The Scots College

Year 12 Mathematics Extension 2

Pre-Trial

April 2010

Name:__________

General Instructions

- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS: 80

THERE ARE 4 QUESTIONS, EACH WORTH 20 MARKS.

WEIGHTING: 30 %

Question 1: (Marks 20)

a)

Let $z = \cos\theta + i\sin\theta$, show that 2

i) $z^n + \frac{1}{Z^n} = 2\cos n\theta$,

ii) $z^n - \frac{1}{Z^n} = 2i \sin n\theta$ 2

b)

For each of the following, find and sketch the locus of the point representing the complex number Z:

i) $Re\,(iz) = -1$ 2

ii) $|z + 1 + i| = 2$ 2

iii) $\left|\frac{z - 1}{z + 1}\right| \leq 2$ 2

iv) $Arg\,(z - i) = \frac{\pi}{4}$ 2

Continue......................

Question 1 continued..........

c)

i) P is a point in an Argand diagram representing a non-zero complex number z such that $\arg\left(\frac{z}{1-i}\right) = \frac{\pi}{2}$

a) Find $\arg(1-i)$ and $\arg\ z$. 2

b) Hence or otherwise sketch the locus of P in an Argand diagram. 1

ii) R is a point on the Argand diagram representing a non-zero complex number w, such that 2

$$w^3 = (\overline{w})^3 \quad and \quad \frac{\pi}{2} < \arg w < \pi$$

Find $\arg\ w$

(Hint: you may let $w = r(\cos\theta + i\ \sin\theta)$

3

i) It is given that in part (i) , $|z| = 2\sqrt{2}$ and in part (ii) , $|w| = 2$, furthermore , $OPQR$ is a parallelogram in an Argand diagram, where O represents the complex number 0.

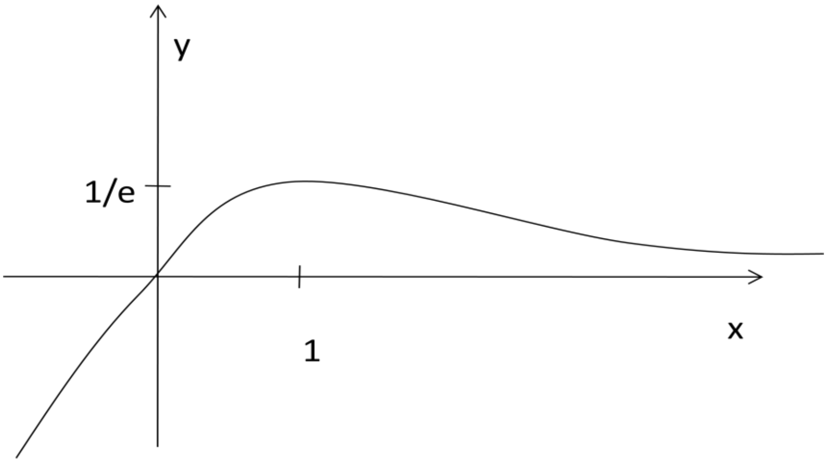
Find the complex number represented by the point Q, giving your answer in standard form.

Continue.....................

Question 2: (Marks 20)

a)

The graph of $y = x\,e^{-x}$ is shown below :



On separate axes, sketch the following curves. Indicate clearly any turning points, asymptotes and intercepts with the coordinate axes.

i) $y = x^2 e^{-2x}$ 2

ii) $y = \log_e(x\,e^{-x})$ 2

iii) $y = \frac{1}{x^2 e^{-2x}}$ 2

iv) $y = e^{x\,e^{-x}}$ 2

b)

Draw a sketch of the following, showing important features :

i) $f(x) = \frac{4}{x} - x$ 2

ii) $y = \sqrt{f(x)}$ 2

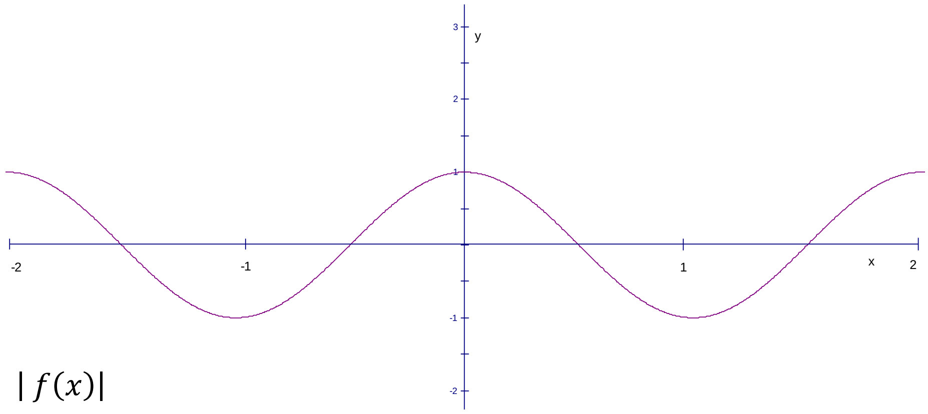
Continue.....

