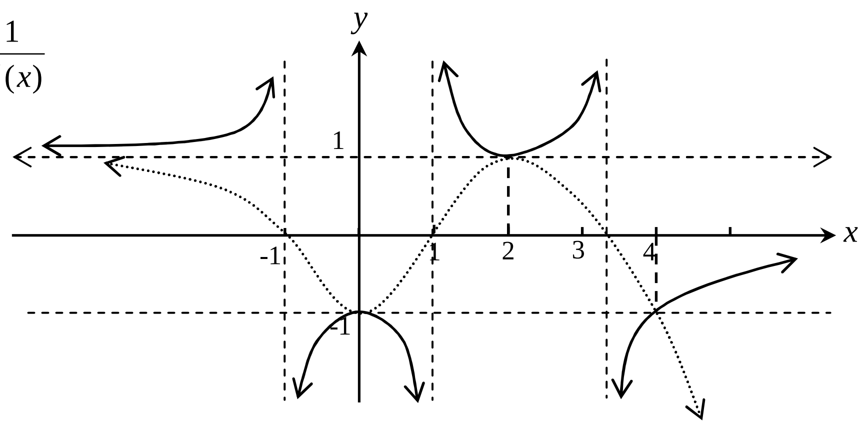
(e) (i) and (ii)



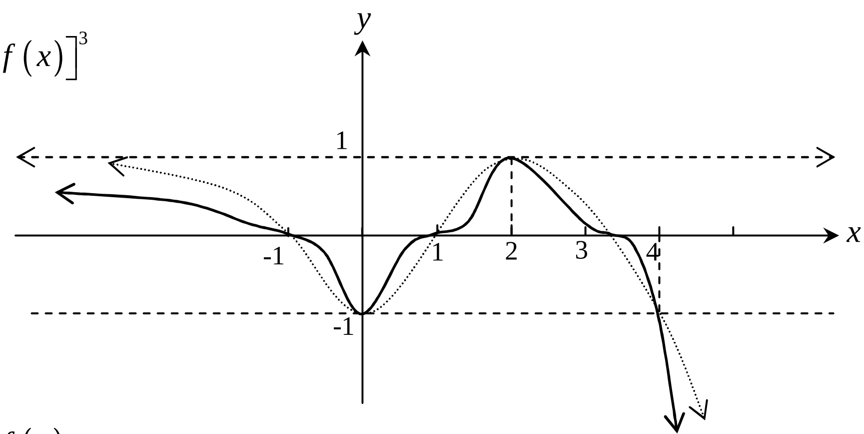
Note: the relationships are indicated by the radii of the circles and the angles indicating arguments – there is no need to calculate the *x* + *iy* forms of the complex numbers.

Question 2:

