The Scots College

Year 12 Mathematics Extension 2

Pre-Trial Assessment

April 2009

General Instructions

- All questions are of equal value
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table is attached

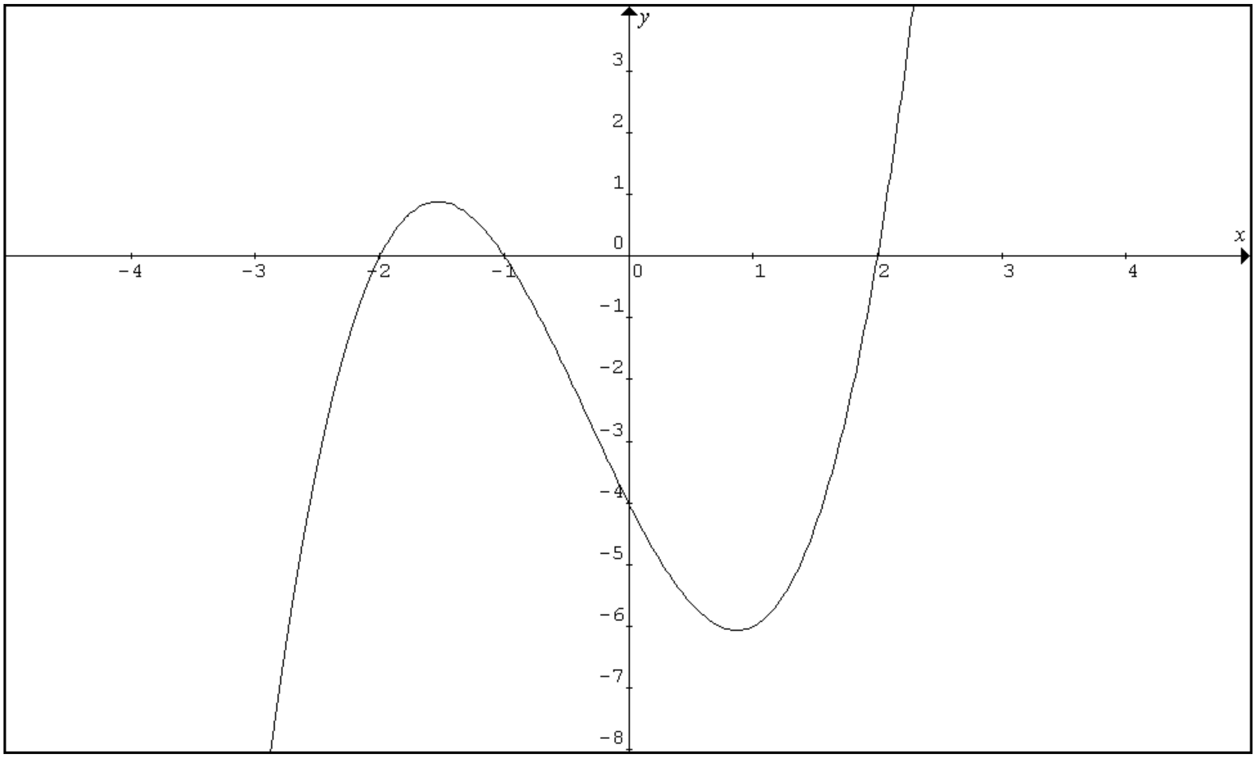
TOTAL MARKS: 75

WEIGHTING: 30 %

- **Start each question in a new booklet**

QUESTION 1 **[15 MARKS]**

(a) In the diagram below, the graph f(x) is drawn.



On separate diagrams, and without the use of calculus, sketch the following graphs. Indicate clearly any asymptotes, any intercepts with the coordinate axes, and any other significant features.

(i) y = $\dfrac{1}{f(x)}$ **2**

(ii) y = $-\sqrt{f(x)}$ **2**

(iii) y = $e^{f(x)}$ **2**

(iv) y = $f(2-x)$ **2**

(v) y = $f'(x)$ **2**

(b) Given the function $f(x)=\sqrt{2-\sqrt{x}}.$

(i) What is the domain of this function **1**

(ii) Show that $f(x)$ is a decreasing function and hence find its range **1**

(iii) Draw a neat sketch of $f(x)=\sqrt{2-\sqrt{x}}.$ **3**

QUESTION 2 **[15 MARKS]**

(a) Find $\int 7x\sqrt{(4x^2-3)}\,dx$ **2**

(b) Find $\int \frac{x+1}{\sqrt{(x-1)}}\,dx$ **2**

(c) Evaluate

(i) $\int_0^{\frac{\pi}{3}} \tan x.\sec^4 x\ dx$ **2**

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x+2}$, by using the substitution $t=\tan\frac{x}{2}$. **2**

(c) Find the values of A and B such that

$$\frac{e^x}{(e^x+2)(e^x+1)}=\frac{A}{(e^x+2)}+\frac{B}{(e^x+1)}$$ **1**

Hence find $\int \frac{e^x}{e^{2x}+3e^x+2}\,dx.$ **2**

(d) $I_n=\int_0^{\frac{\pi}{2}} \cos^n x dx \quad n\ge 0$

(i) Show that $I_n=\frac{n-1}{n}I_{n-2}$ **2**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x dx$ **2**

QUESTION 3 **[15 MARKS]**

(a) Find the equation of the circle which has the points A (3 , -1) and B (9 , 3) which are at opposite ends of a diameter. **2**

(b) State the foci, directrices and eccentricity of $4x^2 + 25y^2 - 100 = 0$ **2**

(c) Prove that for any point P on the Hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the **difference** of its distances from the foci, $S \ and \ S'$ is constant. **2**

Find the value of this constant. **1**

(d)

QUESTION 4 **[15 MARKS]**

(a) Reduce the complex number (2 – i)(8 + 3i)/(3 + i) to the form a + ib, where a and b are real numbers. **2**

(b) The complex number z is given by z = - Root 3 + i.

