PETRUS KY COLLEGE
NEW SOUTH WALES

in partnership with

VIETNAMESE COMMUNITY IN AUSTRALIA
NSW CHAPTER

JULY 2007

MATHEMATICS EXTENSION 2

PRE-TRIAL TEST

HIGHER SCHOOL CERTIFICATE (HSC)

Student Number: | | | | |

Student Name: ______________________________

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Write your Student Number and your Name on all working booklets

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

Marks

Question 1 **12**

(i) $\int \frac{3x^2-6x+1}{(x-3)(x^2+1)}dx$ 2

(ii) $\int_0^1 x.\tan^{-1}x.dx$ 2

(iii) $\int_0^{\pi/2} \sqrt{1+\sin 2x}.dx$ 2

(iv) If $I_n = \int_0^{\pi/2} \frac{\cos(2n+1)\theta}{\cos\theta}d\theta$, show that 4

$I_n + I_{n-1} = 0$ for $n \geq 1$. Hence find the value of I_n for $n \geq 0$

(v) $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}dx$ 2

Question 2 **12**

(A) Express $Z=\sqrt{3}+i$ and $W=1+i$ in the MOD-ARG forms and hence evaluate $\dfrac{Z^{20}}{Z^{30}}$ in the form $a+bi$ 2

(B) If $Z=i-1$, show clearly on an Argand diagrams all the points representing the complex numbers. 2

$$Z, Z^2, Z^3, Z^{-1}, \sqrt{2}.Z, -Z, \bar{Z}, iZ, Z^2-Z, \sqrt{Z}$$

(C) Simplify $Z=\dfrac{1+cos2\theta+i\sin 2\theta}{1+cos2\theta-i\sin 2\theta}$ 2

Hence show that $Z^n=cos2n\theta+i\sin 2n\theta$

(D) Express $cos6\theta$ as a polynomial in terms of $cos\theta$ hence show that 4

$\cos\dfrac{\pi}{12}$, $\cos\dfrac{3\pi}{12}$, $\cos\dfrac{5\pi}{12}$, $\cos\dfrac{7\pi}{12}$, $\cos\dfrac{9\pi}{12}$ and $\cos\dfrac{11\pi}{12}$ are the roots of the equation $32x^6-48x^4+18x^2-1=0$

(E) If w is the complex cube root of unity, $z^3=1$ then simplify 2

$$\frac{1}{3+5w+3w^2}+\frac{1}{7+7w+9w^2}$$

Question 3 **12**

(A)

(i) If $P(x) = x^3 - 6x^2 + 9x + c$ for some real number c, find the value of x for which $P'(x) = 0$. 3

Hence find the values of c for which the equation $P(x)$ has a repeated root.

(ii) Sketch the graphs of $y = P(x)$ with this values of c, hence find the set of values of c for which the equation $P(x) = 0$ has only one real root. 3

(B) Show that the equation $\dfrac{x^2}{36-k} + \dfrac{y^2}{20-k} = 1$, where k is a real number , represents:

(i) an ellipse if $k < 20$ 2

(ii) a hyperbola if $20 < k < 36$ 2

(iii) Show that the foci of the ellipse in (i) or hyperbola in (ii) are independent of the value of k. 2

Question 4 **12**

(A)

(i) The normal at point $P\left(ct, \frac{c}{t}\right)$ on the hyperbola $xy = c^2$ cuts the line $y = x$ at Q. Find the co-ordinates of Q. 2

(ii) Show that OP = PQ and hence show that there is no point on the parabola for which the length of PQ is less than $c\sqrt{2}$ 4

(B) Two points P and Q lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Their parameters are given as θ and $\theta + \frac{\pi}{2}$.

(i) Show that Q has co-ordinates $(-a\sin\theta, b\cos\theta)$. Hence prove: 2

$OP^2 + OQ^2 = a^2 + b^2$

(ii) Find the locus of midpoint M of PQ. 2

(iii) If α is the acute angle between the 2 tangents at P and at Q, show that 2

$$\tan\alpha = \frac{2\sqrt{1-e^2}}{e^2.\sin 2\theta}$$

Question 5 **12**

(A) By using the division of two graphs, or otherwise, sketch the curve 2

$$y = \frac{3x}{x^2 - 4}$$

(B) Find the domain and range of curve $y = \cos^{-1}(e^x)$ and hence sketch the graph of $y = \cos^{-1}(e^x)$ 2

(C) Let $f(x) = (\sin x - \cos x)^2$, find the period and range of $f(x)$, hence sketch the curve of $f(x)$ with $-\pi \le x \le \pi$. 4

From the separated graph, sketch the following curve.

