The Scots College

Year 12 Mathematics Extension 2

Pre-Trial

25th March 2011

Name:__________

General Instructions

- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Standard Integrals Table attached

TOTAL MARKS:

THERE ARE 4 QUESTIONS.

WEIGHTING: 30 %

Question 1: (Marks 20)

a)

Using De Moivre"s Theorem, or otherwise, find the square root of $\sqrt{3}\ + i$. 2

b)

Sketch on separate Argand diagrams, the locus of z defined by:-

i) $|z - 2 - 3i| = 4$ 2

ii) $Arg\left(\frac{z - i}{z + 1}\right) = 0$ 2

iii) $z\bar{z} = z + \bar{z}$ 4

iv) $z + z^{-1}$ *given that* $z + z^{-1}$ *is real.* 4

c)

Given the equation $(5 + 3i)z^2 - (1 - 4i)z + 8 - 2i = 0$ 3

Find:-

i) The product of the roots.
ii) The modulus of your answer in part i)
iii) The argument of your answer in part i)

d)

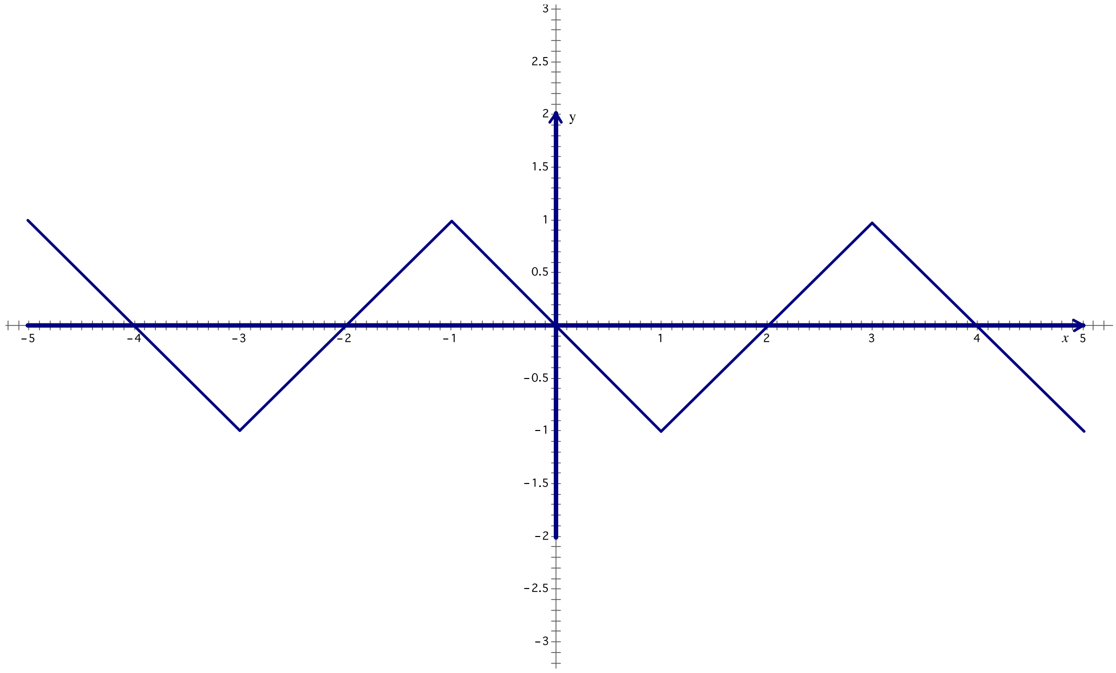
If z is a point on the unit circle with $\arg(z) = \alpha$ 3

i) Prove that $|z^2 - z| = |z - 1|$
ii) Find the arguments of z^2 and $z^2 - z$ in terms of α .

Question 2: (23 Marks) Start a New Booklet

a)

The graph of $y = f(x)$ for $-5 \leq x \leq 5$ is shown below.



On separate axes, sketch the following curves. Indicate clearly any turning points, asymptotes and intercepts with the coordinate axes.

i) $y = f(|x|)$ 2

ii) $y = \frac{1}{f(x)}$ 2

iii) $y = \sqrt{f(x)}$ 2

iv) $y = x\,f(x)$ 2

b)

i) Sketch the graph of $y = \frac{9x - 27}{x^2 - x - 2}$, clearly indicating any asymptotes and any points where the graph intercepts the axes. 4

ii) Find the equation of the normal to the curve $x^3 + 6xy + y^3 = 1$ at $(1, 0)$ 3

c)

i) Show that the function $f(x) = \ln(\ln x)$ is an increasing function for all values of x in its domain. 2

ii) On separate diagrams , sketch the curves

α) $y = f(x)$ 2

β) $y = \ln(|\ln x|)$ 2

γ) $y = |\ln|\ln x||$ 2

Question 3: (20 Marks) Start a New Booklet

a)

Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

i) Find the coordinates of its foci S and S^I. 2

ii) Find the equations of its directrices. 2

b)

i) Find the equation of the tangent to the ellipse $4x^2 + 9y^2 = 36$ at the point $\left(1, \frac{4\sqrt{2}}{3}\right)$. 2

ii) Show that this tangent passes through the point $\left(6, \frac{1}{\sqrt{2}}\right)$. 2

c)

i) If $y = mx + c$ is a tangent to the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then show that $c^2 = a^2 m^2 - b^2$. 3

ii) The point $P\,(a\cos\theta\ ,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that $PS + PS' = 2a$ where, S and S' are the two foci of the ellipse. 3

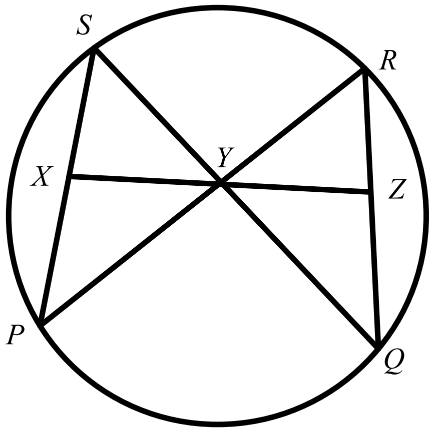
d)

Find the equation of that diameter which bisects the chord $7x + y - 20 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$ 6

Question 4: (15 Marks)

a)

$PQRS$ is a cyclic quadrilateral. The diagonals PR and QS intersect at right angles at Y.
X is the midpoint of PS.
XY produced meets QR at Z.



i) Copy the diagram onto your answer scripts.
Show that $XY = SX$.

ii) Hence show that XZ perpendicular to QR.

b)

i) The equation $2x^3 - x^2 - 6x + 3$ has roots $\alpha,\ \beta\ and\ \gamma$.
Evaluate $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ 2

ii) The roots of the equation $x^3 + 3x^2 + 7x + q = 0$ are in arithmetic progression. 2
Find the value of the constant q.

c)

Prove by mathematical induction $2^{2^n} \geq 5^{2n}$ for $n \geq 5$. 4

END OF EXAM

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\, x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Q1 (a) Let $z = \sqrt{3} + i$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