(i) Write down the values of Arg (z) and !z! **1**

(ii) Hence or otherwise show that z (to the 7) +64 z = 0 **2**

(c) Sketch the following loci on separate Argand diagrams;

(i) Arg(z + 1 + i) = Pi/4 **1**

(ii) !z – 2! = ! z + i! **1**

(d) Given that **z = 3 +4i.** Find **w** so that **O, z** and **w** form a right angled isosceles triangle, (whose right angle is at **z**) on the Argand diagram. **3**

(e) Using any complex number **z** and on separate Argand diagrams, illustrate the geometric properties of the following;

(i) **z squared** **1**

(ii) **z x (1 + i)** **1**

(iii) **z – z(bar)** **1**

(f) Given z (to the 4) = i. Find the four roots of unity **2**

QUESTION 5 **[15 MARKS]**

(a) Using Integration by parts, to show $\int_1^e \sin(\ln x)\,dx = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}$. **4**

(b) **Complex qu**

(c) Sketch the following equation; $|y| = (x-1)(x-2)(x-3)$ **4**

(d) **Conics qu**

END OF PAPER

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

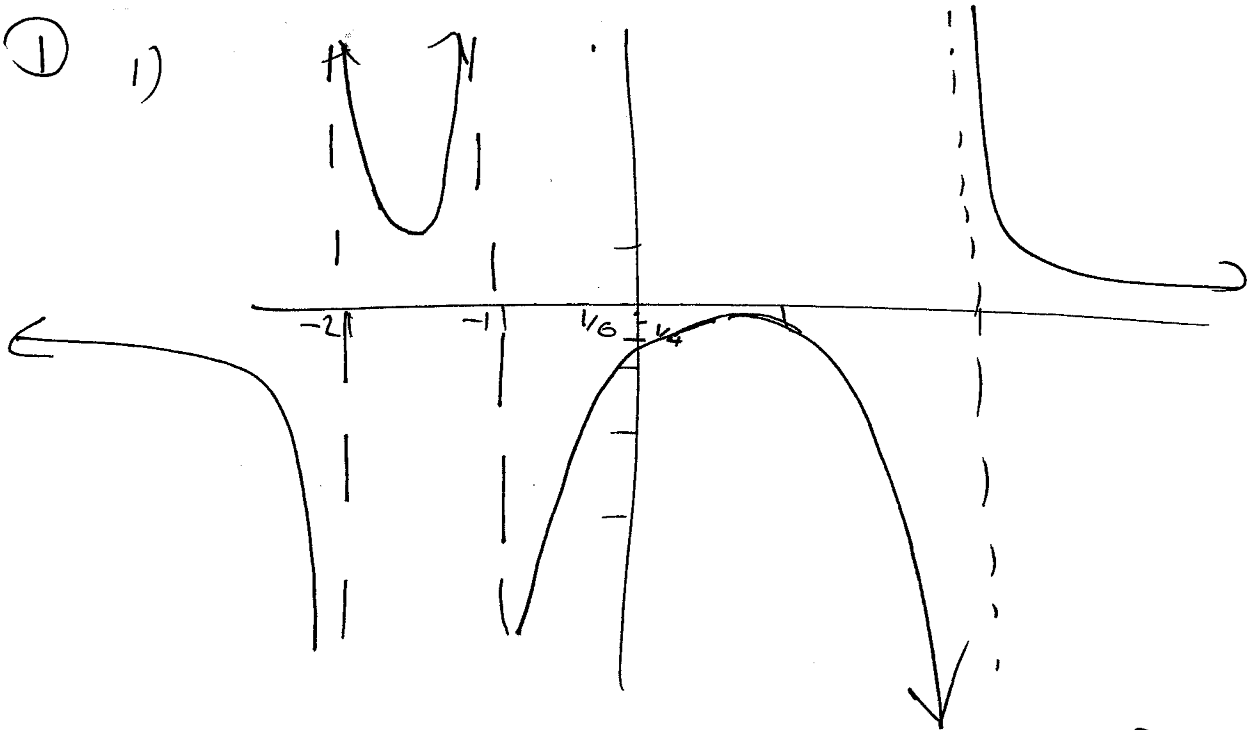
$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x \equiv \log_e x, \quad x > 0$

1 i)