(i) $y = \dfrac{1}{f(x)}$ 1

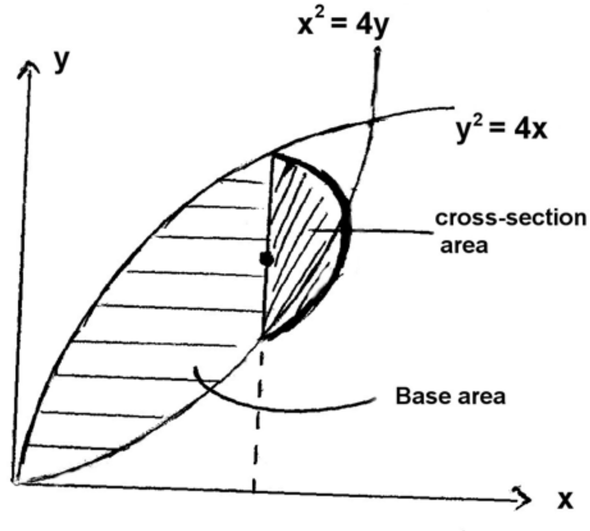
(ii) $y = \sqrt{f(x)}$ 1

(iii) $y = \ell n\left[f(x) \right]$ 1

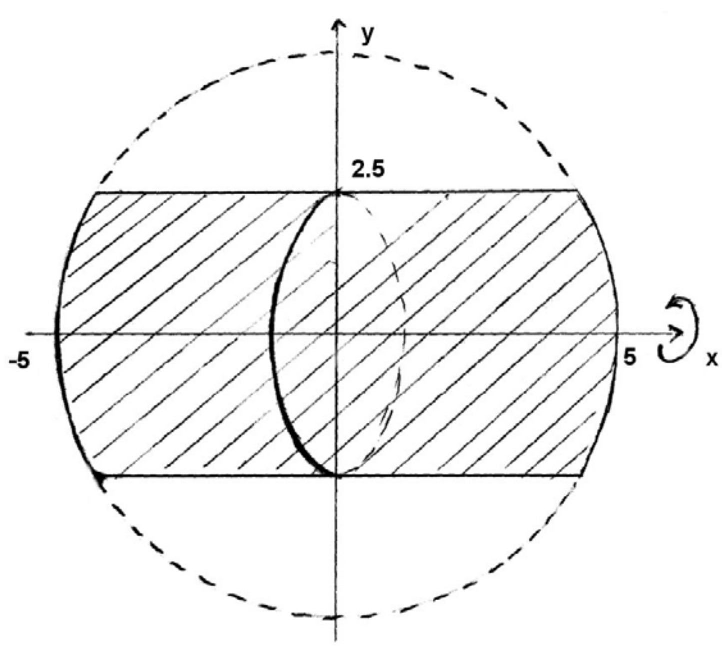
(iv) $y = f\left(|x| \right)$ 1

Question 6 **12**

(A) The base of a certain solid is the region bounded by the curves $y^2 = 4x$ and $x^2 = 4y$, and its cross-sections by planes perpendicular to the x-axis are semi circles. Find the volume of the solid. 6



(B) The area bounded by 2 arcs and 2 chords of a circle as shown in the figure below, is let to rotate about the x-axis. Find the volume of the solid shape. 6



Question 7 **12**

(A) Mice are placed in the centre of a maze which has 5 exits. Each mouse is equally likely to leave the maze through any one of the 5 exits. Four mice A, B, C, D are put into the maze and behave independently.

(i) Find the probability that A, B, C, D all come out the same exit. 1

(ii) What is the probability that A, B and C come out the same exit and D comes out a different exit. 1

(iii) What is the probability that any 3 of 4 mice come out the same exit and the other comes out a different exit. 1

(iv) What is the probability that no more than 2 mice come out the same exit. 1

(B) If $\mu_1 = 1$ and $\mu_n = \sqrt{3 + 2\mu_{n-1}}$ for $n \geq 2$

(i) show that $\mu_n < 3$ for $n \geq 1$ 2

(ii) deduce that $\mu_{n+1} > \mu_n$ for $n \geq 1$ 2

(C) By using induction method , prove that $3^{4n+2} + 2.4^{3n+1}$ is divisible by 17 for $n \geq 1$ 4

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Question 1 **12**

(i) $\int \frac{3x^2-6x+1}{(x-3)(x^2+1)}dx$ 2

$$\int \frac{3x^2-6x+1}{(x-3)(x^2+1)}\,dx = \int \frac{3x^2-6x+1}{x^3-3x^2+x-3}\,dx$$

$$\boxed{I = \log_e(x^3-3x^2+x-3)+C}$$

(ii) $\int_0^1 x.\tan^{-1}x.dx$ 2

$$\int_0^1 x.\tan^{-1}x\,dx = \left[\frac{x^2}{2}.\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\,dx$$

$$= \frac{1}{2}\times\frac{\pi}{4} - \frac{1}{2}\int_0^1 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right)$$

$$\boxed{I = \frac{\pi}{4} - \frac{1}{2}}$$

(iii) $\int_0^{\pi/2}\sqrt{1+\sin 2x}.dx$ 2

$$\int_0^{\pi/2}\sqrt{1+\sin 2x}\,.dx = \int_0^{\pi/2}\sqrt{\sin^2 x+\cos^2 x+2\sin x.\cos x}\;dx$$

$$= \int_0^{\pi/2}\sqrt{(\sin x+\cos x)^2}\,dx$$

$$= \left[-\cos x + \sin x\right]_0^{\pi/2}$$

$$\boxed{I = 2}$$

(iv) If $I_n = \int_0^{\pi/2} \frac{\cos(2n+1)\theta}{\cos\theta} d\theta$, show that 4

$I_n + I_{n-1} = 0$ for $n \geq 1$. Hence find the value of I_n for $n \geq 0$

$$I_n = \int_0^{\pi/2} \frac{\cos(2n+1)\theta}{\cos\theta} d\theta$$

$$I_n + I_{n-1} = \int_0^{\pi/2} \frac{\cos(2n+1)\theta + \cos(2n-1)\theta}{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{2\cos\left(\frac{2n+1+2n-1}{2}\right)\theta \cdot \cos\left(\frac{2n+1-2n+1}{2}\right)\theta}{\cos\theta} d\theta$$

$$= 2\int_0^{\pi/2} \frac{\cos 2n\theta \cdot \cancel{\cos\theta}}{\cancel{\cos\theta}} d\theta$$

$$= \left[\frac{2}{2n} \sin 2n\theta\right]_0^{\pi/2} = 0$$

Hence : $I_0 = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta} d\theta = \frac{\pi}{2}$