Question 2 continued.......

c)

The diagram below shows the graph of the continuous function $y = f(x)$. Critical points occur at $x = -2, -1, 0, 1, 2$

Draw separate sketches for each of the following:



i) $|y| = |f(x)|$ 2

ii) $y = \frac{1}{f(x)}$ 2

iii) $y = \sqrt{f(x)}$ 2

iv) $y = x\,f(x)$ 2

Question 3: (Marks 20)

a)

The hyperbola H has equation $xy = 16$

i) Sketch this hyperbola and indicate on your diagram the positions and co-ordinates of all points at which the curve intersects the axes of symmetry. 2

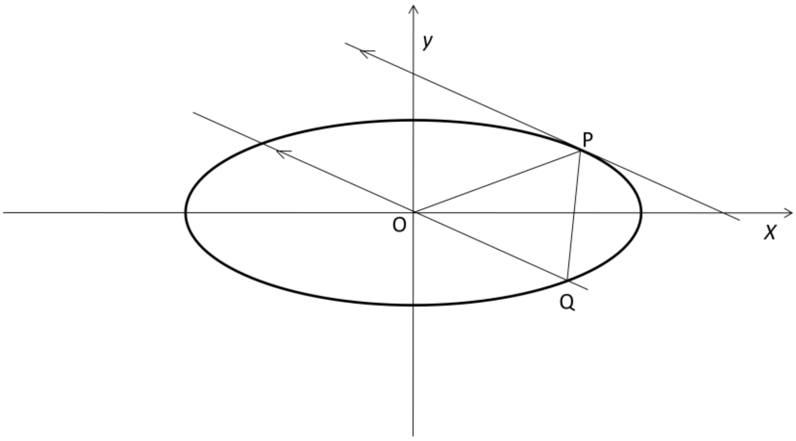
ii) $P\left(4p, \frac{4}{p}\right)$ *and* $Q\left(4q, \frac{4}{q}\right)$ are two distinct arbitrary points on H ($p > 0$, $q > 0$). Find the equation of the chord PQ. 2

iii) Prove that the equation of the tangent at P is $x + p^2 y = 8p$. 2

iv) The tangents at P and Q intersect at T. Find the coordinates of T. 2

Question 3 continued........

b)



In the diagram above $P(a\cos\theta\ ,\ b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where P lies in the first quadrant.
A straight line through the origin parallel to the tangent at P meets the ellipse at the point Q, where P and Q both lie on the same side of the y-axis.

i) Prove that the equation of the line OQ is $x\,b\cos\theta + y\,a\sin\theta = 0$. **2**

ii) Find the coordinates of the point Q given that Q lies in the fourth quadrant. **3**

iii) Prove that the area of the ΔOPQ is independent of the position of P. **3**

c)

i) Express $\frac{x^2-2x-3}{(x+2)(x^2+1)}$ as $\frac{A}{x+2} + \frac{Bx+C}{x^2+1}$ **2**

ii) Hence evaluate $\int_0^1 \frac{x^2-2x-3}{(x+2)(x^2+1)}\,dx$. **2**

Continue......................

Question 4: (Marks 20)

a)

$P(x)$ is a cubic polynomial with real coefficients . One zero of $P(x)$ is $1 + 2i$, the constant term is -15 and $P(2) = 5$. 5

Write $P(x)$ with real coefficients.

b)

The equation $x^3 - 4x + 5 = 0$ has roots α , β and γ.

i) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. 2

ii) Find the equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. 2

c)

Factorise $x^4 - 2x^2 - 15$ over the Complex number field. 2

d)

Solve the equation $x^4 - 5x^3 + 4x^2 + 3x + 9$ over complex field , given it has a root of multiplicity 2. 3

e)