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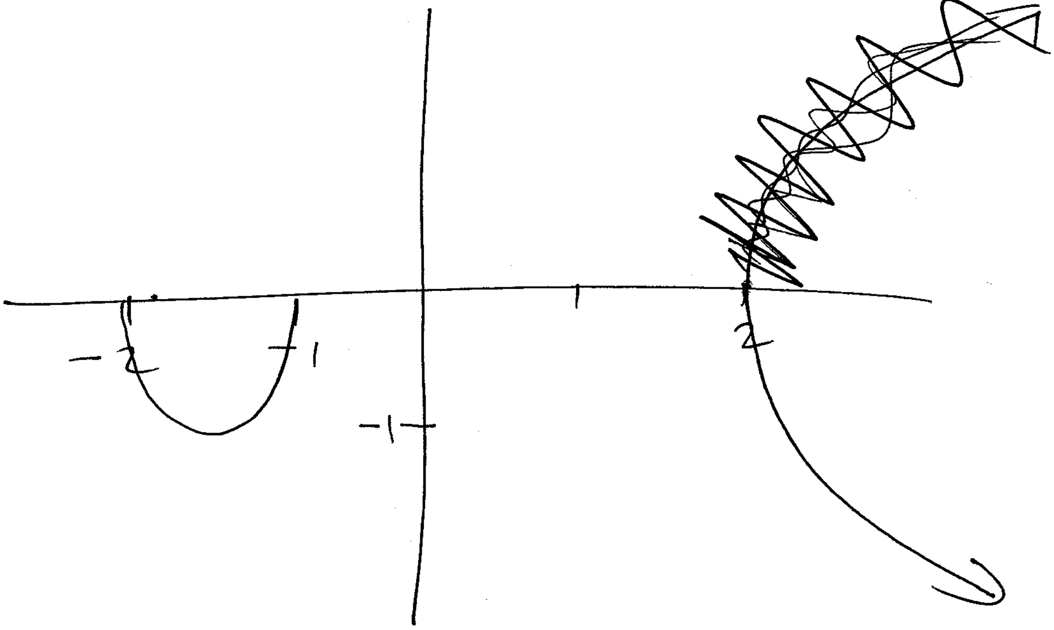
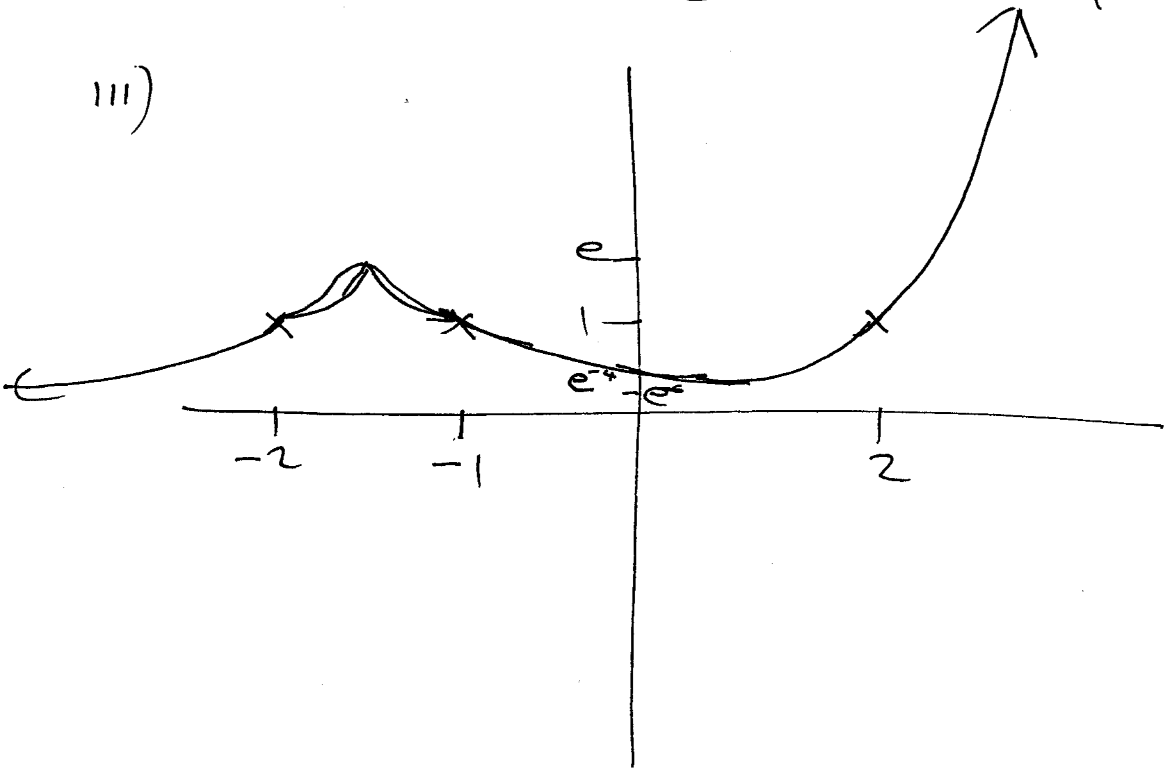


iii)

Ext 2 Pre Trial 2004 Question 1

iii)



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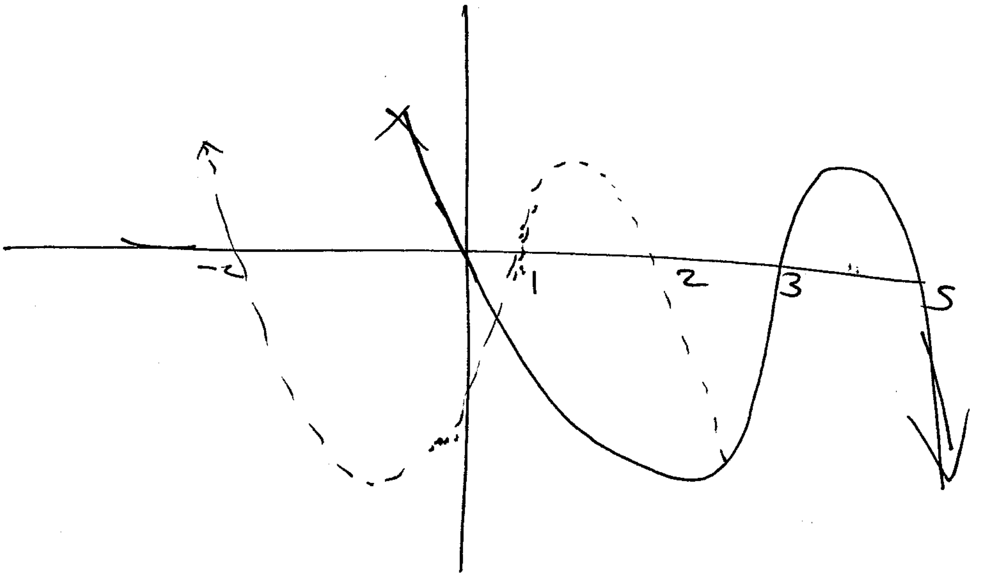

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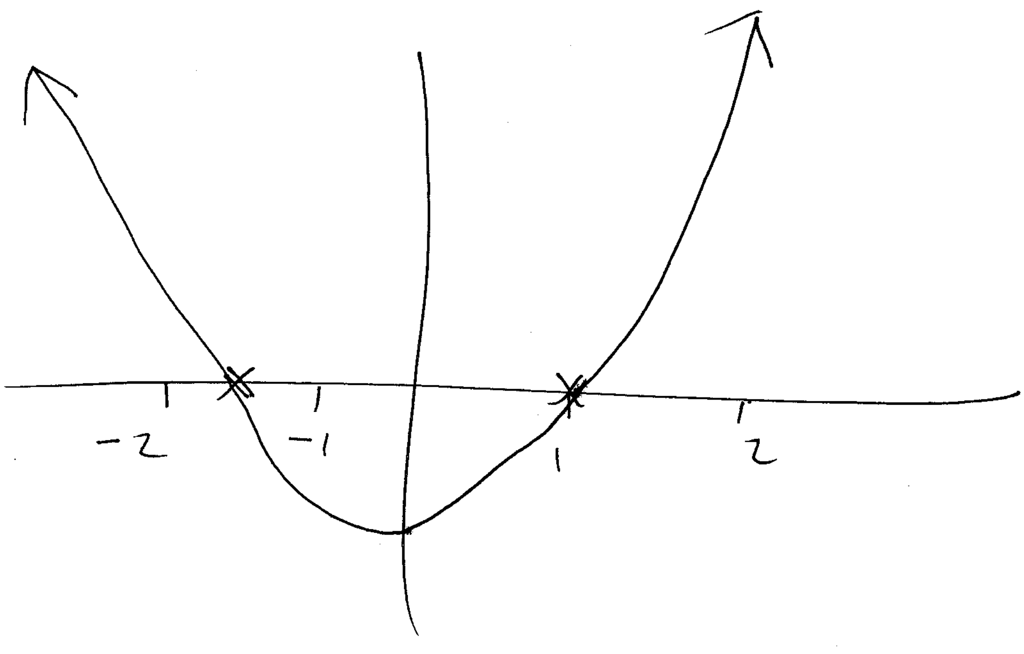