Then : $I_1 = -I_0 = -\frac{\pi}{2}$

$I_2 = -I_1 = \frac{\pi}{2}$

so on, we deduce $\boxed{I_n = (-1)^n \cdot \frac{\pi}{2}}$

(v) $\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}} dx$ 2

$$\int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}} dx$$

Let $x = \sec\theta$, $dx = \sec\theta . \tan\theta . d\theta$

when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$

$x = 2$, $\theta = \frac{\pi}{3}$

$$I = \int_{\pi/4}^{\pi/3} \frac{\sec\theta . \tan\theta . d\theta}{\sec\theta . \sqrt{\sec^2\theta - 1}} = \int_{\pi/4}^{\pi/3} d\theta = \frac{\pi}{3} - \frac{\pi}{4}$$

$$\boxed{I = \frac{\pi}{12}}$$

Question 2 **12**

(A) Express $Z=\sqrt{3}+i$ and $W=1+i$ in the MOD-ARG forms and hence evaluate $\dfrac{Z^{20}}{W^{30}}$ in the form $a+bi$ 2

$z = \sqrt{3} + i = 2 \operatorname{cis} \frac{\pi}{6}$

$w = 1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

Then

$$\frac{z^{20}}{w^{30}} = \frac{2^{20} \operatorname{cis} \frac{20\pi}{6}}{(\sqrt{2})^{30} \operatorname{cis} \frac{30\pi}{4}} = 2^5 \operatorname{cis}\left(\frac{10}{3} - \frac{15}{2}\right)\pi$$

$$= 2^5 \operatorname{cis}\left(-\frac{25}{6}\right)\pi = 2^5 \operatorname{cis}\left(-4\pi - \frac{\pi}{6}\right)$$

$$= 2^5\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

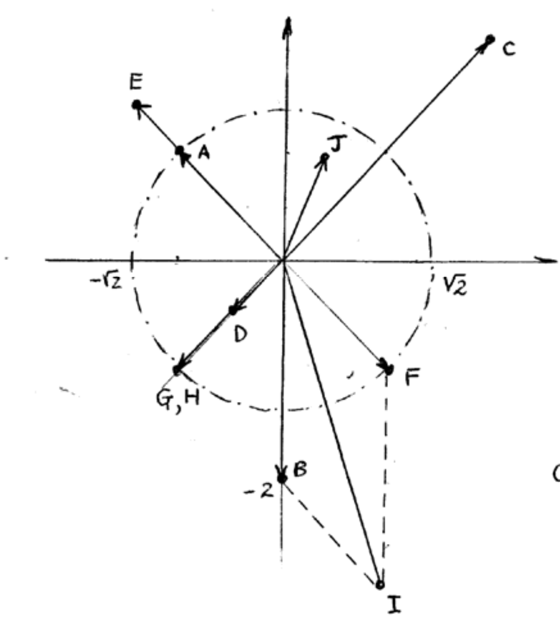
$$= 32\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$\boxed{\frac{z^{20}}{w^{30}} = 16\sqrt{3} - 16i}$$

(B) If $Z=i-1$, show clearly on an Argand diagrams all the points representing the complex numbers. 2

$$Z, Z^2, Z^3, Z^{-1}, \sqrt{2}.Z, -Z, \bar{Z}, iZ, Z^2-Z, \sqrt{Z}$$

ARGAND diagram of $z = i - 1$



List of points :

A : z

B : z^2

C : z^3

D : z^{-1}

E : $\sqrt{2}z$

F : $-z$

G, H : $\bar{z}$, iz

I : $z^2 - z$

J : $\sqrt{z}$

(C) Simplify $Z = \dfrac{1+cos2\theta + i\sin 2\theta}{1+cos2\theta - i\sin 2\theta}$ 2

Hence show that $Z^n = cos2n\theta + i\sin 2n\theta$

Simplify : $z = \dfrac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}$

$= \dfrac{2\cos^2\theta + 2i\cos\theta.\sin\theta}{2\cos^2\theta - 2i\cos\theta.\sin\theta}$

$= \dfrac{2\cancel{\cos\theta}(\cos\theta + i\sin\theta)}{2\cancel{\cos\theta}(\cos\theta - i\sin\theta)}$

$= \dfrac{(\cos\theta + i\sin\theta)}{(\cos\theta - i\sin\theta)} \times \dfrac{(\cos\theta + i\sin\theta)}{(\cos\theta + i\sin\theta)}$

$z = (\cos\theta + i\sin\theta)^2$

$\therefore$ By De Moivres' theorem $\boxed{z = \cos 2\theta + i\sin 2\theta}$

Therefore $z^n = (\cos 2\theta + i\sin 2\theta)$

$\boxed{z^n = \cos 2n\theta + i\sin 2n\theta}$

(D) Express $cos6\theta$ as a polynomial in terms of $cos\theta$ hence show that 4

$\cos\dfrac{\pi}{12}$, $\cos\dfrac{3\pi}{12}$, $\cos\dfrac{5\pi}{12}$, $\cos\dfrac{7\pi}{12}$, $\cos\dfrac{9\pi}{12}$ and $\cos\dfrac{11\pi}{12}$ are the roots of the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

Using binomial expansion

$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6i\cos^5\theta.\sin\theta + 15i^2\cos^4\theta.\sin^2\theta + 20i^3\cos^3\theta.\sin^3\theta$
$+ 15i^4\cos^2\theta.\sin^4\theta + 6i^5\cos\theta.\sin^5\theta + i^6\sin^6\theta$.

Equating the real terms :

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta.\sin^2\theta + 15\cos^2\theta.\sin^4\theta - \sin^6\theta$.

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta)$
$- (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$\boxed{\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}$

given equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$

Let $x = \cos\theta$, then the Left hand side of equation becomes

$\cos 6\theta = 0$

solving equation $\cos 6\theta = 0$

$$6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

Therefore, $\cos\frac{\pi}{12}$, $\cos\frac{3\pi}{12}$, $\cos\frac{5\pi}{12}$, $\cos\frac{7\pi}{12}$, $\cos\frac{9\pi}{12}$ and $\cos\frac{11\pi}{12}$ are the roots of the given equation.

