The Scots College

Year 12 Mathematics Extension 2

Pre-Trial Assessment

April 2007

General Instructions

- All questions are of equal value
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table is attached

TOTAL MARKS: 75

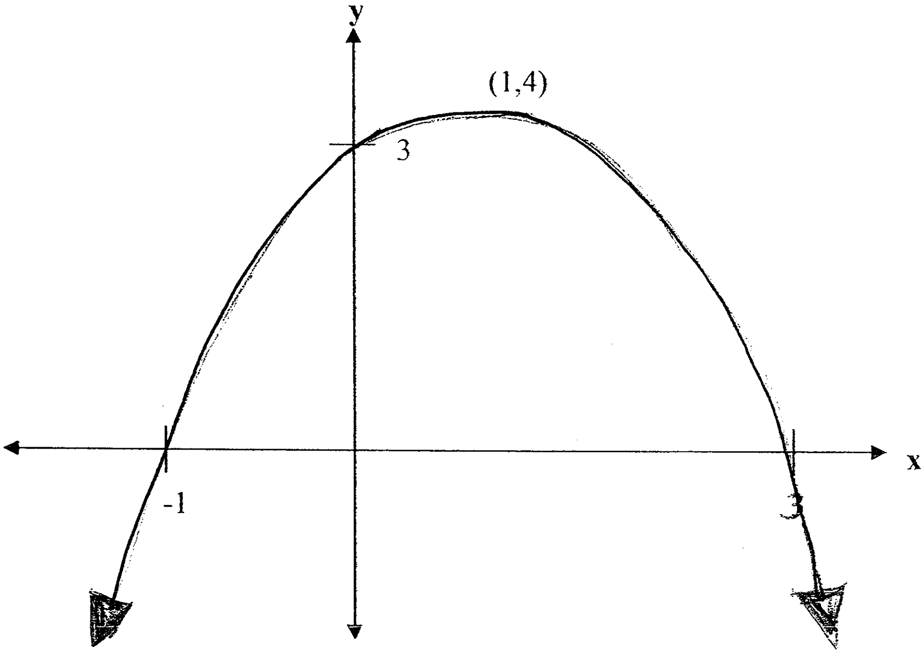
WEIGHTING: 30 %

- **Start each question in a new booklet**

QUESTION 1 **[15 MARKS]**

(a) Let $f(x) = -(x-3)(x+1)$

In the diagram below, the graph is drawn.



On separate diagrams, and without the use of calculus, sketch the following graphs. Indicate clearly any asymptotes, any intercepts with the coordinate axes, and any other significant features.

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y^2 = f(x)$ **2**

(b) Given the two function **g(x) = x** and **h(x) = ln x**.

(i) Sketch showing all important features $f(x) = \dfrac{g(x)}{h(x)}$ **4**

(ii) Explain why $f(x) = \dfrac{|g(x)|}{h(x)}$ does not alter the sketch **1**

(iii) Sketch $f'(x)$ **2**

QUESTION 2 **[15 MARKS]**

(a) Reduce the complex number $\dfrac{(2-i)(8+3i)}{(3+i)}$ to the form **a + ib**,

where **a** and **b** are real numbers. 2

(b) The complex number z is given by $z = -\sqrt{3} + i$

(i) Write down the values of Arg (z) and $|z|$ 1

(ii) Hence or otherwise show that $z^7 + 64z = 0$ 2

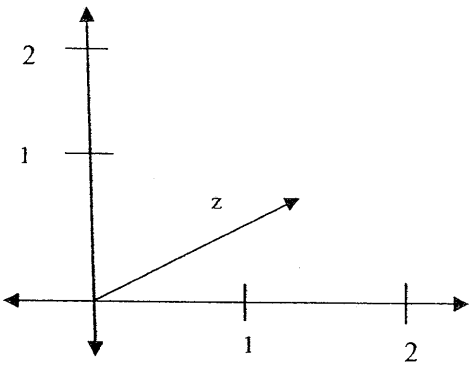
(c) Sketch the following loci on separate Argand diagrams;

(i) $Arg(z+1+i) = \dfrac{\pi}{4}$ 1

(ii) $|z-2| = |z+i|$ 1

(d) Given that **z** = **3 +4i.** Find **w** so that **O, z** and **w** form a right angled isosceles triangle, (whose right angle is at **z**) on the Argand diagram. 3

(e) The complex number **z** shown on the Argand diagram below has $|z| = 1$



On separate Argand diagrams, illustrate the geometric properties of the following;

(i) z^2 1

(ii) $z \times (1+i)$ 1

(iii) $z - \bar{z}$ 1

(f) Given $z^4 = i$ Find the four roots of unity 2

QUESTION 3 [15 MARKS]

(a)

(i) Show that the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ in the first quadrant has the equation $b\,x\cos\theta + a\,y\sin\theta - ab = 0$ 2

(ii) The tangent cuts the x-axis at the point A and the y-axis at the point B. Find the minimum area of ΔAOB and show that when this occurs P is the midpoint of AB 4

(b)

(i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $(2t, \frac{2}{t})$ has the equation $x + t^2 y = 4t$ 2

(ii) This tangent cuts the x-axis at the point Q. Show that the line through Q which is perpendicular to the tangent at T has the equation $t^2 x - y = 4t^3$ 2

(iii) This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has the coordinates $M(2t, -2t^3)$ 3

(iv) Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. 2

QUESTION 4 [15 MARKS]

(a) Evaluate

(i) $\displaystyle\int_0^1 \tan^{-1} x \, dx$ **3**

(ii) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2 + 2x + 5}$ **3**

(b) Find the values of A, B and C such that

$$\frac{3 - x}{(1 + 2x^2)(1 + 6x)} = \frac{Ax + B}{1 + 2x^2} + \frac{C}{1 + 6x}$$ **2**