b) $f(x) = \sqrt{2-\sqrt{x}}$

i) Domain $\sqrt{X} \leqslant 2$

$0 \leqslant X \leqslant 4$

ii) $f(x) = (2 - x^{1/2})^{1/2}$

$\therefore f'(x) = +\frac{1}{2}(2-x^{1/2})^{-1/2} \times -\frac{1}{2}x^{-1/2}$

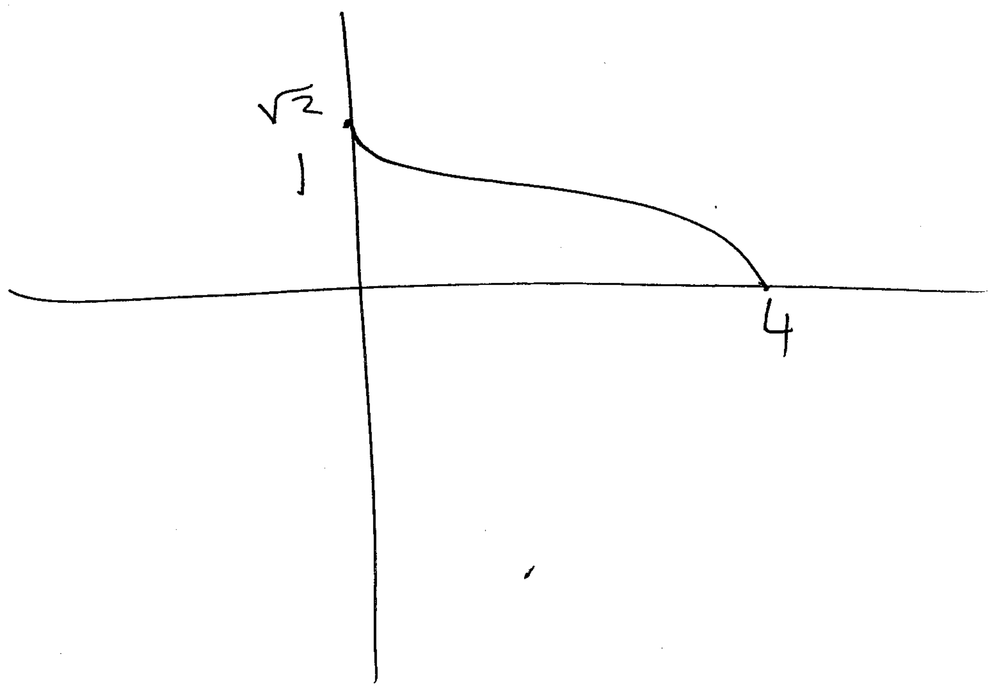
$= \dfrac{-1}{4(2-x^{1/2})^{1/2}x^{1/2}}$ ← Positive for Domain

$\therefore f'(x)$ always $-$ve $\therefore$ Always decreasing

$\therefore f(0) = \sqrt{2} \qquad f(4) = 0$

$\therefore$ Range $0 \leqslant y \leqslant \sqrt{2}$

iii)



Question 2

a) $\int 7x\sqrt{4x^2-3}$

by observation $7x(4x^2-3)^{1/2}$

$\frac{7}{12}(4x^2-3)^{3/2}+C$

b) $\int \frac{x+1}{\sqrt{x-1}}dx$ by sub $u=x-1$

$\frac{du}{dx}=1$

$du=dx$

$\int \frac{u+2}{\sqrt{u}}du$

$\int u^{1/2}du+\int 2u^{-1/2}dx$

$=\frac{2}{3}u^{3/2}+4u^{1/2}+C$

Can tidy up

$\frac{2}{3}(x-1)^{3/2}+4(x-1)^{1/2}$

$(x-1)^{1/2}\left[\frac{2}{3}(x-1)+4\right]$

$\frac{1}{3}(x-1)^{1/2}(2x-10)+c$

C

$$\int_0^{\pi/3} \tan x \sec^4 x \, dx$$

$\tan x \sec^2 x \sec^2 x$

$\tan x (\tan^2 x + 1) \sec^2 x$

$\sec^2 x \tan^3 x + \sec^2 x \tan x$

$$\left[\tfrac{1}{4}\tan^4 x + \tfrac{1}{2}\tan^2 x\right]_0^{\pi/3}$$

$\tfrac{1}{4}(\sqrt{3})^4 + \tfrac{1}{2}(\sqrt{3})^2$

$9/4 + 3/2 = 3\,3/4$

$$\int_0^{\pi/2} \frac{dx}{\cos x + 2}$$

$t = \tan x/2$

$\frac{dt}{dx} = \tfrac{1}{2}\sec^2 x/2$

$\frac{dx}{dt} = \frac{2}{\sec^2 \frac{x}{2}}$

$dx = \frac{2}{1+t^2}\,dt$

$x = \pi/2 \quad t = \tan \pi/4 = 1$

$x = 0 \quad t = \tan 0 = 0$

$$\int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + 2} \times \frac{2}{1+t^2}$$