(E) If w is the complex cube root of unity, $z^3 = 1$ then simplify 2

$$\frac{1}{3+5w+3w^2} + \frac{1}{7+7w+9w^2}$$

If w is one root of the equation $z^3 = 1$, then 1 and w^2 are other roots and $1 + w + w^2 = 0$

Simplify

$$A = \frac{1}{3+5w+3w^2} + \frac{1}{7+7w+9w^2}$$

$$= \frac{1}{3(1+w^2)+5w} + \frac{1}{7(1+w)+9w^2}$$

$$= \frac{1}{-3w+5w} + \frac{1}{-7w^2+9w^2}$$

$$= \frac{1}{2w} + \frac{1}{2w^2}$$

$$= \frac{w+1}{2w^2}$$

$$= \frac{-w^2}{2w^2}$$

$$\boxed{A = -\frac{1}{2}}$$

Question 3 **12**

(A)

(i) If $P(x) = x^3 - 6x^2 + 9x + c$ for some real number c, find the value of x for which $P'(x) = 0$. 3

Hence find the values of c for which the equation $P(x)$ has a repeated root.

$P(x) = x^3 - 6x^2 + 9x + c$

$P'(x) = 3x^2 - 12x + 9$

Let $P'(x) = 0$, $3(x^2 - 4x + 3) = 0$

$x = 1$ or 3

$P(x)$ has repeated root if :

(a) $P(1) = P'(1) = 0$

$\therefore$ $1 - 6 + 9 + c = 0$, $c = -4$

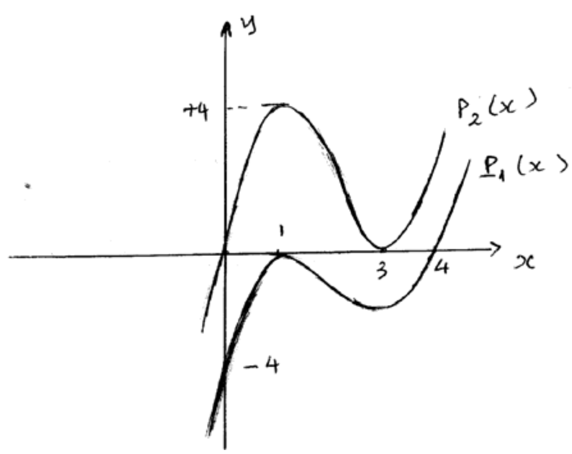
(b) $P(3) = P'(3) = 0$

$\therefore$ $27 - 54 + 27 + c = 0$, $c = 0$

(ii) Sketch the graphs of $y = P(x)$ with this values of c, hence find the set of values of c for which the equation $P(x) = 0$ has only one real root. 3

Sketch 2 curves : $P_1(x) = x^3 - 6x^2 + 9x - 4$

$P_2(x) = x^3 - 6x^2 + 9x$



In order for the curve $P(x)$ has only one x intercept, it has to be lower than $P_1(x)$ or higher than $P_2(x)$. Therfore the equation $P(x)=0$ has only one root if

$$\boxed{c < -4 \quad \text{or} \quad c > 0}$$

(B) Show that the equation $\dfrac{x^2}{36-k}+\dfrac{y^2}{20-k}=1$, where k is a real number, represents:

(i) an ellipse if $k<20$ 2

If $k<20$, the coefficient under y^2 is a positive value then the equation can be expressed as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ it is an ellipse}$$

(ii) a hyperbola if $20<k<36$ 2

If $k>20$, the coefficient under y^2 is a negative value and $k<36$, the coefficient under x^2 is a positive value, then the equation can be expressed as

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ it is a hyperbola}$$

(iii) Show that the foci of the ellipse in (i) or hyperbola in (ii) are independent of the value of k. 2

Foci of ellipse are $S(ae,0)$ and $S'(-ae,0)$

Equation of eccentricity. $b^2=a^2(1-e^2)$

$$= a^2 - a^2e^2$$

$\therefore \quad a^2e^2 = a^2 - b^2$

$$= 36-k-(20-k) = 16$$

$\therefore \quad ae = \pm 4$

Then Foci $S(4,0)$ and $S'(-4,0)$ independent of k

Foci of hyperbola $S(ae,0)$ $S'(ae,0)$

Equation of eccentricity $b^2=a^2(e^2-1)$

$$\therefore \quad a^2e^2 = b^2 + a^2$$
$$= -(20-k) + 36 - k$$
$$a^2e^2 = 16$$
$$ae = \pm 4$$

Foci of hyperbola $S(4,0)$ $S'(-4,0)$ independent of k

Question 4 **12**

(A)

(i) The normal at point $P\left(ct, \frac{c}{t}\right)$ on the hyperbola $xy = c^2$ cuts the line $y = x$ at Q. Find the co-ordinates of Q. 2

Rectangular hyperbola

Normal at P : $y - \frac{c}{t} = t^2(x - ct)$

Intersection point Q with line $y = x$

$$x - \frac{c}{t} = t^2x - ct^3$$

$$x(t^2-1) = \frac{c}{t}(t^4-1)$$

$$x = \frac{c}{t}(t^2+1)$$

$$\therefore \quad Q\left(\frac{c}{t}(t^2+1), \frac{c}{t}(t^2+1)\right)$$

(ii) Show that OP = PQ and hence show that there is no point on the parabola for which the length of PQ is less than $c\sqrt{2}$ 4