Hence show that

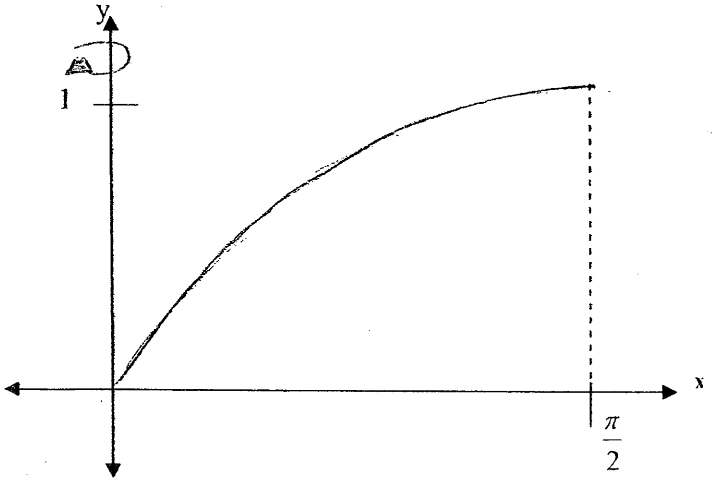
$$\int_0^2 \frac{3 - x}{(1 + 2x^2)(1 + 6x)} dx = \frac{1}{2} \ln \frac{13}{3}$$ **2**

(c) Find $\displaystyle\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$ **2**

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos 2x \, dx$ **3**

QUESTION 5 **[15 MARKS]**

(a) The region under the curve y = sin x, bounded by the x-axis and the ordinate $x = \frac{\pi}{2}$, is rotated about the y-axis. 5



Using the method of **Volume by Slices**, find the volume of the solid generated.

(b) **(i)** Use the principle of mathematical induction to prove that $3^n > n^3$ for all integers $n \geq 4$ 3

(ii) Hence or otherwise show that $\sqrt[3]{3} > \sqrt[n]{n}$ for all integers $n \geq 4$ 2

(c) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

(i) Show that $I_n = \left(\frac{n-1}{n}\right) I_{(n-2)}$ 2

(ii) Hence show that $\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$ 3

(Note $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$)

END OF PAPER

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

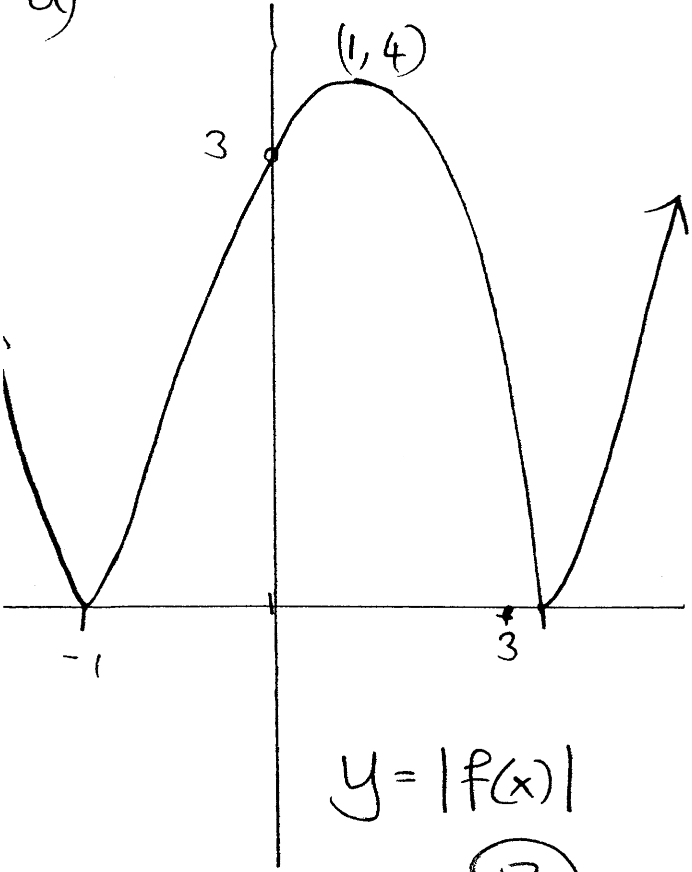
$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

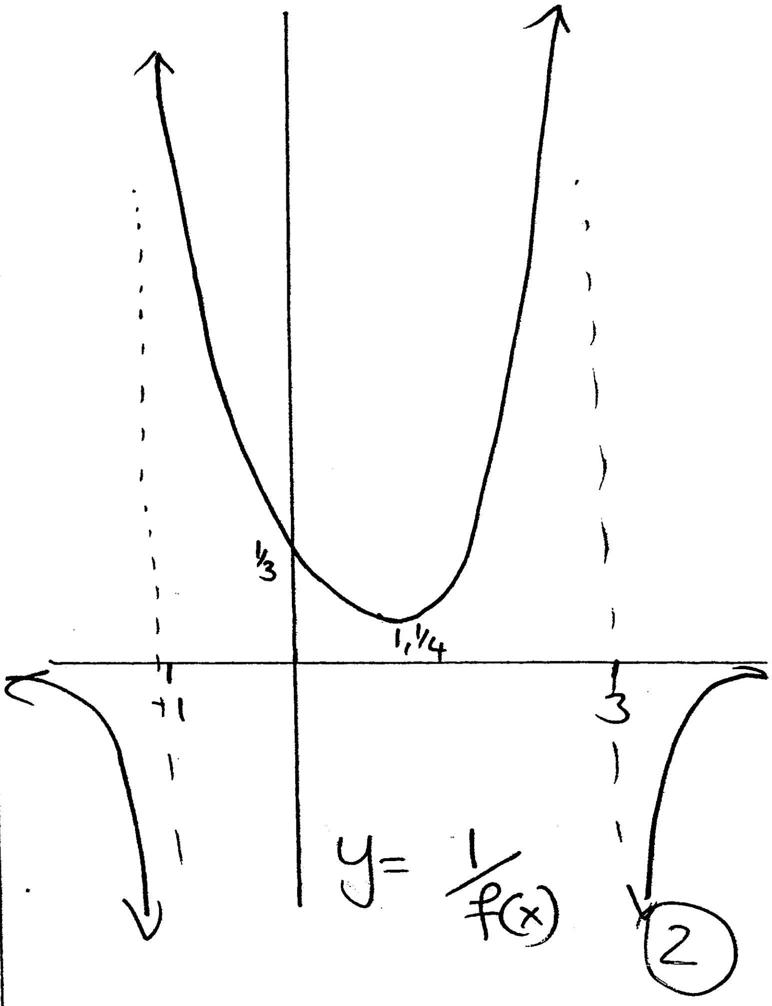
NOTE: $\ln x \equiv \log_e x, \quad x > 0$

