NORTH SYDNEY GIRLS HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 2 Term 1 2013

Name:_______________________ Mathematics Class:________

Time Allowed: 50 minutes + 2 minutes reading time

Available Marks: 40

Instructions:

- Questions are *not* of equal value.
- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1-2	3-5	6	7a,b	7c	8a	8b,c	Total
E4	/2			/8		/2		/12
E6		/3	/10					/13
E9					/4		/11	/15
								/40

Section 1

Multiple Choice Section (5 marks)

Answer **all** the questions for the multiple choice on the answer sheet provided for **Section I**.
Choose the **best** response that is correct for the question.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is provided for any question.

1 The polynomial $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a zero of multiplicity 3 and $x = i$ as another zero. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

(A) $P(x) = (x+1)^3(x-1)(x+1)$

(B) $P(x) = (x-1)^3(x-1)(x+1)$

(C) $P(x) = (x+1)^3(x-i)(x+i)$

(D) $P(x) = (x-1)^3(x-i)(x+i)$

2 The polynomial $P(x) = 2x^4 + 3x^3 - 2x^2 + 7x - 3$ has zeros $\alpha,\ \beta,\ \gamma$ and δ.
Which of the following polynomial equations have roots $\frac{1}{\alpha},\ \frac{1}{\beta},\ \frac{1}{\gamma}$ and $\frac{1}{\delta}$?

(A) $2x^4 - 3x^3 + x^2 - 5x - 4 = 0$

(B) $2x^4 + 3x^3 + x^2 - 5x - 4 = 0$

(C) $3x^4 - 7x^3 + 2x^2 - 3x - 2 = 0$

(D) $3x^4 + 7x^3 + 2x^2 - 3x - 2 = 0$

3 The asymptote(s) of $y = \dfrac{x^3 + 4x^2 - 6}{x^2 + 4}$ is / are?

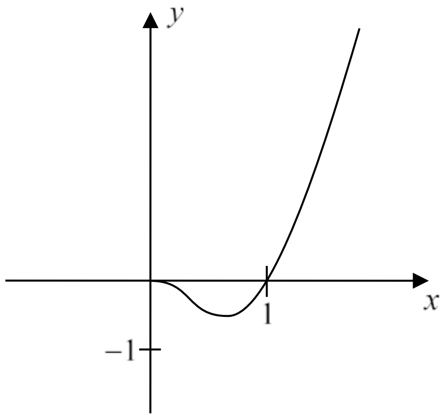
(A) $y = x^2 + 4$

(B) $y = x + 4$

(C) $x = \pm 2$

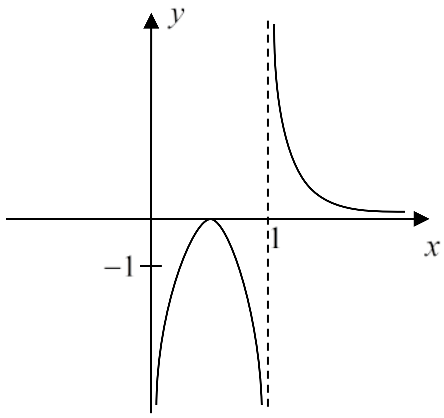
(D) $y = x - 4$

4 The diagram shows the graph of a function $y = f(x)$.

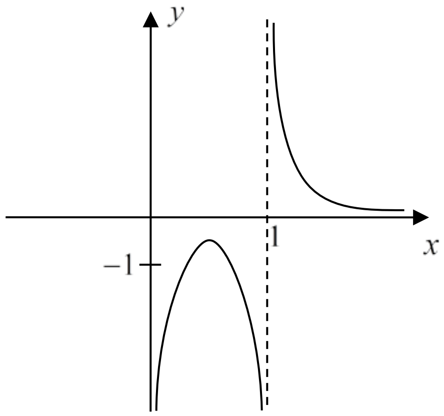


Which of the following could be the graph of $y = \dfrac{1}{f(x)}$?

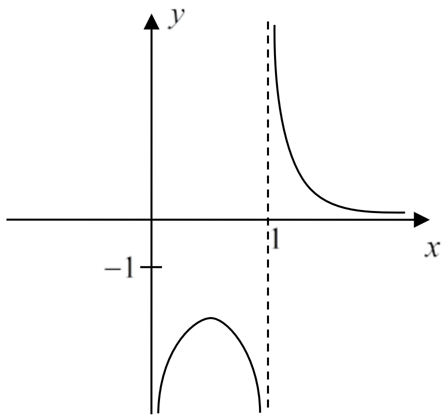
(A)



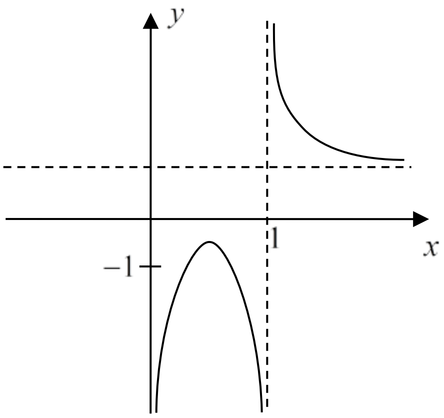
(B)



(C)

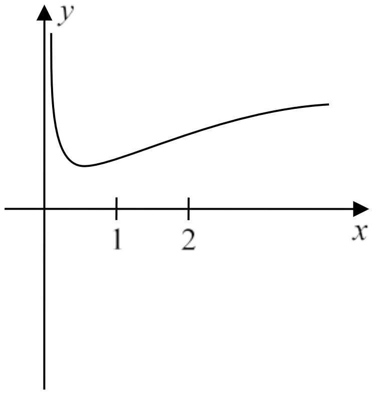


(D)

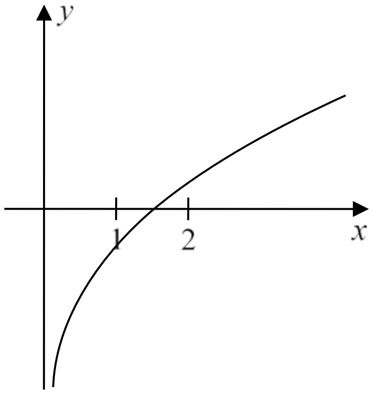


5 Which of the following could be a sketch of $y = \frac{1}{x} + \log_2 x$?

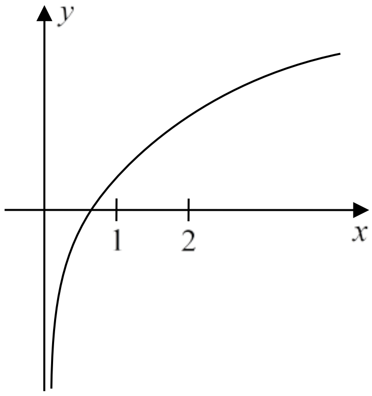
(A)