Show that $OP = PQ$

$$OP^2 = c^2t^2 + \frac{c^2}{t^2} = \frac{c^2}{t^2}(t^4+1)$$

$$PQ^2 = \left(ct - \frac{c}{t}(t^2+1)\right)^2 + \left(\frac{c}{t} - \frac{c}{t}(t^2+1)\right)^2$$

$$= \left(ct - ct - \frac{c}{t}\right)^2 + \left(\frac{c}{t} - ct - \frac{c}{t}\right)^2$$

$$= \frac{c^2}{t^2} + c^2t^2 = \frac{c^2}{t^2}(t^4+1)$$

$$\therefore \quad OP = PQ$$

The shortest distance from O to the hyperbola is the distance from O to P which lies on the hyperbola and the line $y = x$. Coordinates of that point P are (c, c), hence the shortest distance is $OP = c\sqrt{2}$, therefore there is No point P which gives $PQ < c\sqrt{2}$

(B) Two points P and Q lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. Their parameters are given as θ and $\theta+\dfrac{\pi}{2}$.

(i) Show that Q has co-ordinates $(-a\sin\theta, b\cos\theta)$. Hence prove: 2

$OP^2+OQ^2=a^2+b^2$

$P\ (a\cos\theta,\ b\sin\theta)$

$Q\ \left(a\cos\left(\theta+\frac{\pi}{2}\right),\ b\sin\left(\theta+\frac{\pi}{2}\right)\right)$

$$\text{since}:\quad \cos\left(\theta+\frac{\pi}{2}\right) = -\cos\left(\pi-\left(\theta+\frac{\pi}{2}\right)\right) = -\cos\left(\frac{\pi}{2}-\theta\right) = -\sin\theta$$

$$\sin\left(\theta+\frac{\pi}{2}\right) = \sin\left(\pi-\left(\theta+\frac{\pi}{2}\right)\right) = \sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta.$$

Then co-ordinates of Q $(-a\sin\theta,\ b\cos\theta)$

(ii) Find the locus of midpoint M of PQ. 2

Midpoint M

$$x = \frac{a\cos\theta - a\sin\theta}{2} = \frac{a}{2}(\cos\theta-\sin\theta)$$

$$y = \frac{b\sin\theta + b\cos\theta}{2} = \frac{b}{2}(\cos\theta+\sin\theta)$$

$$(\cos\theta - \sin\theta)^2 = \frac{4x^2}{a^2}$$

$$(\cos\theta + \sin\theta)^2 = \frac{4y^2}{b^2}$$

$$1 - 2\sin\theta.\cos\theta = \frac{4x^2}{a^2}$$

$$+ \quad 1 + 2\sin\theta.\cos\theta = \frac{4y^2}{b^2}$$

Equation of Locus of M

$$\boxed{\frac{x^2}{\frac{a^2}{2}} + \frac{y^2}{\frac{b^2}{2}} = 1}$$

(iii) If α is the acute angle between the 2 tangents at P and at Q, show that 2

$$\tan\alpha = \frac{2\sqrt{1-e^2}}{e^2.\sin 2\theta}$$

Differentiate equation of ellipse :

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Gradient of tangent at P $\quad m_P = -\frac{b^2.a\cos\theta}{a^2.b\sin\theta}$

$$= -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$$

gradient of tangent at Q $\quad m_Q = +\frac{b^2.a\sin\theta}{a^2.b\cos\theta}$

$$= \frac{b}{a}\cdot\frac{\sin\theta}{\cos\theta}$$

Acute angle between 2 tangents

$$\tan\alpha = \left|\frac{m_P - m_Q}{1 + m_P.m_Q}\right|$$

$$= \left|\frac{-\frac{b}{a}\cdot\frac{\cos\theta}{\sin\theta} - \frac{b}{a}\frac{\sin\theta}{\cos\theta}}{1 - \frac{b}{a}\frac{\cos\theta}{\sin\theta}\cdot\frac{b}{a}\frac{\sin\theta}{\cos\theta}}\right|$$

$$= \frac{\frac{b}{a}\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta \cdot \cos\theta}\right)}{1 - \frac{b^2}{a^2}}$$

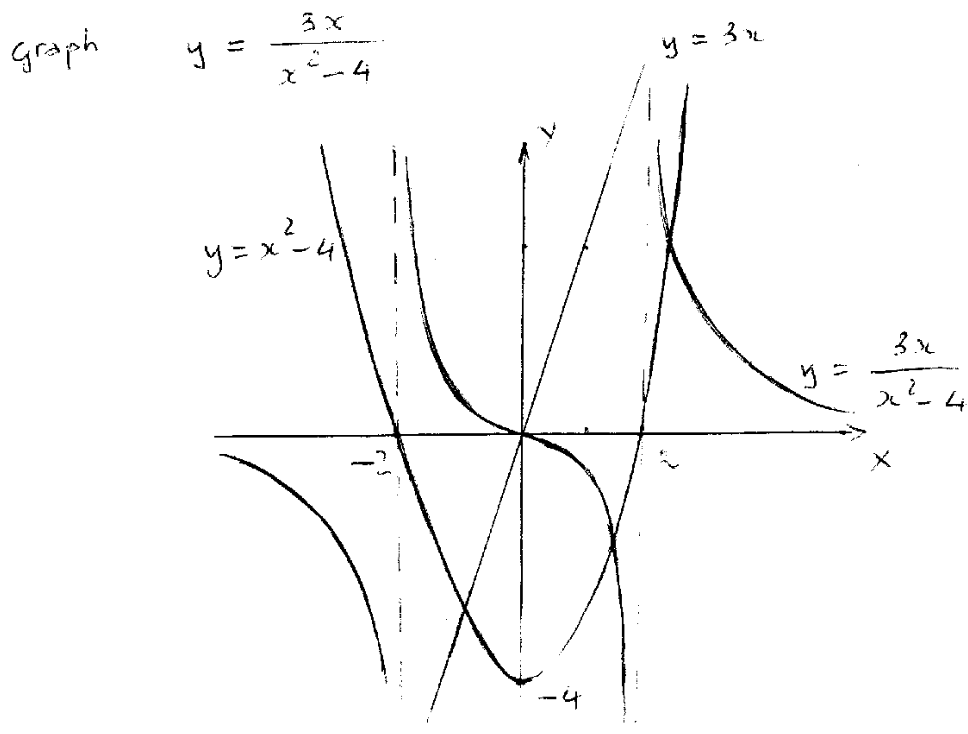
$$= \frac{2}{\sin 2\theta} \times \frac{ab}{a^2 - b^2} = \frac{2}{\sin 2\theta} \times \frac{ab}{a^2 e^2}$$