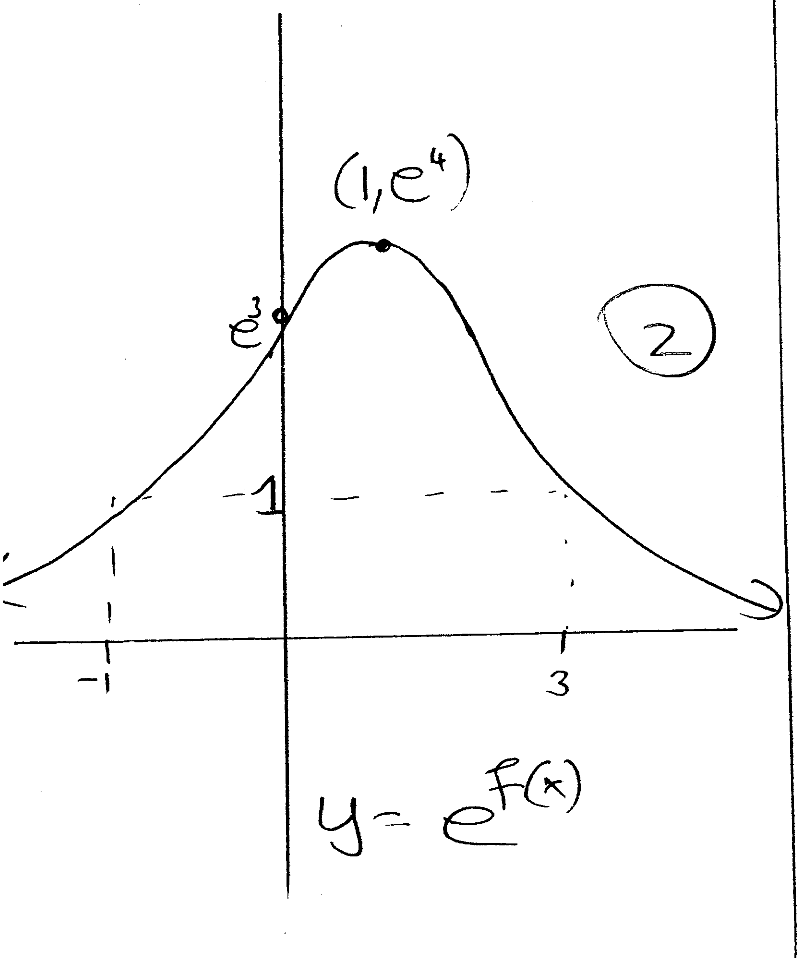
Extension 2 - Pre Trial - 2007

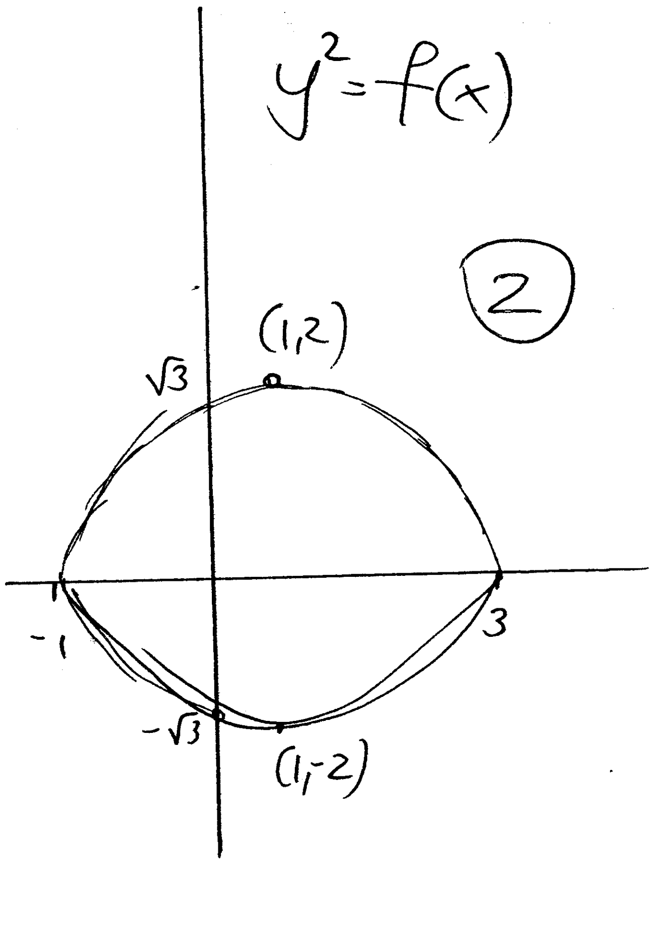
Question 1

a)