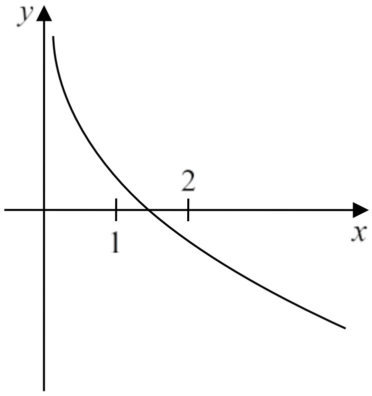
(B)



(C)



(D)

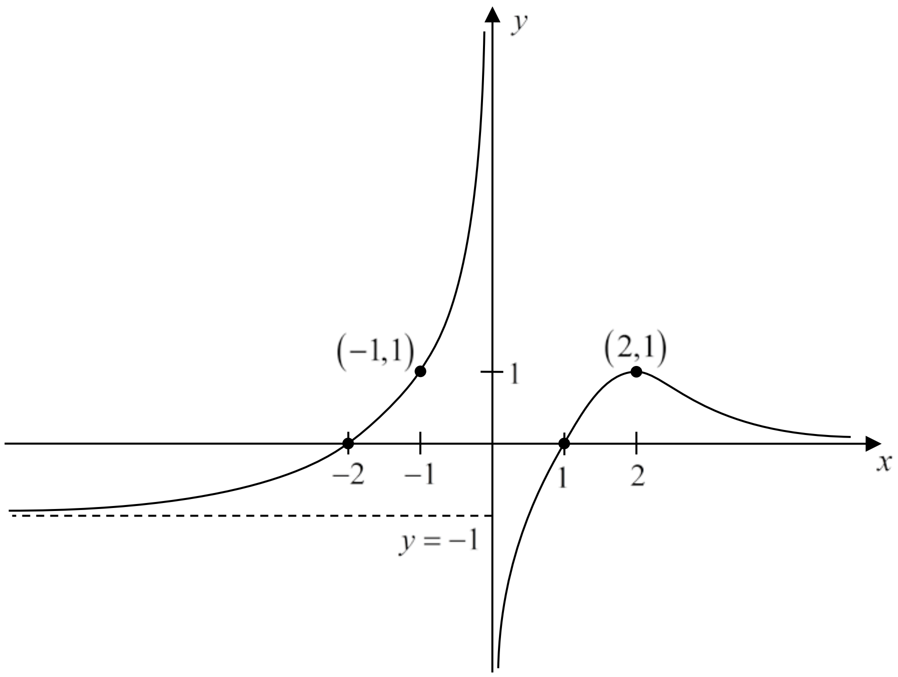


End of Section I

Section 2 **Free Response Section**

Question 6. (10 marks)

(a)



The diagram shows a sketch of $y = f(x)$ with asymptotes at $x = 0$, $y = -1$ and $y = 0$. There is a maximum turning point at $(2,1)$.

Sketch on the answer sheet provided, the graphs of

(i) $y = |f(x)|$ **1**

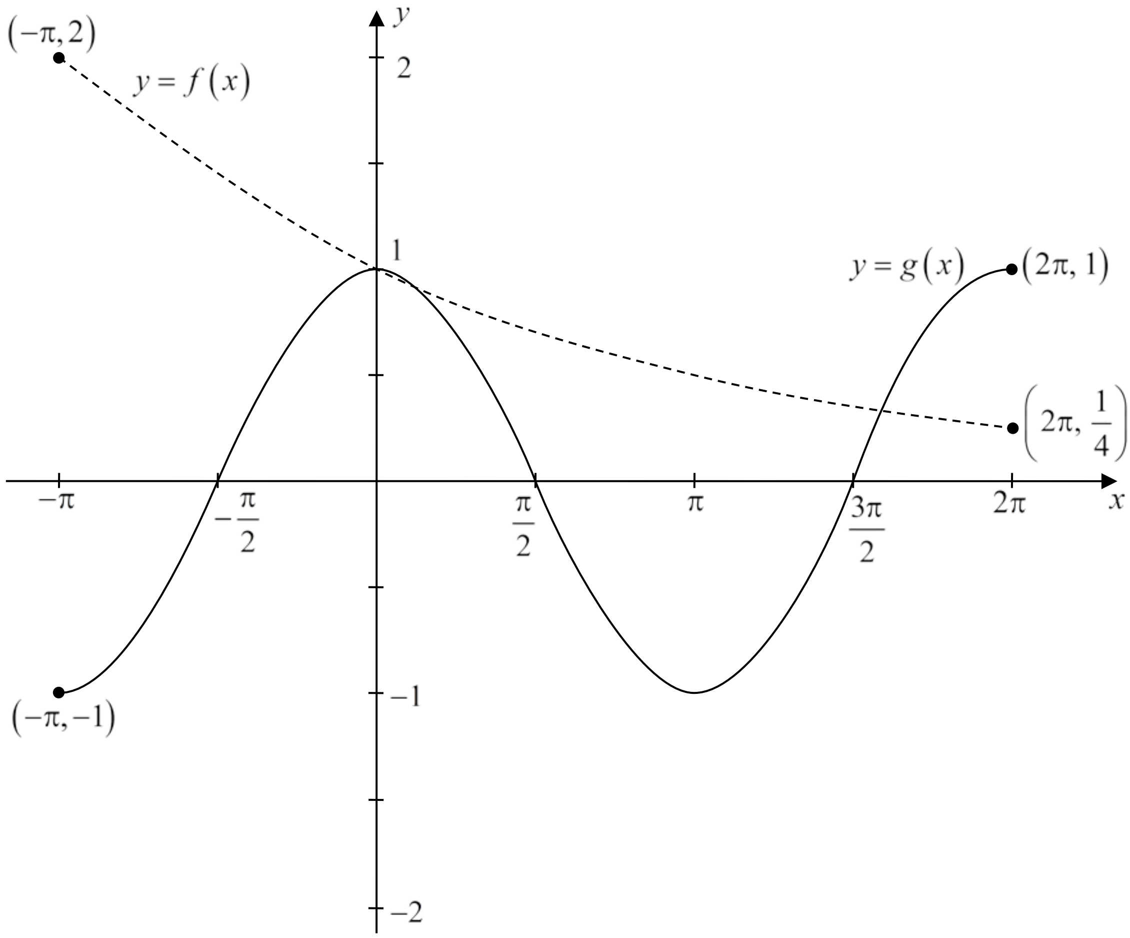
(ii) $y = f(-x)$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

Show any new asymptotes and their equations, and show the coordinates of any new points which can be determined.