$$\boxed{\tan\alpha = \frac{2\sqrt{1-e^2}}{e^2 \sin 2\theta}}$$

Question 5 **12**

(A) By using the division of two graphs, or otherwise, sketch the curve 2

$$y = \frac{3x}{x^2 - 4}$$



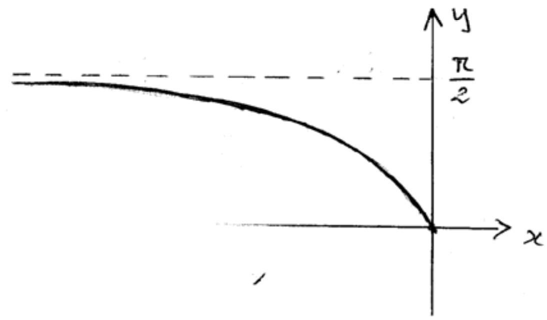
(B) Find the domain and range of curve $y = \cos^{-1}(e^x)$ and hence sketch the graph of $y = \cos^{-1}(e^x)$ 2

$y = \cos^{-1}(e^x)$

Domain $\quad e^x \leq 1 \quad$ then $\quad -\infty < x \leq 0$

Range $\quad 0 \leq y < \frac{\pi}{2}$

The curve :



(C) Let $f(x) = (\sin x - \cos x)^2$, find the period and range of $f(x)$, hence sketch the curve of $f(x)$ with $-\pi \leq x \leq \pi$. 4

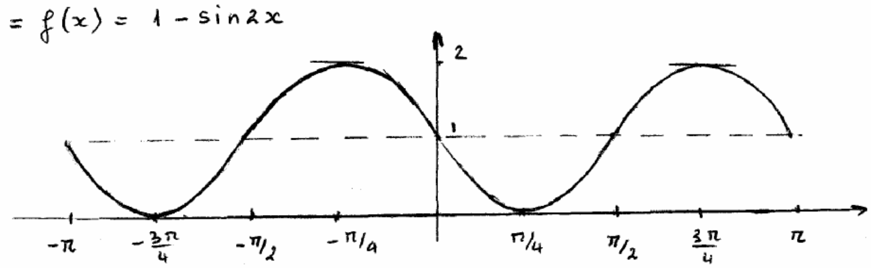
From the separated graph, sketch the following curve. 1

$$f(x) = (\sin x - \cos x)^2$$
$$= \sin^2 x - 2\sin x \cdot \cos x + \cos^2 x$$
$$= 1 - \sin 2x$$