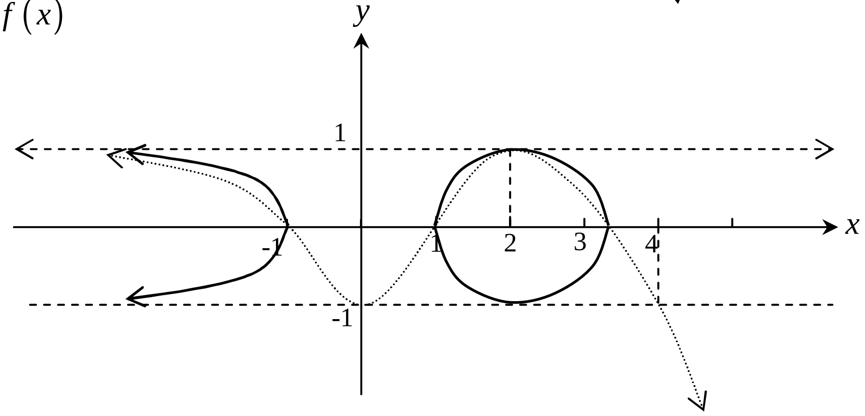
(a) (i) $y=\dfrac{1}{f(x)}$



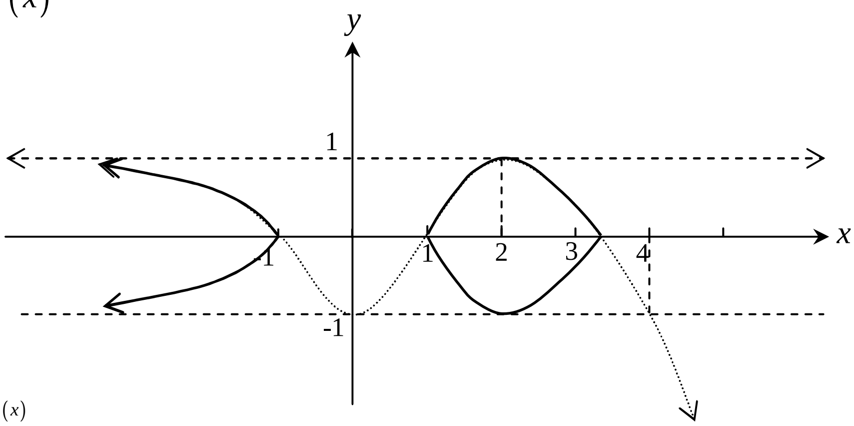
(a) (ii) $y=\left[f\left(x\right)\right]^3$



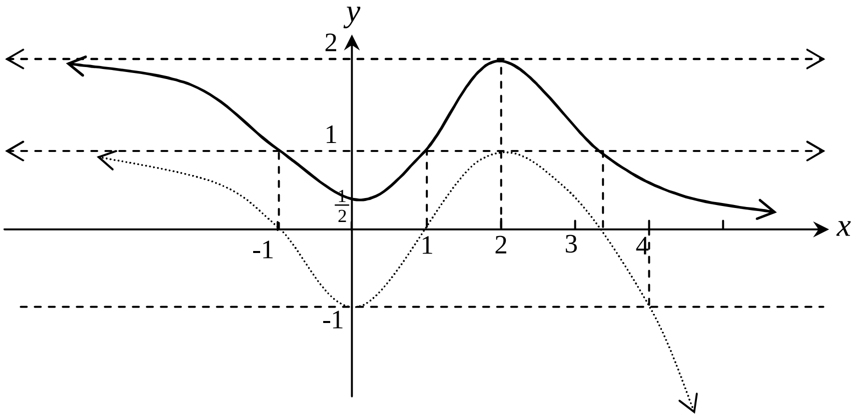
(a) (iii) $y^2=f\left(x\right)$



(a) (iv) $|y| = f(x)$



(a) (v) $y = 2^{f(x)}$

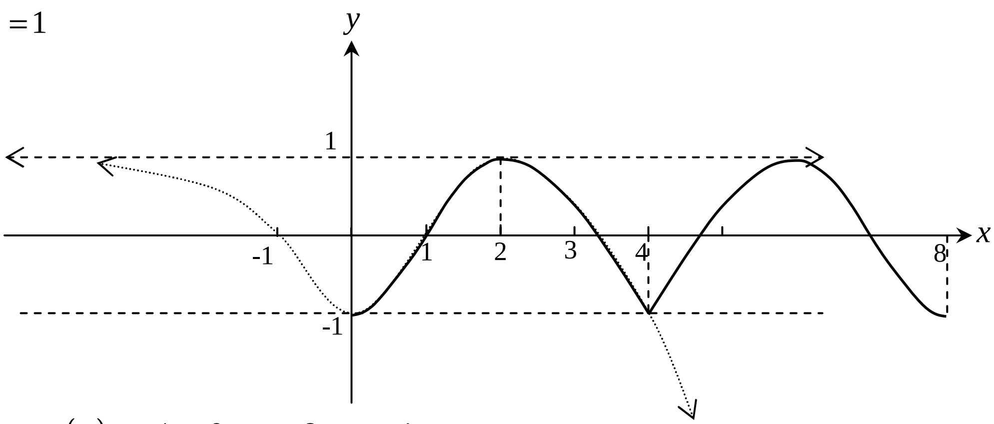


(b) (i) $g(6) = f(8-6)$ as $4 < x \le 8$

$= f(2)$

$= 1$

(b) (ii)



