ST IGNATIUS COLLEGE RIVERVIEW

ASSESSMENT TASK NUMBER 2

SEMESTER ONE EXAMINATION

YEAR 12

2008

EXTENSION 2

Time allowed: 3 hours (+ 5 minutes reading time)

Instructions to Candidates

- Attempt **all** questions
- Show all necessary working.
- Marks may be deducted for missing or poorly arranged work.
- Board approved calculators and templates may be used.
- Each question must be returned in a ***separate*** writing booklet marked Q1, Q2 etc
- **Each booklet must have your name**.

Question 1 [15 marks] Start a new answer booklet.

(a) If $z=\sqrt{3}+i$, find

(i) iz [1]

(ii) $\frac{1}{z}$ [1]

(b) Express the complex number $z=\left(\sqrt{3}-i\right)^2$ in the modulus- argument form. [2]

(c) On separate Argand diagrams, shade the regions:

(i) $-2<\operatorname{Im}(z)\leq 5$, [2]

(ii) $|z|<6$, [1]

(iii) $2<z+\bar{z}<10$, [2]

(iv) $\arg\left(z^2\right)\leq\frac{2\pi}{3}$, [2]

(d)

(i) Find and describe the locus, in the complex plane of a point P which represents the complex number z such that . [3]

$$z\bar{z}-3\left(z+\bar{z}\right)=7$$

(ii) What is the locus of $\bar{z}$? [1]

Question 2 [15 marks] Start a new answer booklet.

(a) Evaluate $\int_0^{\frac{\pi}{2}} \sin^n x \cos x \, dx$. [3]

(b) Evaluate $\int_0^1 \frac{t^2}{t+1} \, dt$. [3]

(c) (i) Express $\frac{1}{16-u^2}$ in the form $\frac{P}{4-u}+\frac{Q}{4+u}$. [1]

(ii) Use the substitution $u=\sqrt{16-x}$ to evaluate the integral [3]

$\int_7^{12} \frac{4}{x\sqrt{16-x}} dx.$

(d) Let $U_m=\int_0^1 x^m e^x dx$

(i) Evaluate U_0. [1]

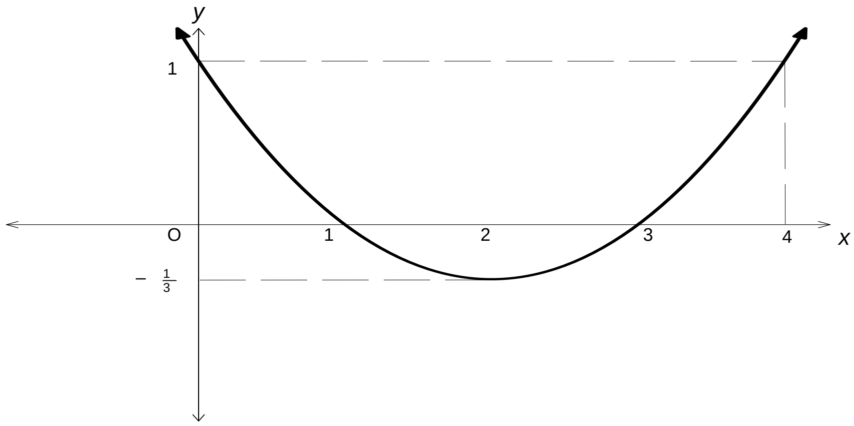
(ii) Show [2]

$U_m=e-mU_{m-1}$, , for $m>1$.

(iii) Hence, evaluate $U_4=\int_0^1 x^4 e^x \, dx$. [2]

Question 3 [15 marks] Start a new answer booklet.

The sketch shows the parabola $y = f(x)$, where $f(x)$ is the quadratic $f(x) = \frac{1}{3}(x-1)(x-3)$.



Without any use of Calculus, draw careful sketches of the following curves, showing all intercepts, asymptotes, and turning points

(a) $y = f(-x)$. [2]

(b) $y = f(2x)$. [2]

(c) $y = \dfrac{1}{f(x)}$. [2]

(d) $y = \sqrt{f(x)}$. [2]

(e) $y = [f(x)]^2$. [2]

(f) $y = \log_e f(x)$. [2]

(g) $y = \tan^{-1} f(x)$. [3]

Question 4 [15 marks] Start a new answer booklet.

(a) Each of the following statements is either TRUE or FALSE. Write TRUE or FALSE for each statement, and give brief reasons for your answer. (Do not find the primitive functions)

(i) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^5 \theta \, d\theta = 0.$ [2]

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 \theta \, d\theta = 0.$ [2]

(iii) $\int_{-1}^{1} e^{-x^3} = 0.$ [2]

(iv) $\int_{0}^{\frac{\pi}{2}} \sin^4 x \, dx = \int_{0}^{\frac{\pi}{2}} \cos^4 x \, dx.$ [2]