$$\int_0^1 \frac{1}{\frac{1-t^2+2+2t^2}{1+t^2}}$$

$$\int_0^1 \frac{\cancel{1+t^2}}{3+t^2} \times \frac{2}{\cancel{1+t^2}}\,dt$$

$$\frac{2}{\cancel{3}}\int_0^1 \frac{1}{3+t^2} \Rightarrow \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$2/\sqrt{3}\left[\tan^{-1} 1/\sqrt{3}\right] - 0 \qquad \frac{\cancel{2}\pi}{3\sqrt{3}}$

d)

$$e^x = A(e^x+1) + B(e^x+2)$$

$$e^x = Ae^x + Be^x + A + 2B$$

$\therefore$ $A+B=1$ $\qquad$ $A+2B=0$

$A = 1-B$

$1-B+2B = 0$

$B = -1$

$A = 2$

$$\int \frac{2}{e^x+2}\,dx + \int \frac{1}{e^x+1}\,dx$$

~~BA~~

let $e^x+2 = U$ $\qquad$ let $e^x+1 = V$

$\therefore \frac{dU}{dx} = e^x$ $\qquad$ $\frac{dV}{dx} = e^x$

$dx = \frac{dU}{e^x}$ $\qquad$ $dx = \frac{dV}{e^x}$

$$\int \frac{2}{U} \times \frac{dU}{U-2} \qquad \int \frac{1}{V} \times \frac{dV}{V-1}$$

$$\int \cancel{\frac{2}{U^2-}} \frac{2}{U(U-2)} = \frac{A}{U} + \frac{B}{(U-2)} \qquad \frac{1}{V(V-1)} = \frac{A}{V} + \frac{B}{V-1}$$

$1 = A(V-1) + BV$

$2 = A(U-2) + BU$

$V=0 \quad A = -1$

$V=1 \quad B = 1$

$U=0 \quad A=-1$

$U=2 \quad B=1$

$$\int -\frac{1}{U} + \frac{1}{U-2}$$

$$\int -\frac{1}{V} + \frac{1}{V-1}$$

$$-\ln U + \ln U-2$$

$$\ln \frac{U-2}{U} - \ln U$$

$$\ln e^x - \ln e^x+2$$

$$1 - \ln(e^x+2)$$

$$-\ln V + \ln V-1$$

$$\ln V-1 - \ln V$$

$$\ln e^x - \ln e^x+1$$

$$1 - \ln(e^x+)$$

$$\ln(e^x+1) - \ln(e^x+2)$$

$$\frac{e^x}{\left(e^x+\frac{3}{2}\right)^2 - \frac{1}{4}}$$

$$\frac{e^x}{e^x+1} - \frac{e^x}{e^x+2}$$

$$\frac{\cancel{e^{2x}}+2e^x-\cancel{e^{2x}}-e^x}{(\quad)(\quad)}$$

$$\frac{e^x}{(\quad)(\quad)}$$

✓

EXT 2 TRIAL 2004 - QUESTION 2

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx \qquad n \geq 0$$

$$\int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x \, dx$$

$dv = \cos x \qquad u = \cos^{n-1} x$

$v = \sin x \qquad du = n-1 \cos^{n-2} x \cdot -\sin x$

$$I_n = \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \sin^2 x \cos^{(n-2)} x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} (1-\cos^2 x) \cos^{n-2} x \, dx$$

$$= n-1 \int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= n-1 \, I_{(n-2)} - (n-1) I_n$$

$$n I_n = n-1 \, I_{(n-2)}$$

$$I_n = \frac{(n-1)}{n} I_{(n-2)}$$

$$I_4 = \frac{3}{4} I_2$$

$$I_2 = \frac{1}{2} I_0$$

$$I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x$$

$$= [x]_0^{\frac{\pi}{2}}$$

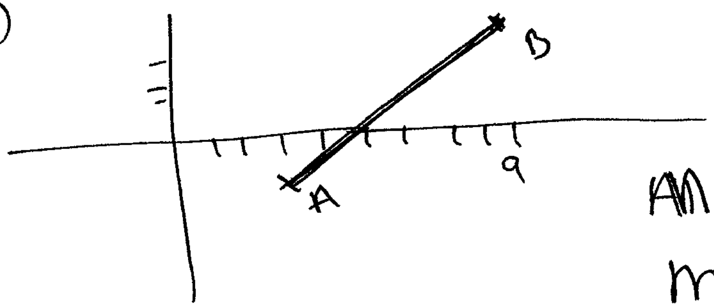
$$= \frac{\pi}{2}$$

$$I_2 = \frac{\pi}{4}$$

$$I_4 = \frac{3\pi}{16}$$

//

a)



Mid Point = Centre

$(6, 1)$

$MB^2 = \sqrt{3^2 + 2^2}$

$MB^2 = 13 = \text{Radie}^2.$

$\therefore (x-6)^2 + (y-1)^2 = 13.$

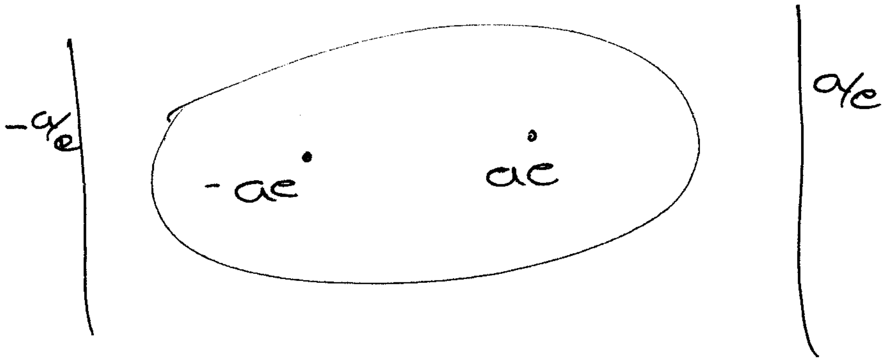
b) $\dfrac{x^2}{5^2} + \dfrac{y^2}{2^2} = 1$ [Ellipse]