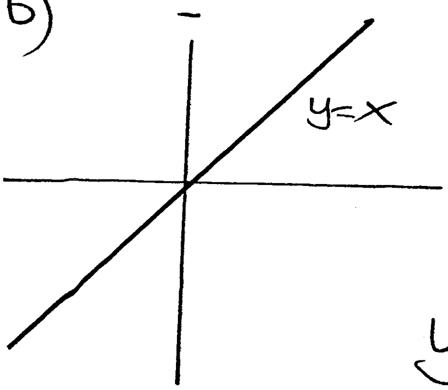


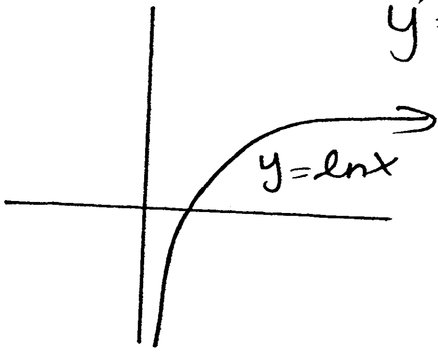




Qu 1 (cont)

b)





$$y = \frac{x}{\ln x}$$

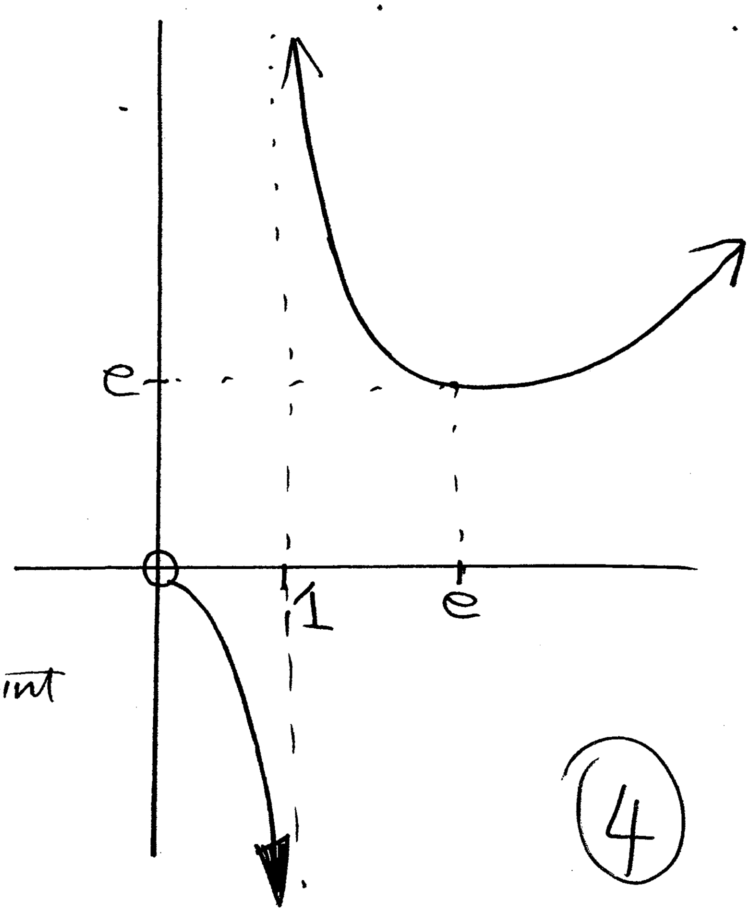
$$y' = \frac{\ln x - 1}{(\ln x)^2}$$

∴ Turning point at

$\ln x = 1$

$\therefore e = x$

(e, e)



4

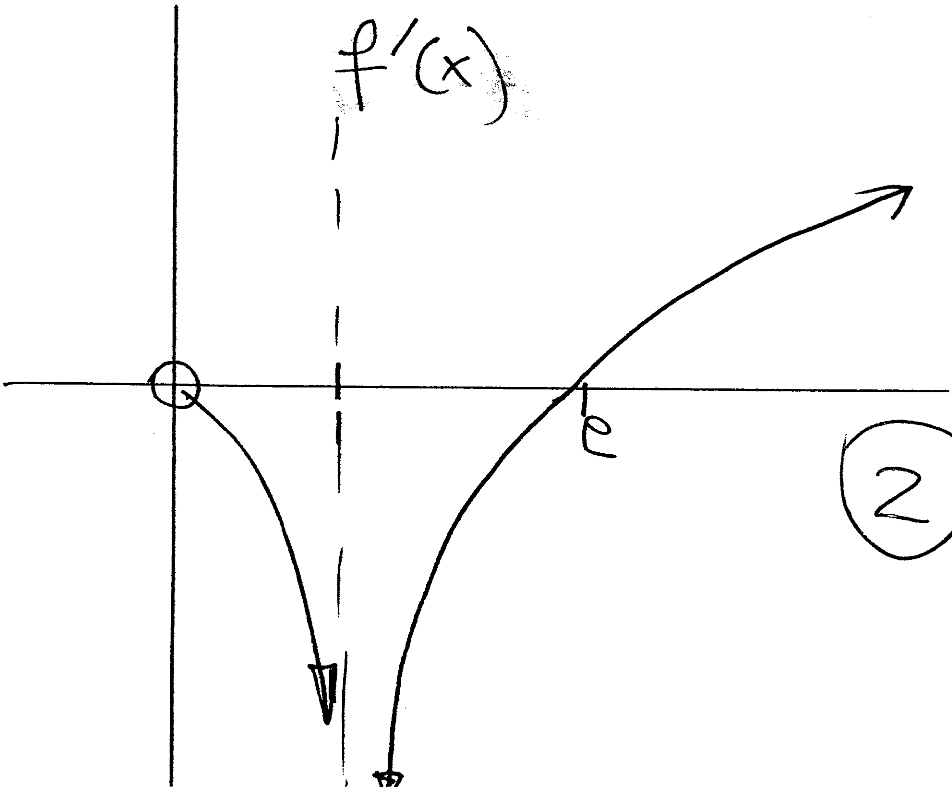
$|g(x)|$ takes effect when $g(x)$ is negative ie $x<0$. However $h(x)$ does not exist when $x<0$, therefore $y = \frac{x}{\ln x}$ does not exist