(b) The diagram below shows the graphs of $y = f(x)$ and $y = g(x)$ for $-\pi \le x \le 2\pi$.



(i) On the answer sheet provided, draw a neat sketch of $y = f(x) \times g(x)$. **2**

(ii) How many real solutions exist for $f(x) \times g(x) - \sqrt[3]{x} = 0$ in this domain? **1**

<u>Question 7.</u> (12 marks) START A NEW BOOKLET

(a) The polynomial $P(x) = x^4 + 3x^3 - x^2 - 13x - 10$ has a zero at $x = -2 - i$.

(i) Explain why $x = -2 + i$ is also a zero. **1**

(ii) Fully factorise $P(x)$ over real numbers. **2**

(b) The roots of the equation $x^3 - 3x^2 + 9 = 0$ are α, β and γ.

(i) Determine the polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence evaluate $\alpha^3 + \beta^3 + \gamma^3$. **3**

(c) Suppose b and d are real numbers and $d \neq 0$. Consider the polynomial

$P(z) = z^4 + bz^2 + d$. The polynomial has a double zero α.

(i) Show that $-\alpha$ is also a double zero. **2**

(ii) Show that $b^2 = 4d$. **2**

Question 8. (13 marks) START A NEW BOOKLET

(a) Find real numbers a, b and c such that

$$\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}.$$ **2**

(b) Consider the curve with equation $3x^2 + 3y^2 + 2xy = 24$

(i) Show that $\frac{dy}{dx} = -\left(\frac{y+3x}{3y+x}\right)$. **2**

(ii) Find the x-coordinates of the point(s) on the curve with horizontal tangents. **2**

(c) Given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$,

(i) Solve the equation $16x^4 - 20x^2 + 5 = 0$ leaving your answers in the form $\cos\alpha$. **3**

(ii) Hence show that $\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} = \frac{5}{4}$. **2**

(iii) Also show that $\cos^2\frac{\pi}{10} = \frac{5+\sqrt{5}}{8}$. **2**

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End of test

NORTH SYDNEY GIRLS HIGH SCHOOL
HSC Assessment Task

Mathematics Extension 2

Answer Booklet – Term 1 2013

Questions 1 to 6

Name: ____________________________

Class: 12MZ_____

Instructions

- Use a separate Writing Booklet to answer each question.
- You may ask for an extra Writing Booklet to answer a question if you need more space.
- If you have not attempted a question, you must still hand in a Writing Booklet, with the words 'NOT ATTEMPTED' written clearly on the front cover.
- Write the number of each question part inside the margin at the beginning of each answer.
- Write using blue or black pen.
- Write on the ruled pages only.

You may NOT take any Writing Booklets, used or unused, from the room.

For marker use only

Q1-2	Q3-5	Q6
/2	/3	/10

Section 1 **Multiple Choice answer sheet.**

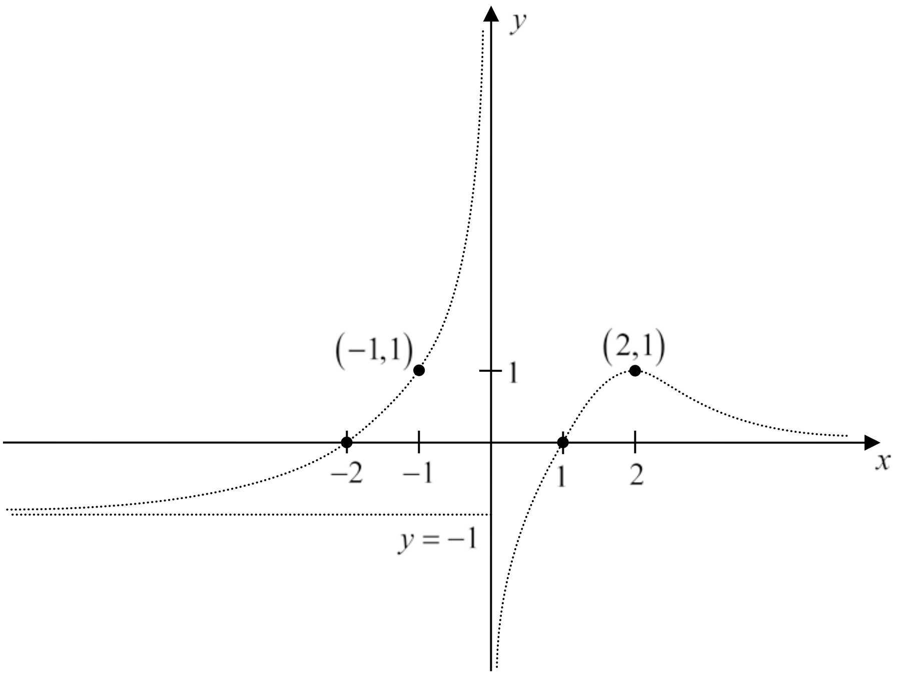
Completely colour the cell representing your answer. Use pencil only.

