SAINT IGNATIUS' COLLEGE
RIVERVIEW

Mathematics Extension 2

Mid Year Examination

2004

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours and 15 minutes
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

QUESTION 1 **[15 marks]** **(Start a new answer booklet)**

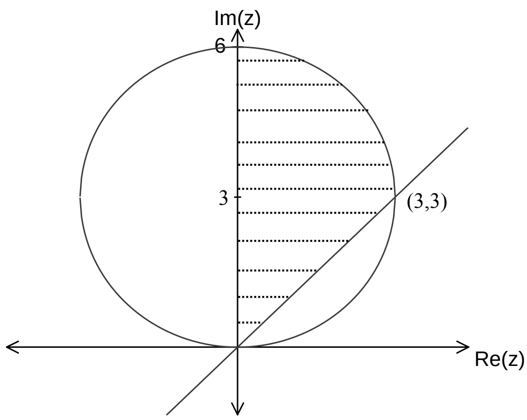
a) Find $\sqrt{9-12i}$ expressing your answer in the form $a+bi$ where a, b are real. [4]

b) Find the modulus and argument of the following complex numbers:

(i) $-4+4\sqrt{3}\,i$ [2]

(ii) $(\sin\theta + i\cos\theta)(\cos\theta - i\sin\theta)$ [3]

c) Write down the set of inequalities, <u>in terms of *z*</u>, if the complex number *z* lies in the shaded region shown on the diagram.



[2]

d) The complex number $\dfrac{z-2}{z+i}$ is purely imaginary. Find the Cartesian equation of the locus of the complex number *z*, and draw the graph of the locus, noting any discontinuities on the graph. [4]

QUESTION 2 **[15 marks]** **(Start a new answer booklet)**

a) Given that $P(x) = ax^3 + bx^2 - 7x + 6$ evaluate a and b if $(x-2)$ and $(x-1)$ are both factors of $P(x)$. [4]

b) Sketch the graph of $y = x(x^2 - 9)$ and hence solve the equation $\dfrac{x^2 - 9}{x} > 0$ [3]

c) Give all solutions for $2\sin x = \tan x$ [4]

d) (i) Sketch the graphs of $y = \sin x$ and $y = x$ on the same set of axes and show that $x = 1.8$ is a good approximation for the x value of the point of intersection of the graphs in the domain $\dfrac{\pi}{4} \le x \le \dfrac{3\pi}{4}$. [2]

(ii) Use Newton's method with 1 application to find a better approximation to the x value of this point of intersection. [2]

QUESTION 3 **[15 marks]** **(Start a new answer booklet)**

a) Consider the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$

(i) Find

α) the eccentricity

β) the foci

γ) the equations of the directices

(ii) Sketch the ellipse.

(iii) Find the gradient of the normal at $\left(1, \dfrac{4\sqrt{2}}{3}\right)$

b) Sketch each of the following pairs of curves on separate axes, clearly indicating all intercepts and the location (you need not show the relevant testing) of any turning points.

(i) $y=\ln x^2$ and $\ln y = x^2$ [3]

(ii) $y=\sin^{-1}|x|$ and $|y| = \sin^{-1} x$ [3]

(iii) $y=\cos^2 x$ and $y=\cos^6 x$ [3]

QUESTION 4 **[15 marks]** **(Start a new answer booklet)**

a) Find the equations of the two tangents to the curve $x^2 - y^2 = 3$ which are parallel to the line $y = 2x$. [5]

b) If α, β and δ are the roots of the equation $x^3 - 7x^2 + 3x + 1 = 0$ evaluate