If the roots of $x^3 - px + q = 0$ are α, β and γ , find in terms of p and q a cubic equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. 2

f)

i) By considering the expansion of $(\cos\theta + i\sin\theta)^5$ and by using De Moivre's theorem, show that 2

$$\cos 5\theta = 16\,cos^5\theta - 20\,cos^3\theta + 5\cos\theta$$

ii) Hence find all the four roots of the equation $16x^4 - 20x^2 + 5 = 0$. 2

END OF EXAM

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

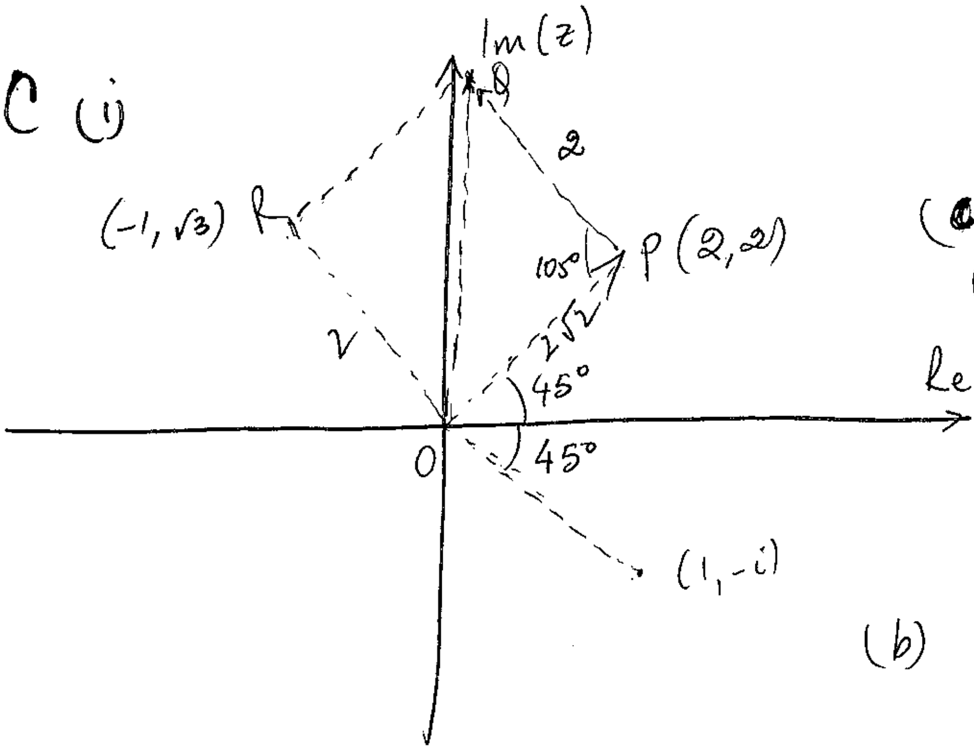
$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

C (i)



(a) (i) $\arg(1-i) = -45°$ or $-\frac{\pi}{4}$

$\arg Z = \frac{\pi}{4}$.

(b) Locus of P

$y = x, \quad x > 0$

(ii) $\omega = r(\cos\theta + i\sin\theta) \quad \bar{\omega} = r(\cos\theta - i\sin\theta)$

$r^3(\cos 3\theta + i\sin 3\theta) = r^3(\cos 3\theta - i\sin 3\theta)$

$2i\sin 3\theta = 0$.

$3\theta = 0, \pi, 2\pi, 3\pi \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi \ldots$

$\arg(\omega) = \frac{2\pi}{3}$ (since $\frac{\pi}{2} < \theta < \pi$)

(iii) $P : 2\sqrt{2}(\cos 45° + i\sin 45°)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= 2 + 2i$

$R : 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -1 + \sqrt{3}i$

$\therefore Q : (2-1) + i(2+\sqrt{3})$

or $Q : 1 + (\sqrt{3}+2)i$

3 (b)

(i) $OQ : x b \cos\theta + y a \sin\theta = 0$

$$y = -\frac{x b \cos\theta}{a \sin\theta}$$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sub : $y = -\frac{x b \cos\theta}{a \sin\theta}$

$$\frac{x^2}{a^2} + \frac{x^2 b^2 \cos^2\theta}{a^2 \sin^2\theta \cdot b^2} = 1$$