$b^2 = a^2(1-e^2)$

$\dfrac{4}{25} = 1 - e^2 \quad \therefore e^2 = 21/25$

$\Rightarrow e = \sqrt{21}/5$

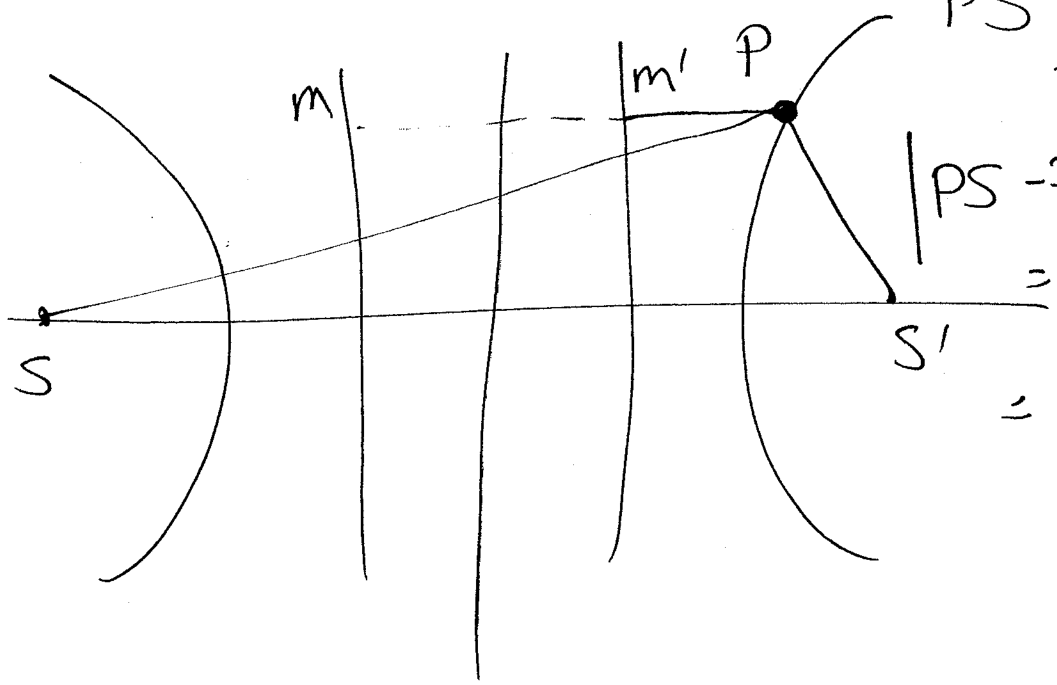
c)



Foci $[\pm\sqrt{21}, 0]$

Direct: $\left[x = \pm \dfrac{25}{\sqrt{21}}\right]$

c)



$PS' = ePM'$

$PS = ePM$

$= e[PM' + 2a/e]$

$|PS - PS'|$

$= e[PM' - PM' + 2a/e]$

$= 2a.$

d) i) $y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

at $P = \frac{-c^2}{c^2p^2}$

$m_{tan} = -\frac{1}{p^2}$

$m_{NORMAL} = p^2$

$y - \frac{c}{p} = p^2(\cancel{x} - cp)$

$yp - c = p^3(x - cp)$

ii) For this to be true

$xy = c^2 \quad -①$

$py - c = p^3(x - cp)$

Rearrange ①

$y = \frac{c^2}{x}$

$py - c = \frac{pc^2}{x} - c$

$\therefore \frac{pc^2}{x} - c = p^3(x - cp)$

$\frac{c^2}{p^2} - \frac{cx}{p^3} = x^2 - cpx$

$x^2 + \frac{c}{p^3}x - cpx - \frac{c^2}{p^2} = 0$

$x^2 - c\left[p - \frac{1}{p^3}\right]x - \frac{c^2}{p^2} = 0$

$\therefore$ P and Q statisfy the equat

Roots of Quadratic are x_Q, x_P.

$\therefore$ Products of Roots $= \dfrac{c^2}{p^2}$

$$x_p = cp$$

$$\therefore x_q = c/p^3$$

Sub in $xy = c^2$

$$y_Q = \frac{c^2}{c/p^3}$$

$$y_q = cp^3$$

$$Q\left(-c/p^3, -cp^3\right)$$

$xy = c^2$ is odd function

$$\therefore R\left(-cp, -c/p\right)$$

$$m_{PR} = \left[\frac{\frac{2c}{p}}{2cp}\right] = 1/p^2$$

$$m_{PR} \left[\frac{-cp^3 - c/p}{-c/p^3 - cp}\right]$$

$$\frac{\cancel{c}}{\cancel{c}}\left[\frac{\frac{p^4-1}{p}}{\frac{\cancel{p^4}-p^4}{p^3}}\right]$$