1. (A) (B) (C) (D)
2. (A) (B) (C) (D)
3. (A) (B) (C) (D)
4. (A) (B) (C) (D)
5. (A) (B) (C) (D)

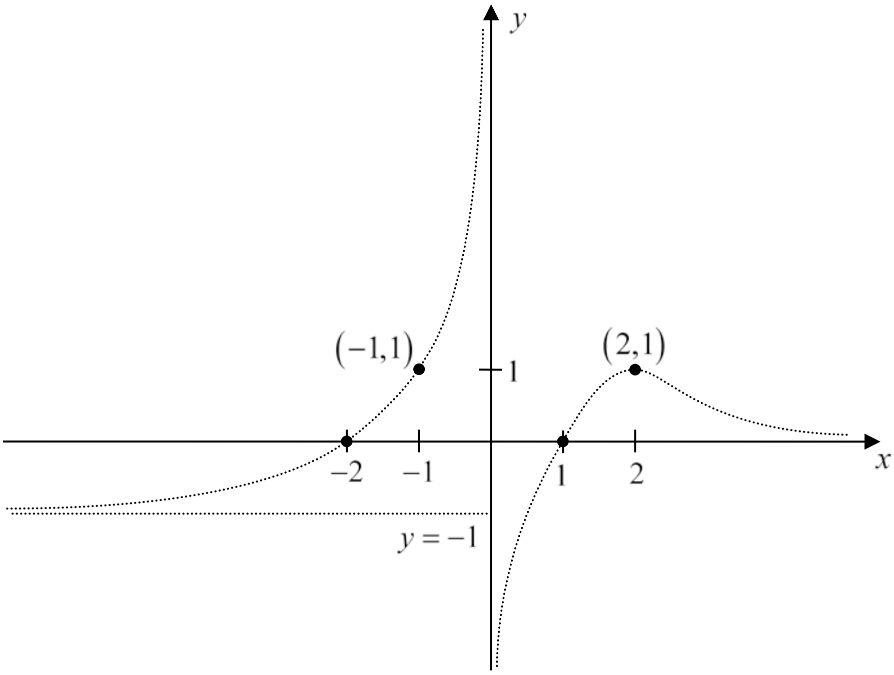
Section 2

Question 6. (a)