~~$x^2(1+\cos^2\theta) = a^2\sin^2\theta$~~

$$x^2 \sin^2\theta + x^2 \cos^2\theta = a^2 \sin^2\theta$$

$$\therefore x^2 = a^2 \sin^2\theta$$

$$\therefore x = + a\sin\theta \quad (\because x > 0 \text{ in } 4^{th} \text{ quadrant})$$

$$y = -\frac{ab\sin\theta\cos\theta}{a\sin\theta}$$

$$= -b\cos\theta$$

$$\therefore Q\ (a\sin\theta, -b\cos\theta)$$

(iii) Perpendicular distance from $P(a\cos\theta, b\sin\theta)$ to OQ $(x b\cos\theta + y a\sin\theta = 0)$ is

$$d = \left| \frac{ab\cos^2\theta + ab\sin^2\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right| = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Distance $OQ = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ $\therefore$ Area $\triangle OPQ = \frac{1}{2} OQ \times d$

$= \frac{1}{2} ab$ which is independent of θ.

Q4

(a) $P(x) = ax^3 + bx^2 + cx + d$

$d = -15 \quad \therefore P(x) = ax^3 + bx^2 + cx - 15$

$P(2) = 8a + 4b + 2c - 15 = 5$

or $8a + 4b + 2c - 20 = 0$

or $4a + 2b + c - 10 = 0$ — (1)

one zero is $1+2i$

$\therefore$ another zero is $1-2i$ ($\because$ $P(x)$ has real coefficients)

$\alpha\beta\gamma = -(-15)/a = 15/a$

$(1+2i)(1-2i)\gamma = \frac{15}{a}$

$5\gamma = \frac{15}{a}$

$\therefore \gamma = 3/a$

$\alpha + \beta + \gamma = 1 + 2i + 1 - 2i + \frac{3}{a} = -\frac{b}{a}$

$2a + 3 = -b$

$2a + b + 3 = 0$ — (2)

$\alpha\beta + \beta\gamma + \gamma\alpha$

$(1+2i)(1-2i) + (1+2i)\frac{3}{a} + \frac{3}{a}(1+2i) = c/a$

$1 + 4 + \frac{3}{a} - \frac{6i}{a} + \frac{3}{a} + \frac{6i}{a} = c/a$

$5a + 6 = c$

$5a - c + 6 = 0$ — (3)

$4a + 2b + c = 10$ — (1)

$2a + b = -3$ — (2)

$5a - c = -6$ — (3)

(1) + (3)

$9a + 2b = 4$ — (4)

(4) − 2× (2)

$5a = 10 \quad \therefore a = 2$

$\therefore b = -3 - 4 = -7$

$c = -6 - 10 = -16$

$\therefore P(x) = 2x^3 - 7x^2 - 16x - 15 = 0$

4(b) (i) $x^3 - 4x + 5 = 0$ $\alpha+\beta+\gamma = 0$, $\alpha\beta+\beta\gamma+\gamma\alpha = -4$

$\alpha\beta\gamma = -5$

$P(\alpha) : \alpha^3 - 4\alpha + 5 = 0$ — ①

$P(\beta) : \beta^3 - 4\beta + 5 = 0$ — ②

$P(\gamma) : \gamma^3 - 4(\gamma) + 5 = 0$ — ③

① + ② + ③

$\alpha^3 + \beta^3 + \gamma^3 - 4(\alpha+\beta+\gamma) + 15 = 0$

$\underline{\alpha^3 + \beta^3 + \gamma^3 = -15}$

(ii) $\alpha + \beta = (\alpha+\beta+\gamma) - \gamma$

$= 0 - \gamma = -\gamma$

$\beta + \gamma = -\alpha$

$\gamma + \alpha = -\beta$

∴ The roots of the new equation are $-\alpha, -\beta, -\gamma$

let $y = -x$

$P(x) = x^3 - 4x + 5 = 0$

$P(-y) : (-y)^3 - 4(-y) + 5 = 0$

$-y^3 + 4y + 5 = 0$

or $\underline{x^3 - 4y - 5 = 0}$

∴ (iii) $(\alpha+\beta)^2(\beta+\gamma)^2(\gamma+\alpha)^2 = [(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)]^2$

$=$ (pdt of roots)^2

$= [-(-5)]^2 = 25$.

(c) Factorise $x^4 - 2x^2 - 15$

(i) $(x^2 - 5)(x^2 + 3)$

(ii) $(x + \sqrt{5})(x - \sqrt{5})(x + \sqrt{3}i)(x - \sqrt{3}i)$