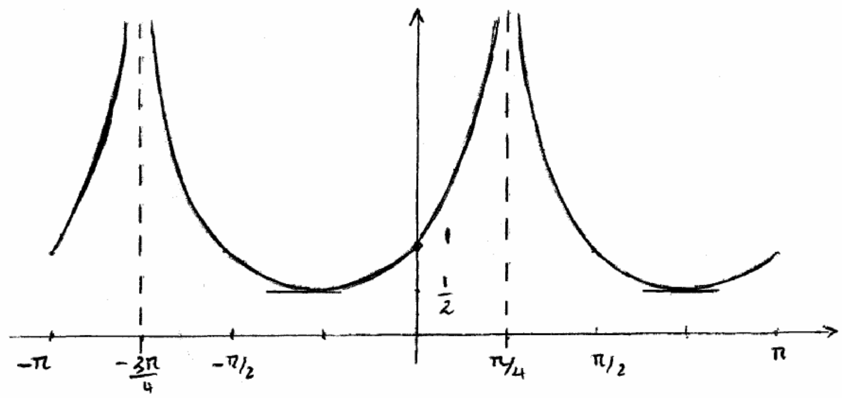
period $\quad \frac{2\pi}{2} = \pi$

Amplitude $= 1$

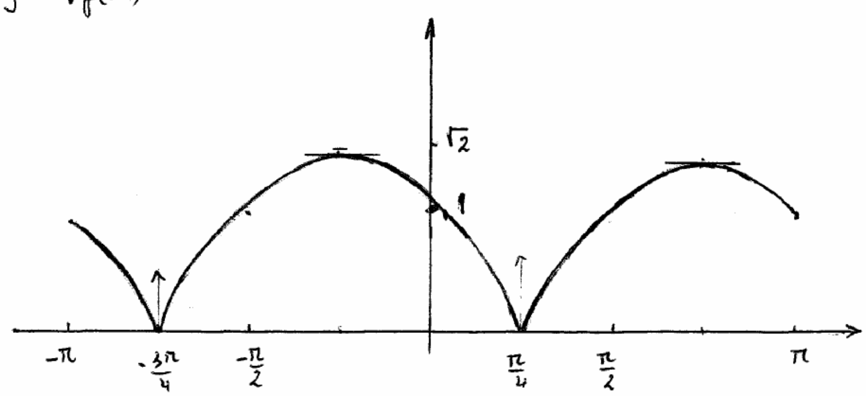
(i) $y = f(x) = 1 - \sin 2x$



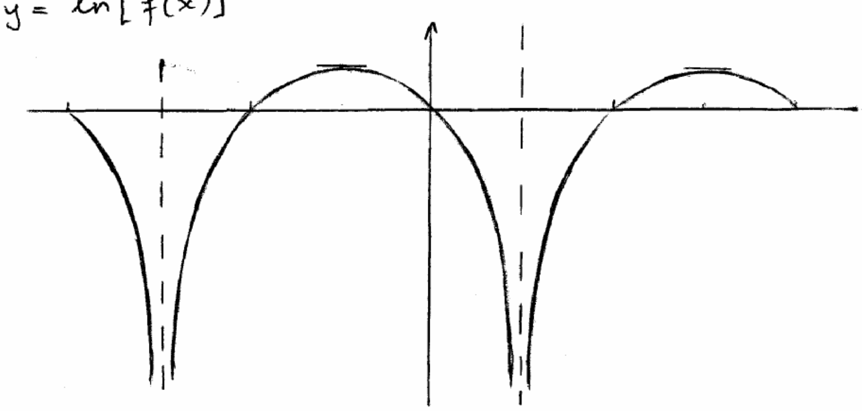
(ii) $y = \dfrac{1}{f(x)}$



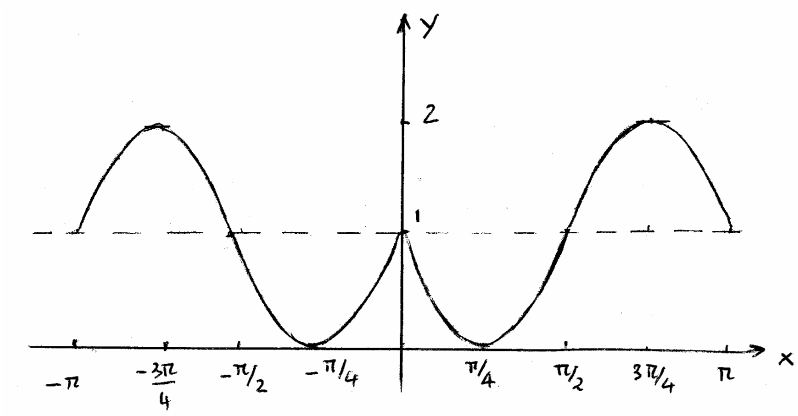
(iii) $y = \sqrt{f(x)}$



(iv) $y = \ln[f(x)]$

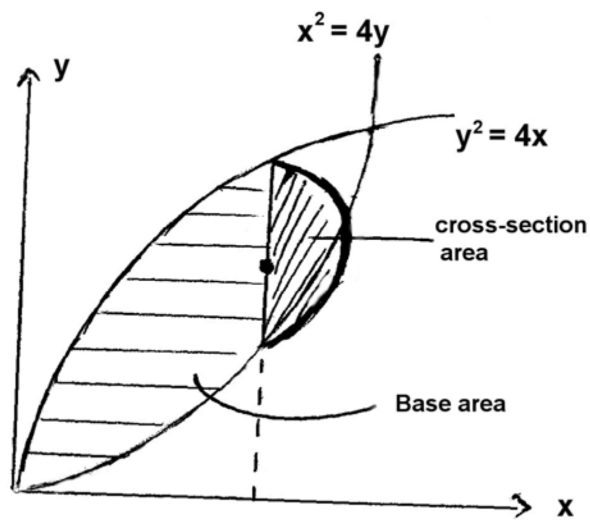


(iv) $y = f\left(|x|\right)$ 1



Question 6 **12**

(A) The base of a certain solid is the region bounded by the curves $y^2 = 4x$ and $x^2 = 4y$, and its cross-sections by planes perpendicular to the x-axis are semi circles. Find the volume of the solid. 6



Slicing method $\quad dV = A.dh$

$$A = \frac{1}{2}\pi R^2$$

of which ; $\quad R = \frac{y_2 - y_1}{2} = \frac{2\sqrt{x} - \frac{x^2}{4}}{2}$

$$R = \sqrt{x} - \frac{x^2}{8}$$

$dh = dx$

$$dV = \frac{1}{2}\pi\left(\sqrt{x} - \frac{x^2}{8}\right)^2 dx$$

$$= \frac{1}{2}\pi\left(x - \frac{x^2\sqrt{x}}{4} + \frac{x^4}{64}\right)dx$$

Intersection point of 2 curves : $x^2 = 4y$
$y^2 = 4x$

$$\therefore \left(\frac{x^2}{4}\right)^2 = 4x$$

$$x^4 - 64x = 0$$

$$x = 0 \text{ or } 4 .$$

Then .

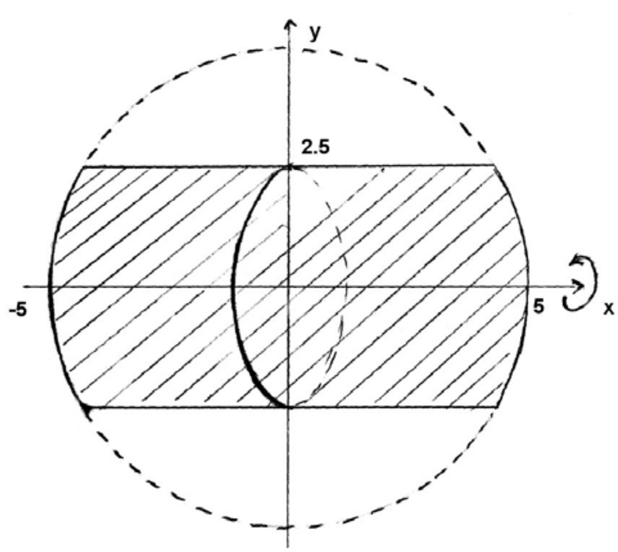
$$V = \lim_{dx \to 0} \sum dV = \int_0^4 \frac{1}{2}\pi\left(x - \frac{x^2\sqrt{x}}{4} + \frac{x^4}{64}\right)dx$$