$$= -p^2$$

$$1/p^2 \times -p^2 = -1$$

$\therefore \perp$

a) $\frac{5+i}{3-2i} \times \frac{3+2i}{3+2i}$

$\frac{15+10i+3i-2}{9+4}$

$\frac{13}{13}+\frac{13i}{13}$

$1+i$

$\sqrt{2}\operatorname{cis}\pi/4$

b) $w^2 = 5-12i$

$(a+bi)^2 = 5-12i$

$a^2-b^2+2abi = 5-12i$

$a^2-b^2=5 \quad 2ab=-12$

$a=\frac{-6}{b}$

$\frac{36}{b^2}-b^2=5$

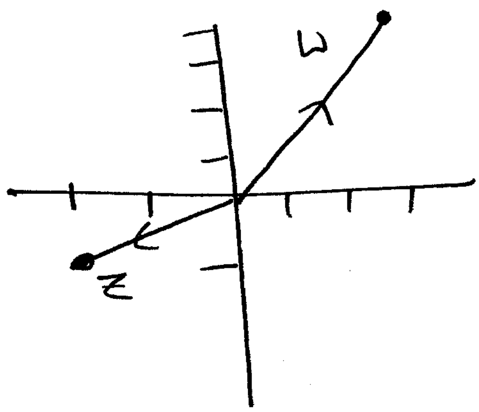
$b^4+5b^2-36=0$

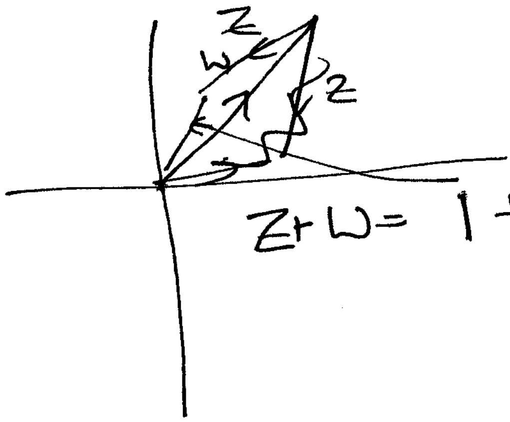
$[b^2+9][b^2-4]=0$

Reject $\quad b=\pm 2$

$a=\mp 3$

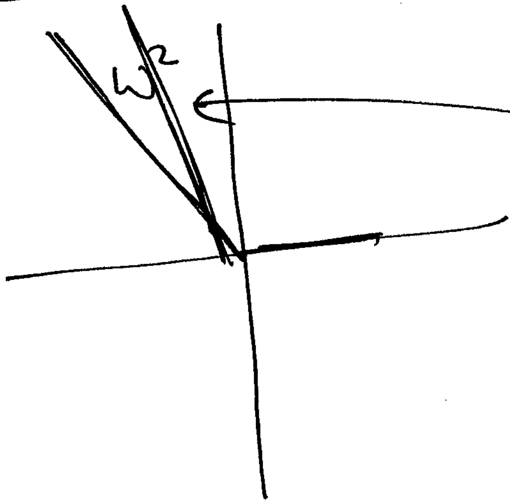
$-3+2i$, $3-2i$





$z+w = 1+3i$

$z.\bar{z}+w^2$



$w^2 = 9-16+24i$

$\therefore -7+24i$

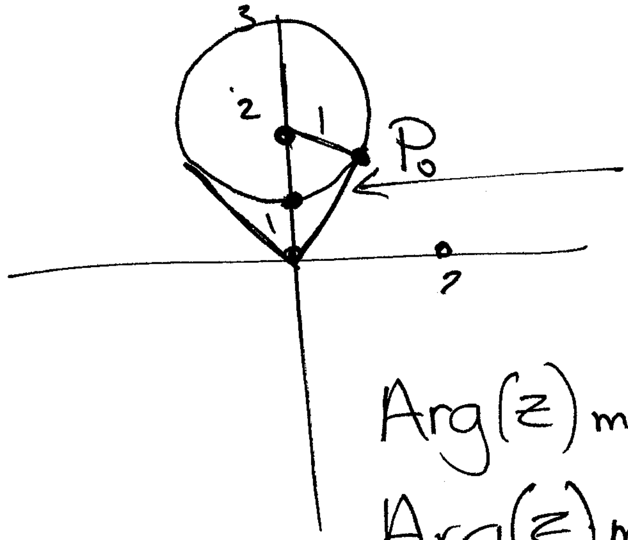
$z\bar{z} = (-2-i)(-2+i)$

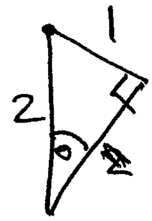
$4+1=5$

$-2+24i$

Ext2 Pre Trial 20?? Question ~~11~~

d)





$\sin\theta = \frac{1}{2}$

$\frac{\pi}{6}.$

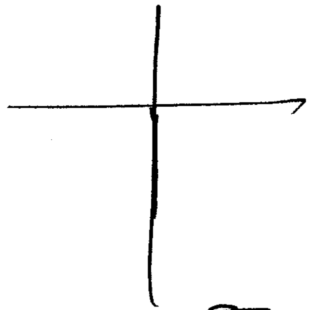
$Arg(z)_{min}$ $\frac{\pi}{3}$

$Arg(z)_{max}$ $\frac{2\pi}{3}.$

$P_0 = \sqrt{3}\ cis \frac{\pi}{3}$

e)

$$z^6 = -1$$



$z^6 = 1 cis -\frac{\pi}{2}.$

$\therefore z_1 = 1 cis -\frac{\pi}{12}.$ $\qquad \frac{2\pi}{6} = \frac{\pi}{3}$

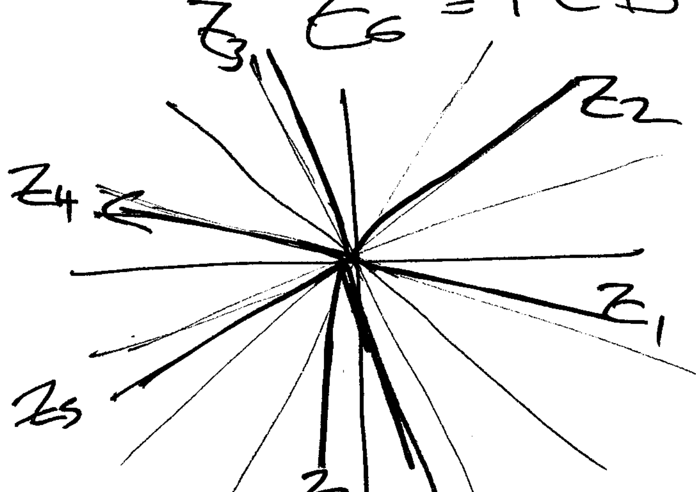
$z_2 = 1 cis\left(-\frac{\pi}{12}+\frac{\pi}{3}\right) = cis\left(\frac{3\pi}{12}\right) = cis\frac{\pi}{4}$ $\left(\frac{4\pi}{12}\right)$

$z_3 = 1 cis\left(\frac{7\pi}{12}\right)$

$z_4 = 1 cis\left(\frac{11\pi}{12}\right)$

$z_5 = 1 cis\left(\frac{15\pi}{12}\right) = 1 cis\left(-\frac{9\pi}{12}\right) = cis\left(\frac{-3\pi}{4}\right)$