1

$$y' = \frac{\ln x - 1}{(\ln x)^2}$$

⇓

Difficult to sketch Therefore best to use approximate gradient of $f(x)$



2

Extension2 Trial - 2007

Question 2

$(2-i)(8+3i)$

$16+6i-8i-3i^2$

$16-2i+3$

$19-2i$

$\Rightarrow \dfrac{(19-2i)}{(3+i)} \times \dfrac{(3-i)}{(3-i)}$

$57-19i-6i-2i^2$

$57-25i+2$

$55-25i$

$(3+i)(3-i)$

$9+1$

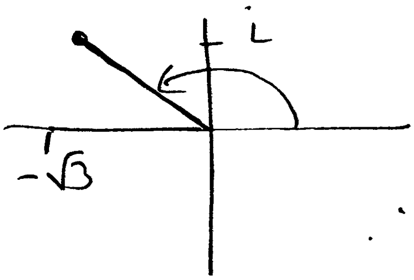
$10.$

$\dfrac{55-25i}{10}$

$\dfrac{11}{2} + \dfrac{5i}{2}.$

2

$z = -\sqrt{3} + i$



$|z| = 2$

$\operatorname{Arg} z = \dfrac{5\pi}{6}.$

1

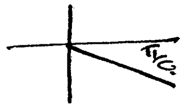
$\therefore z = 2 \operatorname{cis} \dfrac{5\pi}{6}.$

$z^7 = 2^7 \operatorname{cis} \dfrac{5\pi}{6} \times 7$

$= 128 \operatorname{cis} \dfrac{35\pi}{6}.$

$\dfrac{35\pi}{6} = 5\dfrac{5}{6}\pi$

$=$



$\therefore \cos\dfrac{35\pi}{6} = \cos\dfrac{\pi}{6}.$

$\sin\dfrac{35\pi}{6} = -\sin\dfrac{\pi}{6}.$

$\therefore z^7 = 128\cos\frac{\pi}{6} - 128\sin\frac{\pi}{6} i$

$= 128 \times \dfrac{\sqrt{3}}{2} - 128 \times \dfrac{1}{2} i$

$64z = 128(-)\cos\frac{\pi}{6} + 128\sin\frac{\pi}{6}$

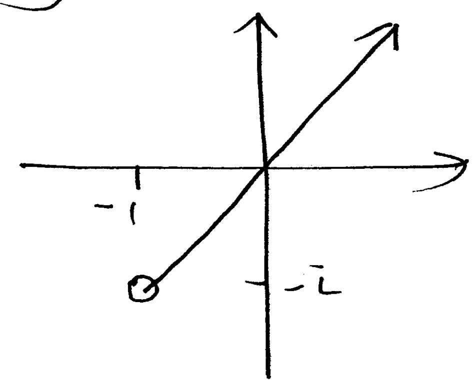
$= -128\dfrac{\sqrt{3}}{2} + 128 \times \dfrac{1}{2} i$

$z^7 + 64z = 0.$

QED.

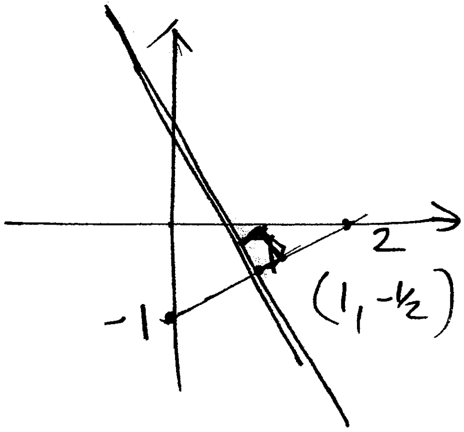
2

$\operatorname{Arg}(z+1+i) = \dfrac{\pi}{4}$



1

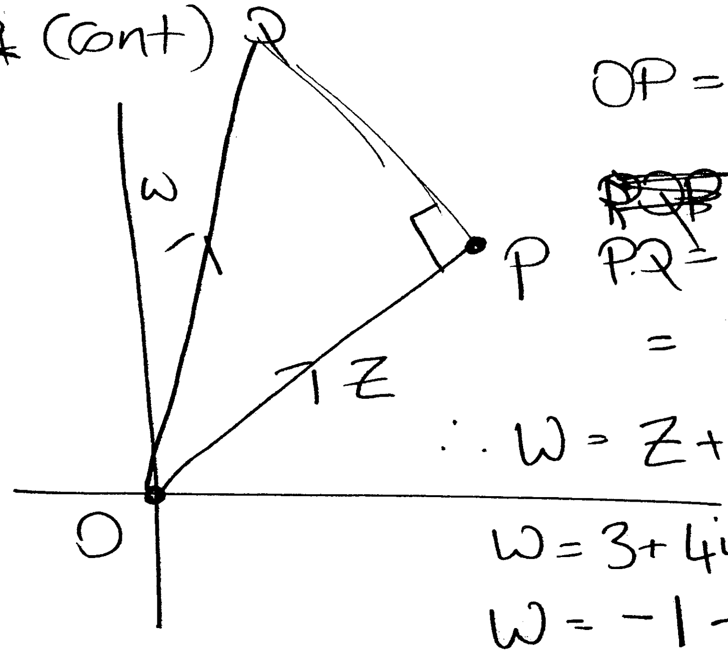
$|z-2| = |z+i|$



1

Extension 2 - The Trial - 2007

Question 4 (cont)