(b) Find

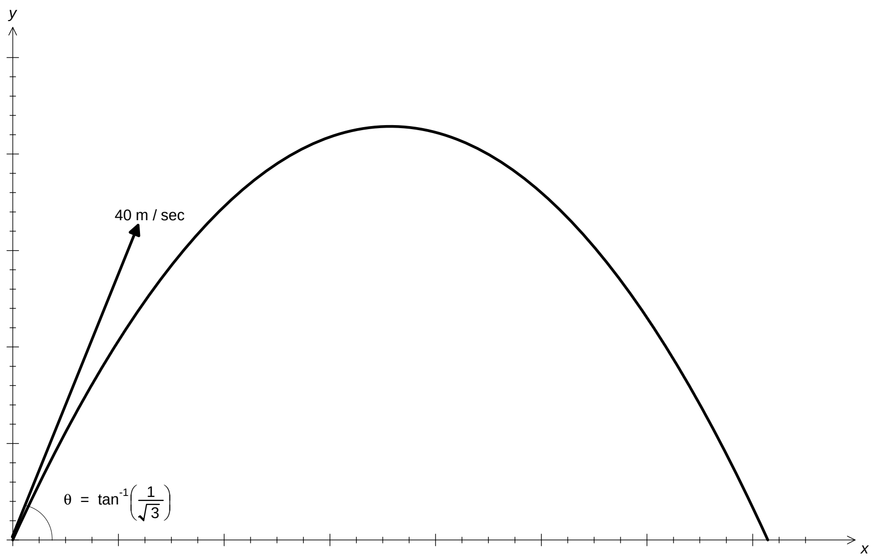
(i) $\int \frac{dx}{\sqrt{2x - x^2}}$. [2]

(ii) $\int \cos^{-1} x \, dx$. [2]

(iii) $\int \sqrt{4 - x^2} \, dx$. [3]

Question 5 [15 marks] Start a new answer booklet.

(a) The diagram below shows the path of an object launched at an angle of $\theta = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ to the horizontal with an initial speed of 40 metres per second from O. The acceleration due to gravity is taken as 9.8 metres per second squared, and air resistance is ignored.



(i) Derive expressions for $x(t)$ and $y(t)$, where $'t'$ is time in seconds. [2]

(ii) Show that the path of the object is parabolic. [2]

(iii) Find the <u>angle</u> and <u>speed</u> of the object at 1.5 seconds. [3]

(iv) Calculate the time of flight. [2]

(v) What is the range of the flight. [1]

(b) The depth of water in a harbour on a particular day is 8.2 metres at low tide and 14.6 metres at high tide. Low tide is at 1:05pm and high tide is at 7:20pm. The captain of a ship drawing 13.3 metres of water wants to leave the harbour on that afternoon. Between what times can he leave. (Assume that the tide changes in SHM) [5]

Question 6 [15 marks] Start a new answer booklet.

(a) (i) Show that $(1+i)$ is a root of the equation $z^4 - 8z^3 + 14z^2 - 12z = 0$ [2]

(ii) Hence, solve this equation over the field of complex numbers [3]

(b) The roots of the equation $x^3 + 7x^2 - 5x - 1 = 0$ are α, β, γ.

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$. [2]

(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$. [2]

(iii) Find the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$ [2]

(c) Consider the polynomial $P(x) = (x+c)^4 - 32x$ where ‘c’ is a constant. If $P(x) = 0$ has a double root at $x = \alpha$

(i) Prove that $\alpha = 2 - c$. [2]

(ii) Find the numerical values of $'\alpha'$ and $'c'$ [2]

Question 7 [15 marks] Start a new answer booklet.

(a) Find the equation of the tangent to the curve $x^3+y^3-8y+7=0$ at the point $(1,2)$. [3]

(b) Given $y=\dfrac{x^3}{x^2-4}$

(i) Find the co-ordinates of all the stationary points. [2]

(ii) Find the points of intersection with the co-ordinate axes and the position of all asymptotes. [2]

(iii) Hence, sketch the curve $y=\dfrac{x^3}{x^2-4}$. [2]

(iv) Use the graph of $y=\dfrac{x^3}{x^2-4}$ to find the number of roots of the equation $x^3-k\left(x^2-4\right)=0$ for varying values of k. [3]

(c) Draw a neat sketch of the relation $y^2=x\left(x-2\right)\left(x-3\right)$. [3]
You must show all critical points.

Question 8 [15 marks] Start a new answer booklet.

(a) (i) Show that the function defined by $f(x) = 2x^3 + 2x - 1$ has a zero between $x = 0$ and $x = 1$. [1]

(ii) By considering $f'(x)$, explain why this is the only zero. [2]

(iii) Taking $x = 0.5$ as the first approximation to the solution of the equation $f(x) = 0$, use Newton's method once to find a closer approximation. Give your answer correct to 2 decimal places. [2]

(b) The polynomial $P(x) = x^3 + mx + n$ has zeros for $P(x) = 0$, at $\alpha, \beta,$ and $\frac{1}{\beta}$.

(i) Explain why $\alpha = -n$. [2]

(ii) By considering the sum of the roots in pairs, show that [2]

$$\frac{\beta^2 + 1}{\beta} = \frac{1 - m}{n}.$$

(iii) Hence, or otherwise, show that such roots only exist for this polynomial provided $(m-1)^2 \geq 4n^2$. [2]

(iv) The roots of the equation $4x^3 - 24x^2 + 45x - 26 = 0$ form an arithmetic sequence. Solve the equation. [4]

End of the Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$