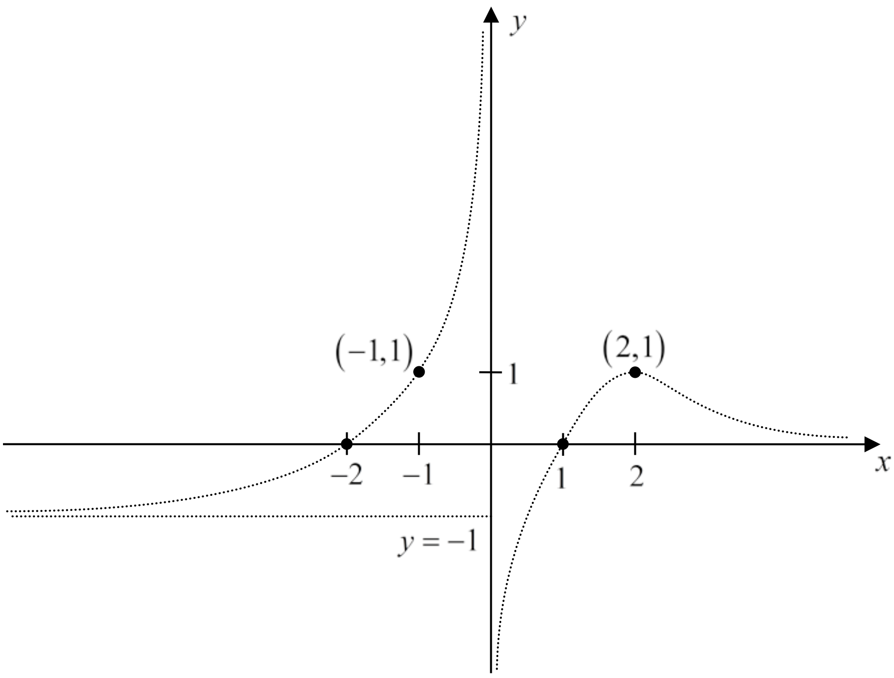
(i) $y = |f(x)|$



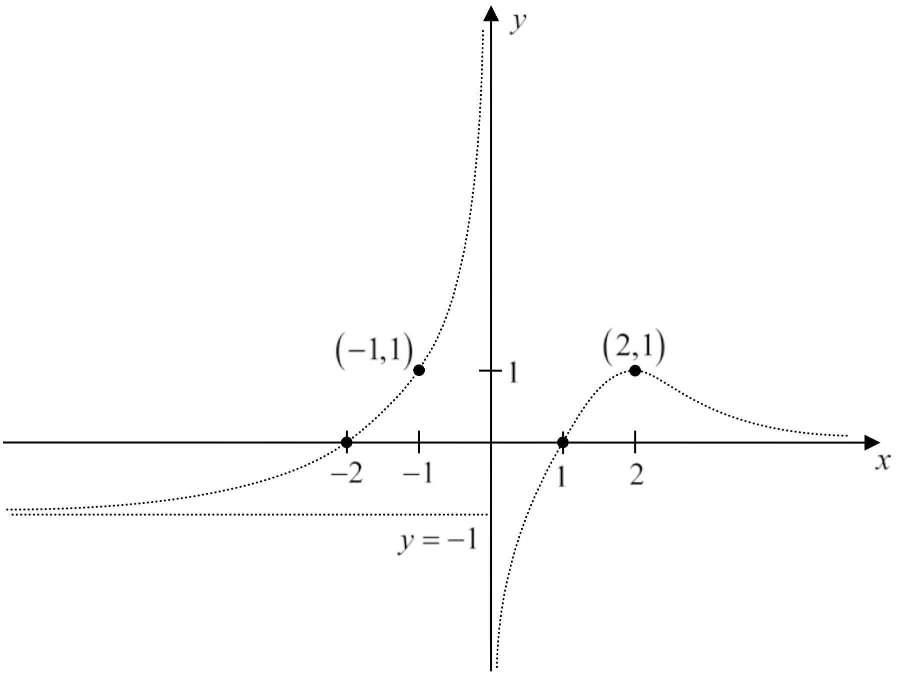
(ii) $y = f(-x)$



(iii) $y^2 = f(x)$

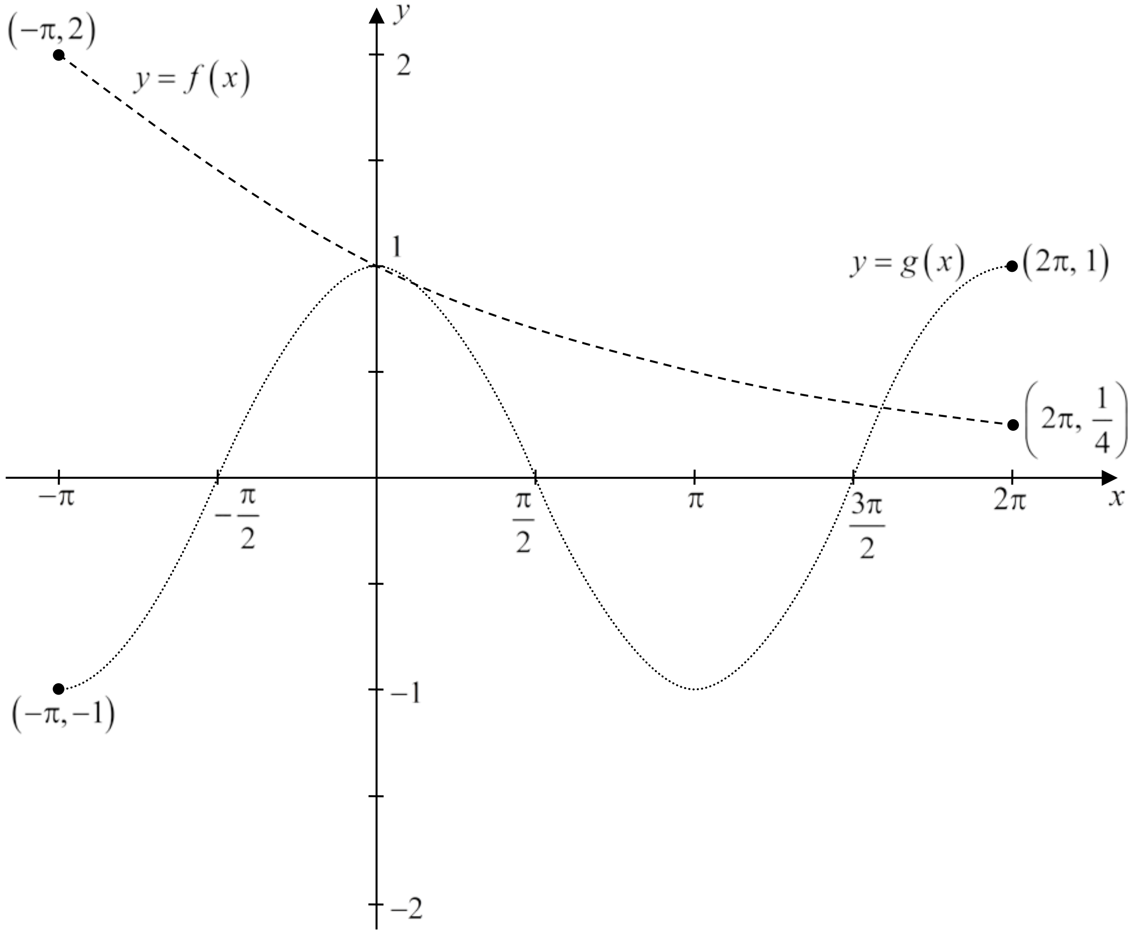


(iv) $y = e^{f(x)}$



Question 6. (b)

(i)



(ii) ______________________________

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Section 1 **Multiple Choice answer sheet.**

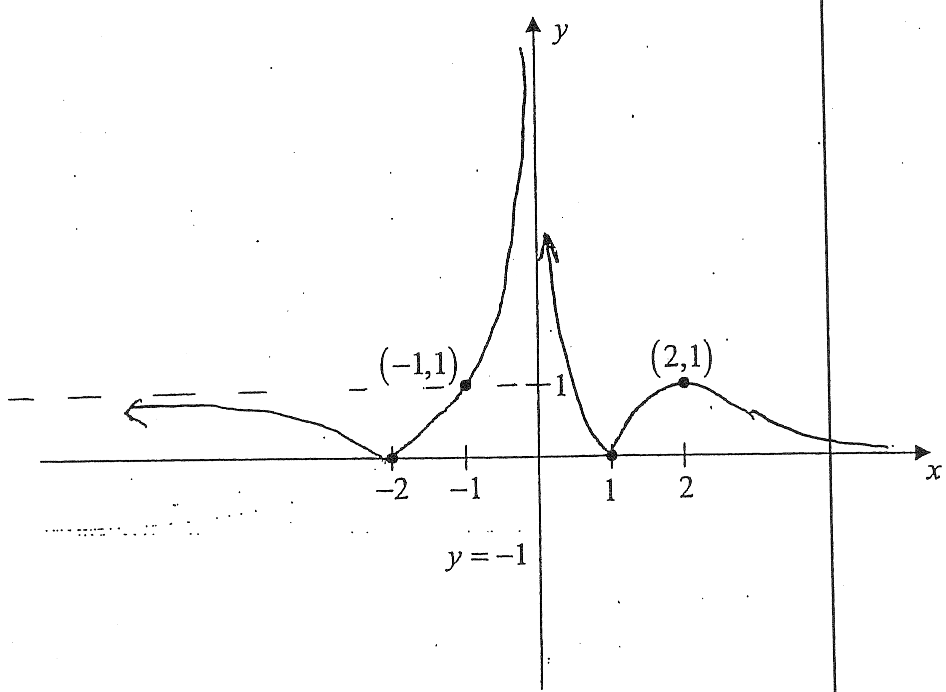
Completely colour the cell representing your answer. Use pencil only.