$OP = PQ$ (Isosceles)

$\cancel{PQP} =$

$PQ = w - z$

$= iz$ [$90°$ Rotation Same Size]

$\therefore w = z + iz$

$w = 3 + 4i + 3i - 4$

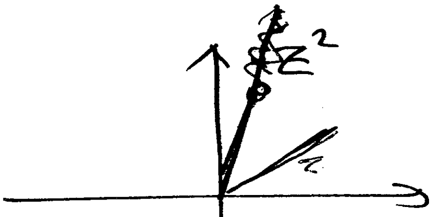
$w = -1 + 7i$.

or

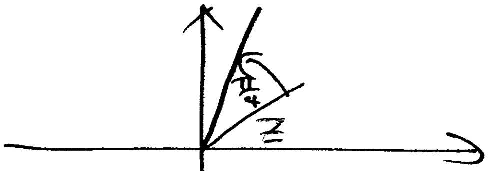
$3 + 4i - (3i - 4)$

$w = 7 + i$.

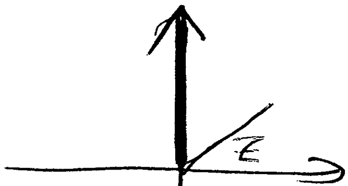
3



z^2 [Rotate by Arg, $| |^2$] 1

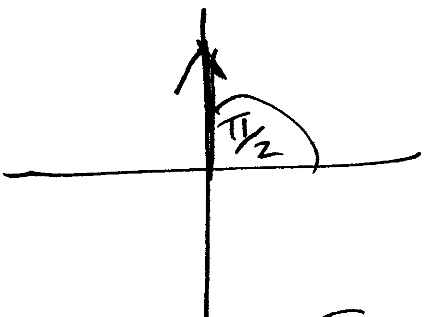


Rotate by $\pi/4$, Mult $| |$ by $\sqrt{2}$ 1



No real component, Twice Imaginary component. 1

$z^4 = i$



$z^4 = 1\,\text{cis}\,\pi/2$

$\therefore z = 1\,\text{cis}\,\pi/8$

By De Moivre.

$z_1 = \text{cis}\,\pi/8$

$\frac{2\pi}{4} = \frac{\pi}{2}$.

$z_2 = \text{cis}\,\frac{5\pi}{8}$

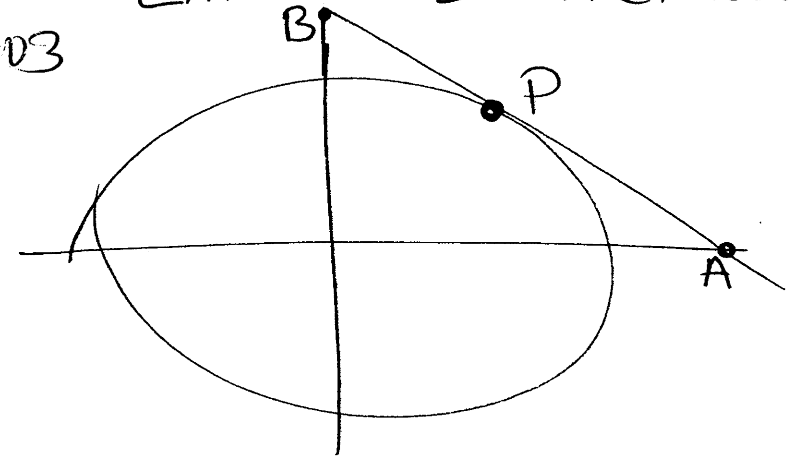
$z_3 = \text{cis}\,\frac{-3\pi}{8}$

$z_4 = \text{cis}\,\frac{-7\pi}{8}$.

[$\text{cis}\,\frac{13\pi}{8}$, $\text{cis}\,\frac{9\pi}{8}$]

2

P03



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Implicit Differentiate

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

2

$$m \text{ Tangent} = \frac{-b^2 . a\cos\theta}{a^2 . b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$$

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta - ab(\sin^2\theta + \cos^2\theta) = 0$$