(b) (iii) Solve $g(x) > -1$: $0 < x < 8, \quad x \neq 4$

Question 3:

(a)(i)

$$x^3 + y^3 = 3xy + 3$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x\frac{dy}{dx} + y(3)$$

$$3y^2 \frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2 \qquad .$$

$$3\frac{dy}{dx}\left(y^2 - x\right) = 3\left(y - x^2\right)$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

$$\therefore \text{ at } (1,2) \quad \frac{dy}{dx} = \frac{1-2^2}{1^2-2} = 3$$

(a) (ii) The derivative is undefined when $y^2 - x = 0$

Then $x = y^2$ and substituting into the function gives:

$$\left(y^2\right)^3 + y^3 = 3\left(y^2\right)y + 3$$

$$y^6 - 2y^3 - 3 = 0$$

$$\left(y^3 - 3\right)\left(y^3 + 1\right) = 0$$

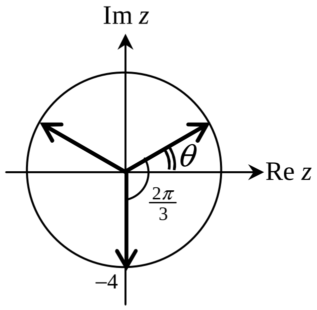
$$y^3 = 3 \text{ or } y^3 = -1$$

$y = \sqrt[3]{3}$ or $y = -1$ ie. at $\left(\sqrt[3]{9}, \sqrt[3]{3}\right)$ and $(1, -1)$

(b) (i) $\left(-4i\right)^3 = -64i^3$

$$= 64i$$

(b) (ii) If $z^3 = 64i$ then $-4i$ is one solution as demonstrated above. The other solutions lie evenly spaced around the circle passing through $-4i$ as illustrated in the diagram below.



Now $\theta = \dfrac{2\pi}{3} - \dfrac{\pi}{2} = \dfrac{\pi}{6}$

Hence the roots are:

$z_1 = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $z_2 = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

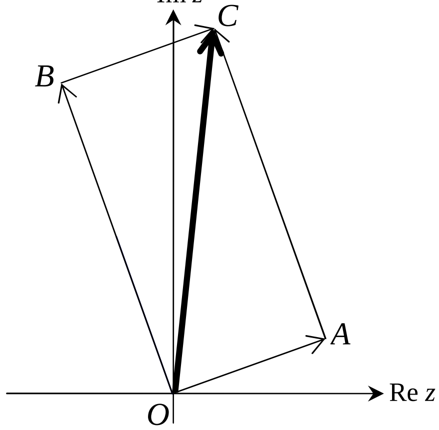
and $z_3 = -4i = 4\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$

(b) (iii) Let α and β be two of the roots of $z^3 = 64i$

$$\begin{aligned}
\alpha^{12n} + \beta^{12n} &= \left(\alpha^3\right)^{4n} + \left(\beta^3\right)^{4n} \\
&= \left(64i\right)^{4n} + \left(64i\right)^{4n} \\
&= 2 \times \left(2^6 i\right)^{4n} \\
&= 2 \times \left(2^{24n} i^{4n}\right) \\
&= 2 \times 2^{24n} \times \left(i^4\right)^n \\
&= 2^{24n+1}
\end{aligned}$$

(c) (i) $w = 2iz = i\left(2z\right)$ Hence w is the rotation of $2z$, 90° anticlockwise about O.

w is also twice the length of the vector z. These results are illustrated in the diagram below.