1. (A) (B) (C) (D)
2. (A) (B) (C) (D)
3. (A) (B) (C) (D)
4. (A) (B) (C) (D)
5. (A) (B) (C) (D)

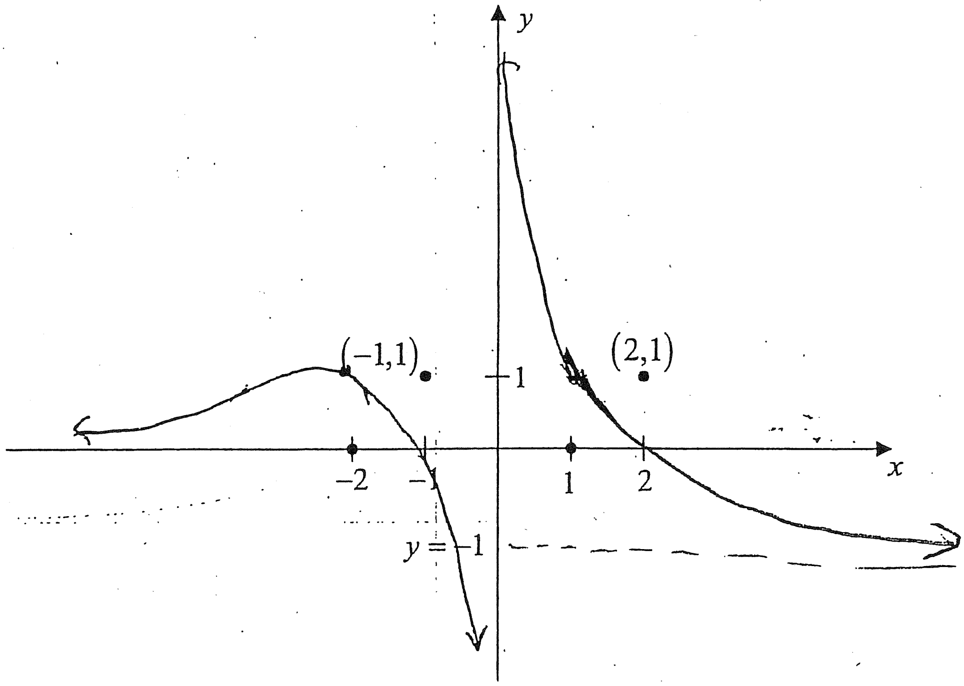
Section 2

Question 6. **(a)**

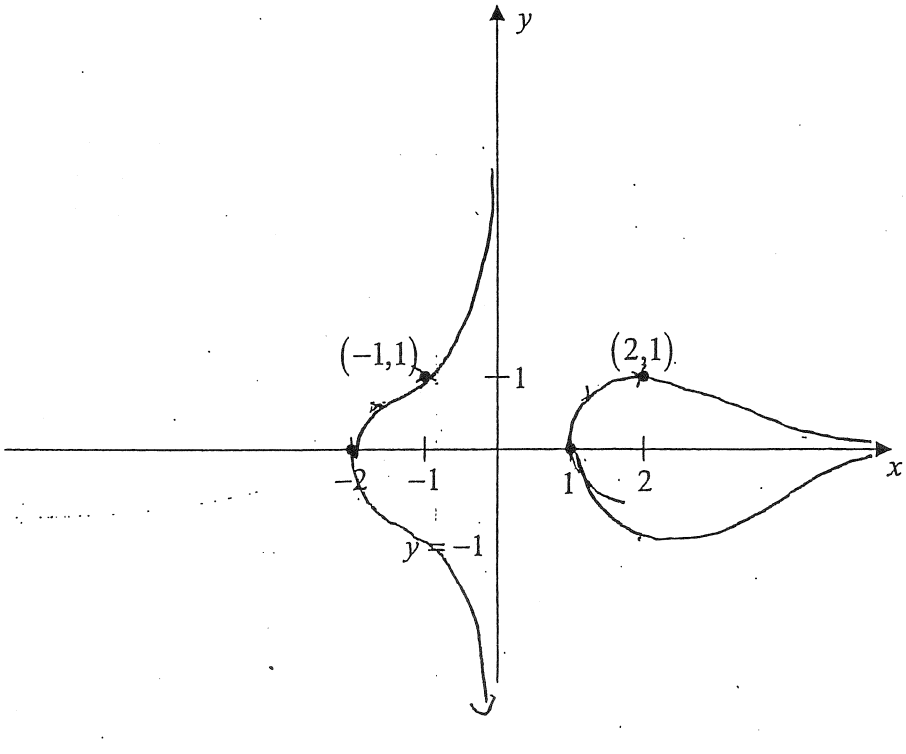
(i) $y = |f(x)|$



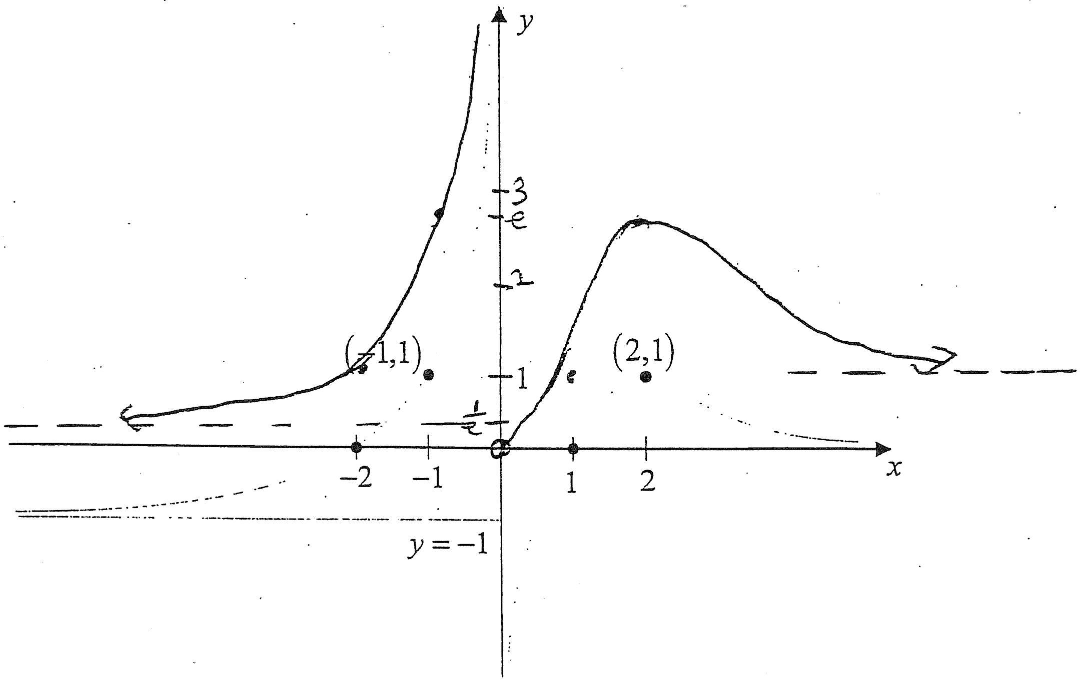
(ii) $y = f(-x)$



(iii) $y^2 = f(x)$

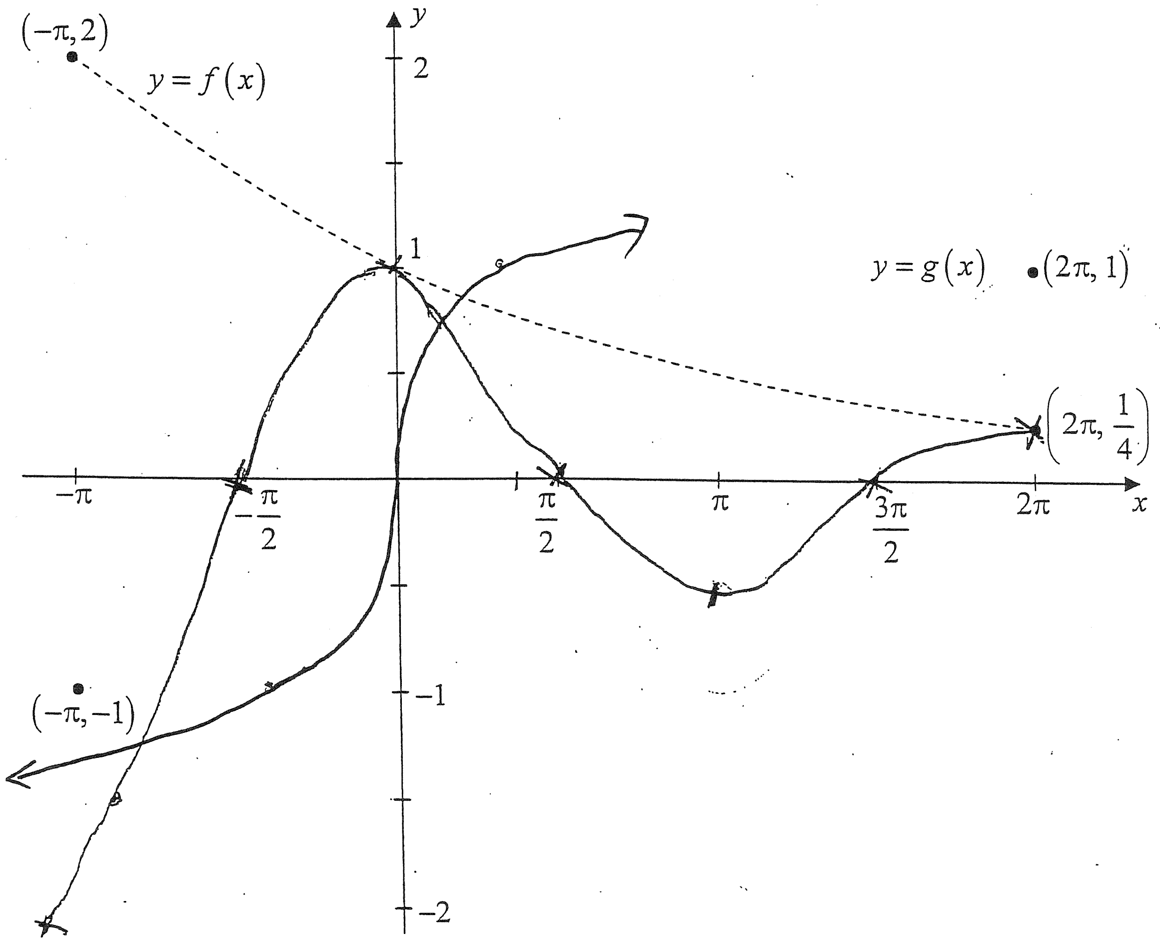


(iv) $y = e^{f(x)}$



Question 6. **(b)**

(i)



(ii) 2 solutions (points of intersection

March 2013 Assessment Task 2 Extension 2 Solutions

Section 1

1.

$x = i$ is a zero
$\therefore x = i$ is a zero
$\therefore \left(x^2+1\right)$ a factor.
constant term $= -1$
$\therefore$ constant term of
multiple factor $= -1$
i.e. D

2.

Roots satisfy

$$P\left(\frac{1}{x}\right) = \frac{2}{x^4} + \frac{3}{x^3} - \frac{2}{x^2} + \frac{7}{x} - 3 = 0$$

i.e. $2 + 3x - 2x^2 + 7x^3 - 3x^4 = 0$
$3x^4 - 7x^3 + 2x^2 - 3x - 2 = 0$
i.e. C

3.

On division,

$$y = \frac{x^3 + 4x^2 - 6}{x^2 + 4}$$

$$= x + 4 - \frac{4x + 22}{x^2 + 4}$$

Domain all real x
(no vertical asymptotes)
As $x \to \infty$, $y \to x + 4$
i.e. asymptote $y = x + 4$
i.e. B

4.

$x \to \infty$, $f(x) \to \infty$.