1

QED

At A $(y=0)$

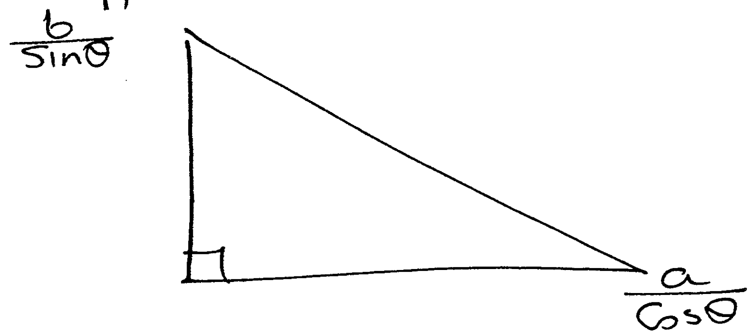
$b x\cos\theta = ab.$

$x = \frac{a}{\cos\theta}$

At B $(x=0)$

$ay\sin\theta = ab$

$y = \frac{b}{\sin\theta}$



$$\text{Area} = \tfrac{1}{2} \times \frac{ab}{\cos\theta\sin\theta}$$

$$A = \frac{ab}{\sin 2\theta}$$

CHECK FOR MINIMUM

4

$$\frac{dA}{d\theta} = \frac{-2ab\cos 2\theta}{(\sin 2\theta)^2}$$

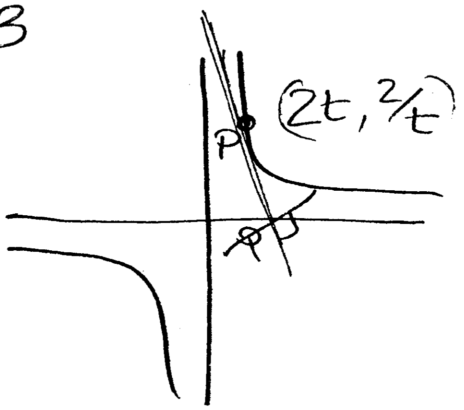
$$\frac{dA}{d\theta} = 0 \quad \therefore \quad \cancel{-2ab}\cos 2\theta = 0$$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

Extension 2 - Pre Trial - 2007

Q3
i)



$Y = \frac{4}{x}$

$\frac{dy}{dx} = -\frac{4}{x^2}$

$m = \frac{-4}{4t^2} = -\frac{1}{t^2}$

$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$

$yt^2 - 2t = -x + 2t$

(2)

ii) $l_P \quad \underline{x + yt^2 = 4t}$

Q on x axis $\therefore y = 0$ $\quad Q(4t, 0)$

$m = t^2$

$y - 0 = t^2(x - 4t)$

$l_Q: \underline{t^2x - y = 4t^3}$

(2)

iii) Sub $y = 4/x$ into l_Q

$t^2x - \frac{4}{x} = 4t^3$

(3)

$t^2x^2 - 4t^3x - 4 = 0$

~~$x = 4t^3 \pm \sqrt{\ }$~~ Roots are x co-ordinates

$\therefore$ Midpoint is $\frac{\text{Sum of Roots}}{2}$ or $\frac{-b/a}{2}$

$\frac{4t^3}{2t^2} \Rightarrow 2t$

Sub into l_Q ~~$y = 4/x \therefore y = 2/t$~~ $t^2 \times 2t - 4t^3 = y$

$y = -2t^3$

$M = [2t, -2t^3]$

(2)

iv) $x = 2t$

$y = -2t^3 \Rightarrow y = \frac{-x^3}{4}$ $\quad x \neq 0$

Question ~~2~~4

c) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx \Rightarrow \sin^{-1} e^x + C$ (2)

a) $\displaystyle\int_0^1 \tan^{-1} x \, dx$

$u = \tan^{-1} x \qquad dv = 1$

$du = \dfrac{1}{1+x^2} \qquad v = x$

$\displaystyle\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$\displaystyle\left[x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1$ (3)

$\left[\tan^{-1} 1 - \tfrac{1}{2}\ln 2\right] - \left[0 - \tfrac{1}{2}\ln 1\right]$

$\dfrac{\pi}{4} - \ln\sqrt{2}$

$\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$x^2+2x+1+4$

$(x+1)^2+4$

$\displaystyle\int_{-1}^{1} \frac{1}{4+(x+1)^2}$

(3)

$\displaystyle\left[\frac{1}{2}\tan^{-1}\frac{(x+1)}{2}\right]_{-1}^{1}$

$\tfrac{1}{2}\left[\tan 1 - \tan^{-1} 0\right]$

$\dfrac{\pi}{8}$

Extension 2 - PreTrial - 2007

Question 4 (cont)