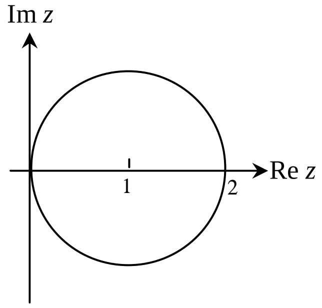
$\therefore \angle AOB = 90°$

(c) (ii) If $OACB$ is a rectangle, the complex number represented by C is located as illustrated in the diagram.

$$\begin{aligned}
\therefore \overrightarrow{OC} &= z + w \\
&= z + 2iz \\
&= 2 + i + 2i(2 + i) \\
&= 5i
\end{aligned}$$

Question 4:

(a) (i) $z+\bar{z}=|z|^2$ let $z=x+iy$



$z+\bar{z}=|z|^2$ becomes

$x+iy+(x-iy)=\left(\sqrt{x^2+y^2}\right)^2$

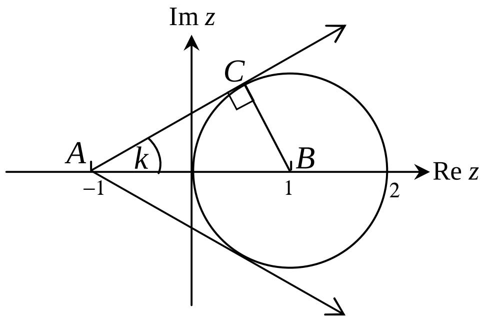
$2x=x^2+y^2$ and hence $x\geq 0$

$x^2-2x+y^2=1$

$(x-1)^2+y^2=1$

This is circle with centre $(1,0)$, radius 1 as illustrated in the diagram above.

(a) (ii) $\arg(z+1)$ is the angle made with the positive real by vectors with tails at -1.
For there to be at least one solution to the pair of equations, these vectors must intersect the circle obtained in (i) above. The case when there is only one solution is illustrated below.



Now AB = 2 and CB = 1

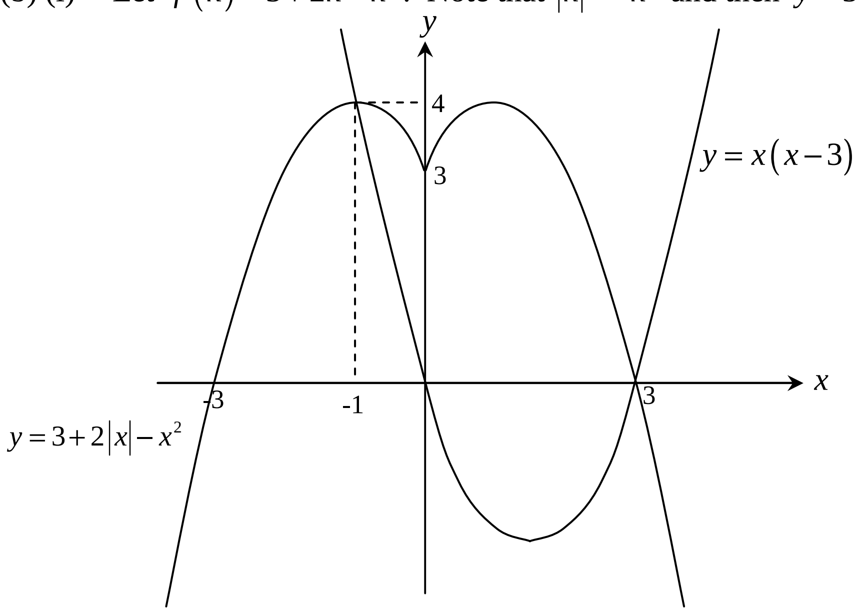
$\angle ACB = 90°$ (angle between tangent and radius)

$\therefore \quad \sin k=\dfrac{1}{2}$

$k=\dfrac{\pi}{6}$

$\therefore$ for at least one solution $-\dfrac{\pi}{6}\leq k\leq\dfrac{\pi}{6}$

(b) (i) Let $f(x)=3+2x-x^2$. Note that $|x|^2=x^2$ and then $y=3+2|x|-x^2=f(|x|)$



(b) (ii) If $\dfrac{3+2|x|-x^2}{x(x-3)}\leq 1$ then the functions must have different signs (for then the quotient will be negative) or, if they have the same sign, then $y=3+2|x|-x^2$ must lie closer the x-axis than $y=x(x-3)$. Also, $x\neq 0,3$. These conditions are all met when: $x\leq -1$, $x>0$ and $x\neq 3$.

End of Solutions