$$\therefore \frac{1}{f(x)} \to 0$$

$f(0.5) \approx -0.5$

$$\therefore \frac{1}{f(0.5)} \approx -2$$

i.e. C

5.

As $x \to \infty$, $y \to \infty$
As $x \to 0$, $y \to \infty$
(let $x = 2^{-30}$)
i.e. A

Section 2

Question 6 see attached sheet

Question 7

(a) (i) Since all the coefficients are real the complex roots occur as conjugate pairs.
i.e. $\bar{\alpha} = \left(-2+i\right)$ is also a zero.

(ii)

$x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$ is a factor
i.e. $x^2 + 4x + 5$ is a factor
$P(x) = x^4 + 3x^3 - x^2 - 13x - 10$
$= (x^2 + 4x + 5)(x^2 - x - 2)$
on comparing coefficients.
i.e. $P(x) = (x^2 + 4x + 5)(x - 2)(x + 1)$
for reals.

(b) (i)

Required equation has the same roots as $P(\sqrt{x}) = 0$.

$P(\sqrt{x}) = (\sqrt{x})^3 - 3(\sqrt{x})^2 + 9$
$\therefore P(\sqrt{x}) = x\sqrt{x} - 3x + 9$
$x\sqrt{x} - 3x + 9 = 0$
$x\sqrt{x} = 3x - 9$
$x^3 = 9x^2 - 54x + 81$
$\therefore$ polynomial equation is
$x^3 - 9x^2 + 54x - 81 = 0$

(ii)

$\alpha^2, \beta^2, \gamma^2$ are the solutions to (i)

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a} = -\frac{-9}{1} = 9$$

α, β, γ satisfy original equation.