$$= \frac{1}{2}\pi\left[\frac{x^2}{2} - \frac{1}{14}x^{7/2} + \frac{x^5}{320}\right]_0^4$$

$$= \frac{4}{2}\pi\left(\frac{16}{2} - \frac{128}{14} + \frac{16}{5}\right)$$

$$\boxed{V = 3.231 \text{ unit cube}}$$

(B) The area bounded by 2 arcs and 2 chords of a circle as shown in the figure below, is let to rotate about the x-axis. Find the volume of the solid shape. 6



Cylindrical shell : $dV = 2\pi R h \, dR$

$$R = y$$
$$h = 2x$$
$$dR = dy$$

$$dV = 4\pi \, x y \, dy.$$

Equation of circle : $x^2 + y^2 = 25$

$$\therefore \quad x = \sqrt{25 - y^2}$$

$$V = \lim_{dy \to 0} \sum dV = 4\pi \int_0^{2.5} y\sqrt{25 - y^2} \, dy$$

$$= 4\pi \left[-\frac{1}{3} (25 - y^2)^{3/2} \right]_0^{2.5}$$

$$= 4\pi \left(-\frac{1}{3} (81.19 - 125) \right)$$

$$\boxed{V = 183.5 \text{ unit cube}}$$

Question 7 **12**

(A) Mice are placed in the centre of a maze which has 5 exits. Each mouse is equally likely to leave the maze through any one of the 5 exits. Four mice A, B, C, D are put into the maze and behave independently.

(B)

(i) Find the probability that A, B, C, D all come out the same exit. 1

$P(\text{4 mice out same exit}) = \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \boxed{\frac{1}{125}}$

(ii) What is the probability that A, B and C come out the same exit and D comes out a different exit. 1

$P(\text{A, B, C out same exit but different for D}) = \frac{1}{5} \times \frac{1}{5} \times \frac{4}{5} = \boxed{\frac{4}{125}}$

(iii) What is the probability that any 3 of 4 mice come out the same exit and the other comes out a different exit. 1

P (3 mice out same exit and other out different exit)

$$= 4 \times \frac{4}{125} = \boxed{\frac{16}{125}}$$

(iv) What is the probability that no more than 2 mice come out the same exit. 1

P(no more than 2 mice out same exit)

$$= 1 - \left[P(\text{3 mice same exit}) + P(\text{4 mice same exit}) \right]$$

$$= 1 - \left(\frac{1}{125} + \frac{16}{125} \right)$$

$$= \boxed{\frac{108}{125}}$$

(B) If $\mu_1 = 1$ and $\mu_n = \sqrt{3 + 2\mu_{n-1}}$ for $n \geq 2$

<u>In- equality</u> : show by induction method

(i) show that $\mu_n < 3$ for $n \geq 1$ 2

Show $u_n < 3$, $u_1 = 1$

Test true for $n = 2$: $u_2 = \sqrt{3 + 2u_1}$

$= \sqrt{3 + 2 \times 1}$

$u_2 = \sqrt{5} < 3$

The statement is true for $n = 2$

Assuming true for $n = k$, ie $u_k < 3$

Prove true for $n = k+1$, i.e, $u_{k+1} < 3$

Proof :

Since $u_k < 3$

Then $2u_k < 6$

Hence $3 + 2u_k < 9$

Therefore $\sqrt{3+2u_k} < \sqrt{9}$

So $u_{k+1} < 3$

Conclusion :

Since the statement is true for $n=1$, it is proved also true for $n=2$ and so on it is true for every integers n.

(ii) deduce that $\mu_{n+1} > \mu_n$ for $n \geq 1$ 2

Deduce $u_{n+1} > u_n$.

- Since $3 > u_n$
  Then $3u_n > u_n^2$
- And $3 > u_n$
  $2u_n + 3 > 3u_n$
- So $2u_n + 3 > u_n^2$
  $u_{n+1}^2 > u_n^2$

$\therefore \boxed{u_{n+1} > u_n}$

(C) By using induction method , prove that $3^{4n+2} + 2.4^{3n+1}$ is divisible by 17 for $n \geq 1$ 4

Induction method, prove $3^{4n+2} + 2.4^{3n+1}$ is divisible by 17.

- Test true for $n = 1$

$3^{4+2} + 2 \times 4^{3+1} = 1241$

$= 73 \times 17$ is divisible by 17

- Assuming true for $n = k$

$$3^{4k+2} + 2 \times 4^{3k+1} = 17p \quad (p \text{ is integer})$$

- Prove true for $n = k+1$

$$3^{4(k+1)+2} + 2 \times 4^{3(k+1)+1} = 17q \quad (q \text{ is integer})$$

Since :

$$3^{4k+2} = 17p - 2 \times 4^{3k+1}$$

Then

$$3^{4k+4+2} + 2 \times 4^{3k+3+1} = 3^4(17p - 2 \times 4^{3k+1}) - 2 \times 4^{3k+1} \times 4^3$$

$$\begin{aligned} LHS &= 17 \times 81p - 2 \times 4^{3k+1}(81 - 64) \\ &= 17(81p - 2 \times 4^{3k+1}) \\ &= 17q \quad \text{divisible by } 17 \end{aligned}$$

- Since the statement is true for $n = 1$, it is also true for $n = 2$, and so on it is true for any values of integer n.