$z_6 = 1 cis\left(-\frac{5\pi}{12}\right)$



Multiply to make an imaginary number

$\therefore$ no conjugate

a) $\int_1^e \sin(\ln x)\,dx$

$dv = 1 \quad u = \sin(\ln x)$

$v = x \quad du = \frac{1}{x}\cos(\ln x)$

$$I = \left[x\sin(\ln x)\right]_1^e + \int_1^e -\cos(\ln x)\,dx$$

$dv = 1 \quad u = -\cos(\ln x)$

$v = x \quad du = \frac{1}{x}\sin(\ln x)$

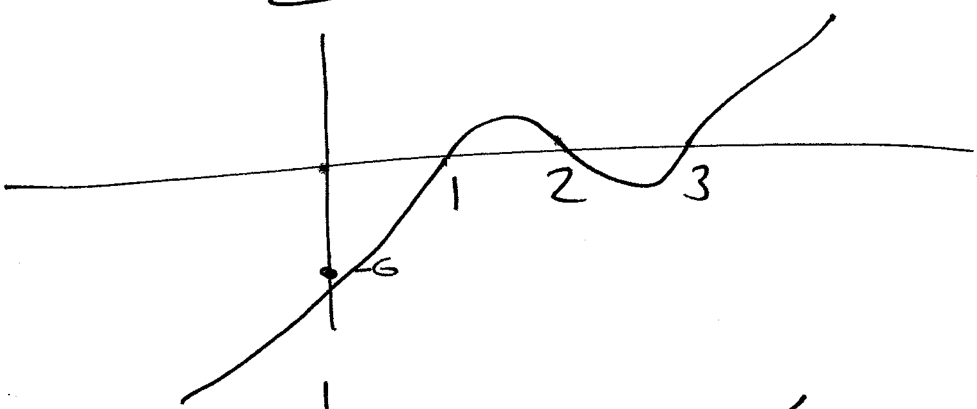
$$I = \left[x\sin(\ln x)\right]_1^e + \left[-x\cos(\ln x)\right]_1^e - I$$

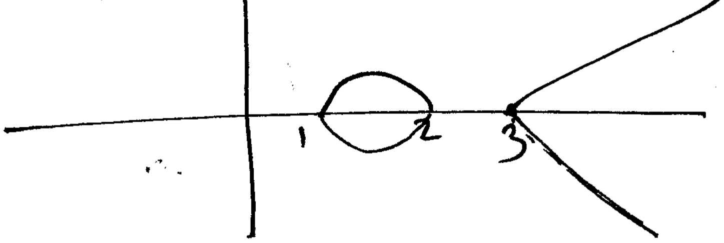
$$2I = e\sin 1 - 0 + -e\cos 1 + 1$$

$$2I = e[\sin 1 - \cos 1] + 1$$

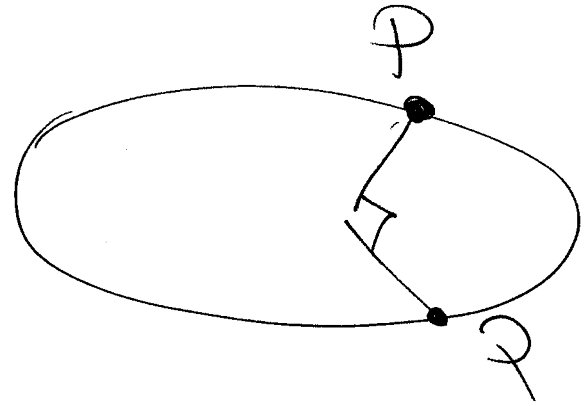
$$I = \tfrac{e}{2}[\sin 1 - \cos 1] + \tfrac{1}{2}$$

b)





c)



$$4x^2+9y^2=36$$

$$8x+18y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{8x}{18y}$$

at $P_{TAN}=\dfrac{-24\cos\alpha}{36\sin\alpha}$

$\therefore$ $P_{NORMAL}=+\dfrac{3\sin\alpha}{2\cos\alpha}$

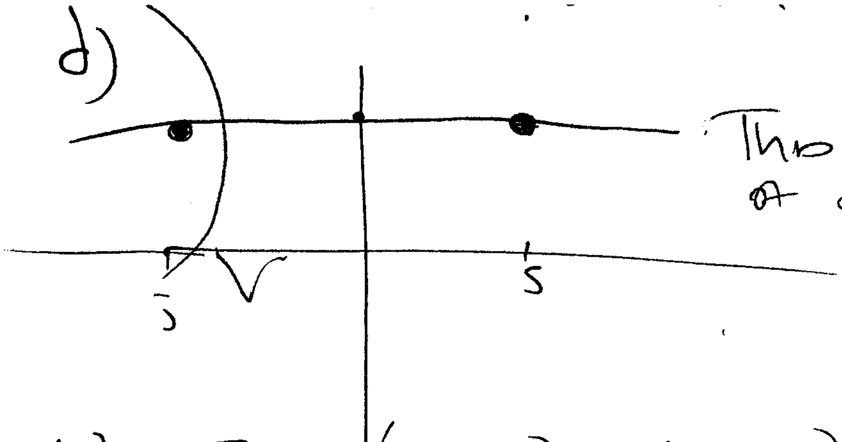
Similarly $Q_{NORMAL}=\dfrac{3\sin\beta}{2\cos\beta}$

we know $\dfrac{3\sin\alpha}{2\cos\alpha}\cdot\dfrac{3\sin\beta}{2\cos\beta}=-1$

$$\therefore \frac{9}{4}\tan\alpha\tan\beta=-1$$

$$\therefore 4\cot\alpha\cot\beta=-9$$

d)



This is the definition of a Hyperbola $= 2a$

ii) Foci $(5,0)$ $(-5,0)$

$2ae = 10$

$a = 2$

$\therefore e = \frac{5}{2}$

$b^2 = 21$

$b^2 = a^2(e^2 - 1)$

$b^2 = 4\left(\frac{25}{4} - 1\right)$

$\frac{b^2}{4} = \frac{21}{4}$

$$\frac{x^2}{4} - \frac{y^2}{21} = 1$$

~~Same shape redrawn to show shift~~

$x = 5$

$\frac{25}{4} - \frac{y^2}{21} = 1$

$\frac{21}{4} = \frac{y^2}{21}$

$\frac{21^2}{4} = y^2$

$\pm \frac{21}{2}$

$\therefore$ latus $= \frac{42}{2}$

$= 21$