$\alpha^3 - 3\alpha^2 + 9 = 0$
$\beta^3 - 3\beta^2 + 9 = 0$
$\gamma^3 - 3\gamma^2 + 9 = 0$
On adding,
$\alpha^3 + \beta^3 + \gamma^3 - 3(\alpha^2 + \beta^2 + \gamma^2) + 27 = 0$
$\alpha^3 + \beta^3 + \gamma^3 - 3 \times 9 + 27 = 0$
$\therefore \alpha^3 + \beta^3 + \gamma^3 = 0$

(c) (i)

March 2013 Assessment Task 2 Extension 2 Solutions

$P(z) = z^4 + bz^2 + d$

$P'(z) = 4z^3 + 2bz$

α a double root.

$\therefore P(\alpha) = \alpha^4 + b\alpha^2 + d = 0$

$P'(\alpha) = 4\alpha^3 + 2b\alpha = 0$

Test for $z = -\alpha$

$P(-\alpha) = (-\alpha)^4 + b(-\alpha)^2 + d$

$= \alpha^4 + b\alpha^2 + d = 0$

$P'(-\alpha) = 4(-\alpha)^3 + 2b(-\alpha)$

$= -4\alpha^3 + 2b\alpha$

$= -(4\alpha^3 + 2b\alpha) = 0$

$\therefore P(-\alpha) = P'(-\alpha) = 0$

$\therefore z = -\alpha$ is a double root.

(ii)

$z^4 + bz^2 + d = (x-\alpha)^2(x+\alpha)^2$

$= (x^2 - \alpha^2)^2$

$= x^4 - 2\alpha^2 x + \alpha^4$

Comparing coefficients,

$b = -2\alpha^2 \quad i.e.\ \alpha^2 = \dfrac{-b}{2}$

$d = \alpha^4$

$\therefore d = \left(\dfrac{-b}{2}\right)^2$

$i.e.\ b^2 = 4d$

Question 8.

(a)

$$\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

$4x+3 \equiv (x+2)(ax+b) + c(x^2+1)$

coefficient $x^2 \rightarrow a + c = 0$

$x = -2 \rightarrow -8 + 3 = 5c$

$\therefore c = -1$

$\therefore a = 1$

$x = 0 \rightarrow 3 = 2b + c$

$\therefore b = 2$

(b) (i)

$3x^2 + 3y^2 + 2xy = 24$

$6x + \dfrac{d}{dy}\left(3y^2\right)\dfrac{dy}{dx} + \dfrac{d}{dx}\left(2xy\right) = 0$

$6x + 6y\dfrac{dy}{dx} + 2y + 2x\dfrac{dy}{dx} = 0$

$\left(6y + 2x\right)\dfrac{dy}{dx} = -\left(6x + 2y\right)$

$\dfrac{dy}{dx} = \dfrac{-\left(6x+2y\right)}{\left(6y+2x\right)} = \dfrac{-\left(y+3x\right)}{\left(3y+x\right)}$

(ii)

For horizontal tangents

$\dfrac{dy}{dx} = 0$

i.e. $3x + y = 0$

$y = -3x$

Sub. into equation

$3x^2 + 3\left(-3x\right)^2 + 2x\left(-3x\right) = 24$

$3x^2 + 27x^2 - 6x^2 = 24$

$24x^2 = 24$

$x = \pm 1$

Sub. into $y = -3x$

Points $\left(1,-3\right), \left(-1,3\right)$

(c) (i)

$16x^4 - 20x^2 + 5 = 0$

are the solutions to

$16x^5 - 20x^3 + 5x = 0, \quad x \neq 0$

$16x^5 - 20x^3 + 5x = 0$....(*)

let $x = \cos\theta$

(*) becomes

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

i.e. $\cos 5\theta = 0$

$\therefore 5\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \dfrac{9\pi}{2}$.......

$\theta = \dfrac{\pi}{10}, \dfrac{3\pi}{10}, \dfrac{5\pi}{10}, \dfrac{7\pi}{10}, \dfrac{9\pi}{10}$.....

$x = \cos\theta = \cos\dfrac{\pi}{10}, \cos\dfrac{3\pi}{10}, \cos\dfrac{5\pi}{10},$

$\cos\dfrac{7\pi}{10}, \cos\dfrac{9\pi}{10}$ for 5 solutions

$\cos\dfrac{5\pi}{10} = 0$

$\therefore$ solutions to

$16x^4 - 20x^2 + 5 = 0$ are

$x = \cos\dfrac{\pi}{10}, \cos\dfrac{3\pi}{10}, \cos\dfrac{7\pi}{10}, \cos\dfrac{9\pi}{10}$

March 2013 Assessment Task 2 Extension 2 Solutions

(ii)

$\cos\frac{9\pi}{10} = -\cos\frac{\pi}{10}$

$\cos\frac{7\pi}{10} = -\cos\frac{3\pi}{10}$

Solutions can be written

Sum of roots in pairs:

$x = \pm\cos\frac{\pi}{10}, \pm\cos\frac{\pi}{10} = \pm\alpha, \pm\beta$

$-\alpha^2 + \alpha\beta - \alpha\beta - \alpha\beta + \alpha\beta - \beta^2 = \frac{c}{a}$

$-\alpha^2 - \beta^2 = \frac{-20}{16}$

$\therefore \cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10} = \frac{5}{4}$

(iii)

$16x^4 - 20x^2 + 5 = 0$

is a quadratic in x^2

$x^2 = \frac{20 \pm \sqrt{20^2 - 4\times16\times5}}{32}$

$= \frac{20 \pm \sqrt{80}}{32}$

$= \frac{5 \pm \sqrt{5}}{8}$

These solutions must be

$\cos^2\frac{\pi}{10}, \cos^2\frac{3\pi}{10}.$

Since $\cos\frac{\pi}{10} > \cos\frac{3\pi}{10} > 0,$

then $\cos^2\frac{\pi}{10} = \frac{5+\sqrt{5}}{8}$

Note: Part (ii) could have been done by considering the sum of the roots of the quadratic.