b) $(3-x) \equiv (Ax+B)(1+6x) + C(1+2x^2)$

$x=0 \quad B+C=3 \quad -①$

$x=1 \quad 7A+7B+3C=2 \quad -②$

$x=-1 \quad +5A-5B+3C=4 \quad -③$

Rearrange ① ⇒ Put int ② an ③

$7A-4C=-19$ and $19=5A+8C$

⇓

$A=-1,\ B=0,\ C=3$

2

$$\int_0^2 \frac{-1x}{1+2x^2} + \frac{3}{1+6x}\,dx$$

$$\left[-\tfrac{1}{4}\ln(1+2x^2) + \tfrac{1}{2}\ln(1+6x)\right]_0^2$$

$$\tfrac{1}{2}\ln 13 - \tfrac{1}{2}\ln 9^{1/2}$$

$$\tfrac{1}{2}\ln\tfrac{13}{3}$$

2

d) $\displaystyle\int_0^{\pi/2} \sin x \cos 2x\,dx$

$\cos 2x = 2\cos^2 x - 1$

$\therefore \sin x\cos 2x = 2\sin x\cos^2 x - \sin x$

$\int 2\sin x\cos^2 x\,dx \qquad \int \sin x\,dx$

$\left[-\tfrac{2}{3}\cos^3 x\right] \qquad [-\cos x]$

$$\left[-\tfrac{2}{3}\cos^3 x + \cos x\right]_0^{\pi/2}$$

$$\left[0 - \tfrac{1}{3}\right]$$

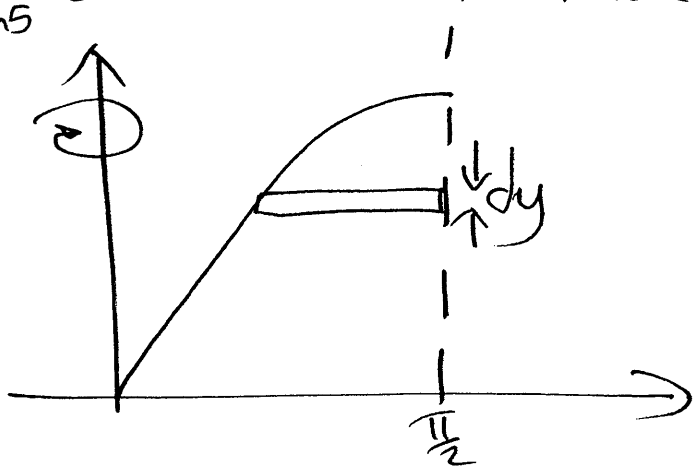
$$-\tfrac{1}{3}$$

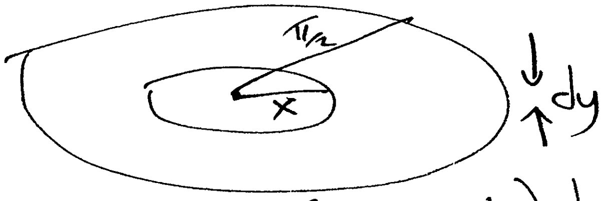
3

Extension 2 - Pre Trial - 2004

Question 5

a)





$$dv = \left(\pi\left(\frac{\pi}{2}\right)^2 - \pi x^2\right)dy$$

$$V = \pi\int_0^1 \left(\frac{\pi}{2}\right)^2 - x^2\,dy$$

$y = \sin x$
$\therefore x = \sin^{-1} y$
(Tough integration)
$\therefore$ change to dx

$\Rightarrow$ $y = \sin x$
$\frac{dy}{dx} = \cos x$ $\Rightarrow$
$dy = \cos x\,dx$

Change Limit
$1 \to \frac{\pi}{2}$
$0 \to 0$

$$V = \pi\int_0^{\frac{\pi}{2}} \frac{\pi^2}{4}\,dy \cancel{\sin^2 x \cos x\,dx} - \pi\int_0^{\frac{\pi}{2}} x^2\cos x\,dx$$

$u = x^2$ $dv = \cos x$
$du = 2x$ $v = \sin x$

$$= \pi\left[\frac{\pi^2}{4}y \cancel{\frac{1}{3}\sin^3 x}\right]_0 - \pi\left\{\left[x^2\sin x\right]_0^{\frac{\pi}{2}} - \int 2x\sin x\,dx\right\}$$

$u = 2x$ $dv = \sin x$
$du = 2$ $v = -\cos x$

$$= \pi\left[\left(\frac{\pi^2}{4} - \cancel{\frac{1}{3}}\right) - (0 - 0)\right]$$

$$= \pi\left[\frac{\pi^3}{4}\right] - \pi\left[\left[x^2\sin x\right]_0^{\frac{\pi}{2}} - \left[-2x\cos x\right]_0^{\frac{\pi}{2}} - \int_0 2\cos x\,dx\right]$$

$$= \pi\left\{\frac{\pi^3}{4} - \left(\frac{\pi^2}{4} - 0\right) - (0 - 0) - \left[2\sin x\right]_0^{\frac{\pi}{2}}\right\}$$

$$= \pi\left\{\frac{\pi^2}{4} - \frac{\pi^2}{4} + 2\right\}$$

$$= 2\pi$$

5

Question 5 (cont)

b) Step 1 $n=4$ $\quad 3^4=81 \quad 4^3=64$

$3^n > n^3$ is true for $n=4$

Step 2 Assume $3^K > K^3$

Prove $3^{K+1} > (K+1)^3$

$3^{K+1} - (K+1)^3 > 0$

$3.3^K - K^3 - 3K^2 - 3K - 1$

$3(3^K - K^3) + 2K^3 - 3K^2 - 3K - 1$

$3(3^K - K^3) + (K^3 - 3K^2 + 3K - 1) + (K^3 - 6K)$

$3(3^K - K^3) + (K-1)^3 + K(K^2 - 6)$

Assume >0 $\quad$ true as $K>3$ $\quad$ true as $K>3$

3

$\therefore 3^{K+1} > (K+1)^3$.

Proved true for $n=4$, $n=K+1$ $\therefore$ $n=5$ . . .

$3^n > n^3$

$(3^n)^{1/3n} > (n^3)^{1/3n}$ $\quad$ Take $3n^{th}$ Root

2

$3^{1/3} > n^{1/n}$

$\sqrt[3]{3} > \sqrt[n]{n}$.

Extension 2 - Pre Trial - 2007

Question 5 (cont)

c $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx$

$= \left[-\sin x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

$= n-1 \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x)$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x$

$I_n = (n-1) I_{n-2} - (n-1) I_n$

$n I_n = (n-1) I_{n-2}$

$I_n = \frac{(n-1)}{n} I_{n-2}$.

2

$I_{2n} = \left(\frac{2n-1}{2n}\right) I_{(2n-2)}$

$I_{2n-2} = \left(\frac{2n-3}{2n-2}\right) I_{2n-4}$

$I_{2n-4} = \left(\frac{2n-5}{2n-4}\right) I_{2n-6}$

(Notice Pattern $\frac{a}{b} I_c$ where order b, a, c go to end at sequence 2, 1, 0.

$I_2 = \frac{1}{2} I_0$ *

$I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx = \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$I_2 = \frac{2n}{2n} \times \frac{2n-1}{2n} \times \frac{2n-2}{2n-2} \times \frac{2n-3}{2n-2} \times \cdots \times \frac{1}{2} \times \frac{\pi}{2}$

Insert.

$\boxed{\text{Top Line } 2n!}$

Bottom Line $4n \times 4 \times (n-1) \times 4 \times (\ldots$

$\boxed{4^n \times (n!)^2}$

3

$\frac{2n!}{(n!)^2} \times \frac{\pi}{4^n \times 2}$

2^{2n+1}

QED.