(i) $\alpha + \beta + \delta$

(ii) $\alpha\beta\delta$

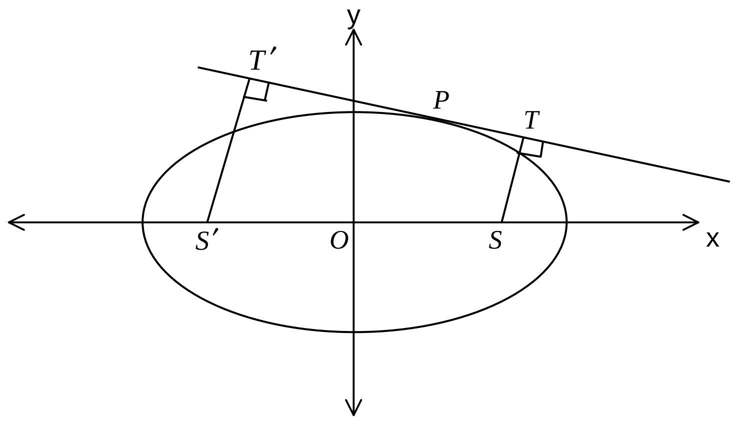
(iii) $\alpha^2 + \beta^2 + \delta^2$

(iv) $\dfrac{1}{\alpha^2\beta^2} + \dfrac{1}{\beta^2\delta^2} + \dfrac{1}{\delta^2\alpha^2}$

Hence form the equation whose roots are $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$, $\dfrac{1}{\delta^2}$

[5]

c)



$P(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with foci S, S'.

The equation of the tangent at P is $bx\cos\theta + ay\sin\theta - ab = 0.$

The perpendicular to the tangent from S meets it at T, and the perpendicular from S' meets the tangent at T'.

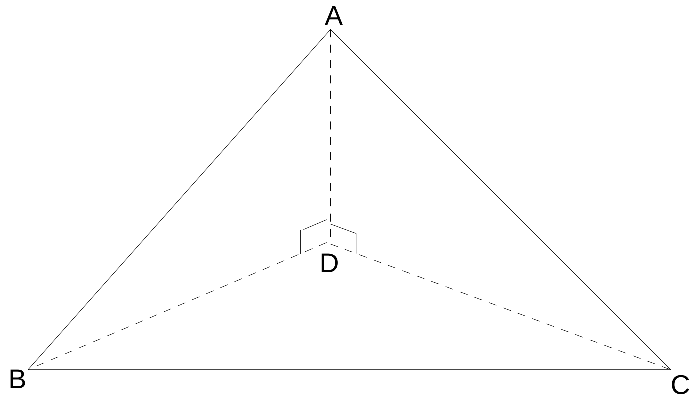
(i) Find the length of α) ST [1]

β) $S'T'$ [1]

(ii) Prove that $ST \times S'T' = b^2$ [3]

QUESTION 5 **[15 marks]** **(Start a new answer booklet)**

a)



The diagram shows a triangular pyramid ABCD. The base is the triangle BCD and AD is the perpendicular height. The perpendicular height is *h* units and base edge BC is *x* units.
Given that $A\hat{B}D = 30^\circ$, $A\hat{C}D = 45^\circ$ and $B\hat{C}D = 30^\circ$, [5]

show that $\dfrac{h}{x} = \dfrac{\sqrt{11}-\sqrt{3}}{4}$

b)
(i) $P\left(4p, \dfrac{4}{p}\right)$ and $Q\left(4q, \dfrac{4}{q}\right)$ are two different variable points on the hyperbola $xy = 16.$ Find the equation of the tangent at $P.$ [3]

(ii) Show that if $pq = 4$, the locus of the point of intersection of the tangents at P and Q lies on a straight line. [4]

(iii) Determine the domain and range of the locus. [3]

QUESTION 6 **[15 marks]** **(Start a new answer booklet)**

a) Find all integers n such that $(\sqrt{3}+i)^n - (\sqrt{3}-i)^n = 0$ [3]

b) (i) Sketch $y = 1 + x^2$ [1]

(ii) On separate diagrams, and without the use of calculus, sketch

(α) $y = \dfrac{1}{x^2+1}$ [1]

(β) $y = \dfrac{1}{x^2-2x+2}$ [2]

(iii) Express $y = \dfrac{x}{x^2+1}$ as a quadratic equation in x. [1]

(iv) Use the result of (iii) to find the range, turning points and asymptotes of the function $y = \dfrac{x}{x^2+1}$ [3]

(v) Hence sketch $y = \dfrac{x}{x^2+1}$ [1]

(vi) Also sketch on separate diagrams, clearly showing turning points and asymptotes,

(α) $y = \left|\dfrac{x}{x^2+1}\right|$ [1]

(β) $y^2 = \dfrac{x}{x^2+1}$ [2]

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$