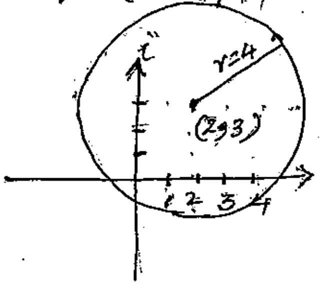
$= 2 \text{ cis } \frac{\pi}{6}$ ✓

$z^{1/2} = 2^{1/2} \text{ cis } \frac{\pi}{12}$ ✓

$= \pm\sqrt{2} \text{ cis } \pi/12$

(b)(i) $|z-2-3i| = \cancel{16}$

$|z-(2+3i)| = \cancel{16}$

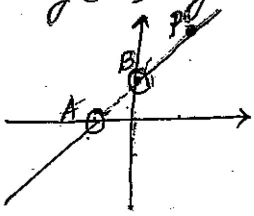


4

(ii) $\text{Arg}\left(\frac{z-i}{z+1}\right) = 0$

A B P
i, -1 & z

$\text{Arg}(z-i) = \text{Arg } z+1$



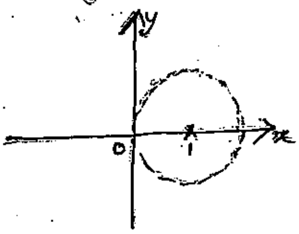
(iii) $z\bar{z} = z + \bar{z}$

$(x+iy)(x-iy) = x+iy+x-iy$

$x^2+y^2 = 2x$

$x^2-2x+1+y^2 = 1$

$(x-1)^2+y^2 = 1$



(iv) $z + z^{-1}$

$(x+iy) + \frac{1}{x+iy}$

$x+iy + \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

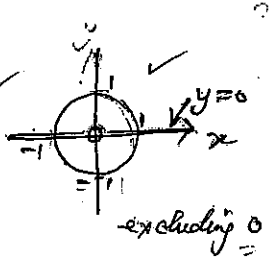
$x+iy + \frac{x-iy}{x^2+y^2}$

$x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$ ✓

$\because$ Real $\therefore$ $y - \frac{y}{x^2+y^2} = 0$

$y(x^2+y^2-1) = 0$ ✓

$y = 0$ or $x^2+y^2 = 1$



excluding 0 ✓

(c) $(5+3i)z^2 - (1-4i)z + 8-2i = 0$

(i) product of roots $= \frac{8-2i}{5+3i} \times \frac{5-3i}{5-3i}$

$= \frac{40-24i-10i-6}{25+9}$

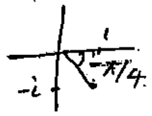
$= \frac{34-34i}{34}$ ✓

$= 1-i$

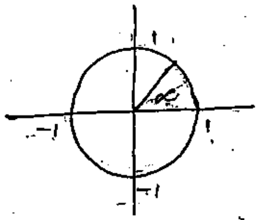
(ii) $|1-i| = \sqrt{1+(-1)^2}$ ✓

$= \sqrt{2}$

(iii) $\text{Arg}(1-i) = -\frac{\pi}{4}$ ✓



(d)



(i) $|z^2-z| = |z(z-1)|$

$= |z||z-1|$ $\quad \because |z| = 1$ unit circle

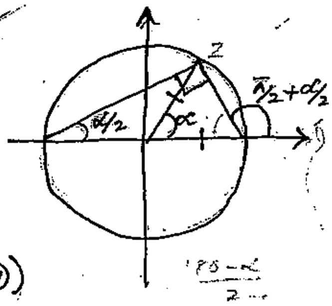
$= 1|z-1|$

$= |z-1|$ ✓

(ii) $\text{Arg}(z^2) = \text{Arg}(zz)$

$= 2\text{Arg}(z)$

$= 2\alpha$ ✓



and

$\text{Arg}(z^2-z) = \text{Arg}(z(z-1))$

$= \text{Arg } z + \text{Arg}(z-1)$

$= \alpha + \frac{\pi}{2} + \frac{\alpha}{2}$

$= \frac{3\alpha+\pi}{2}$ ✓

$\frac{180-\alpha}{2}$

$90 - \frac{\alpha}{2}$

$180 - (90 - \alpha/2)$

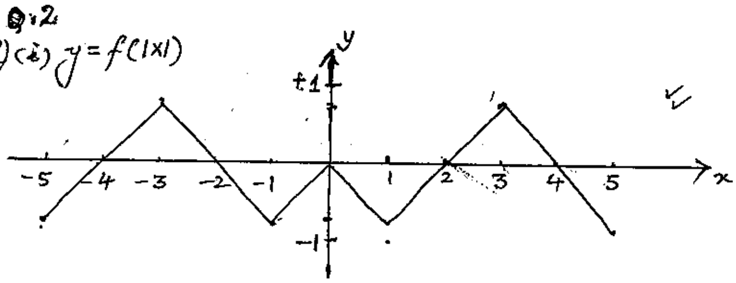
$90 - \alpha/2$

7

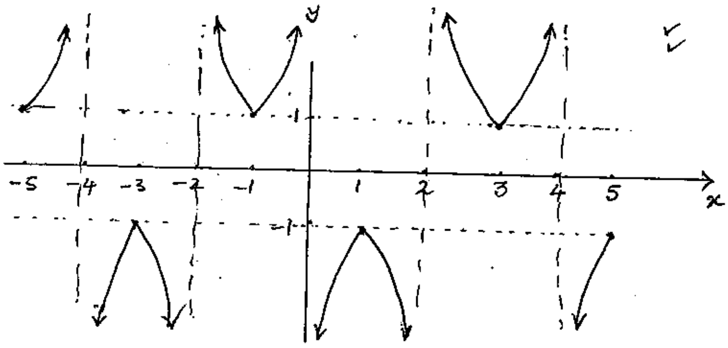
Marks

Q.2

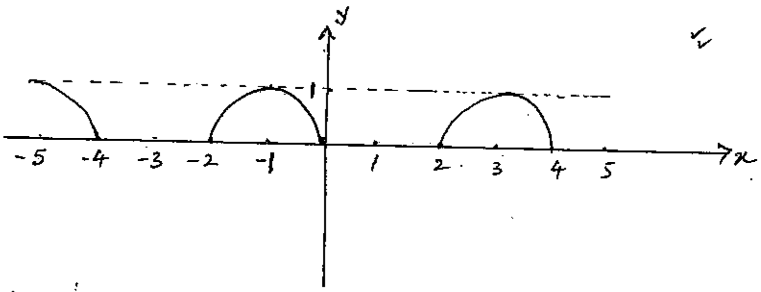
(a) (i) $y = f(|x|)$



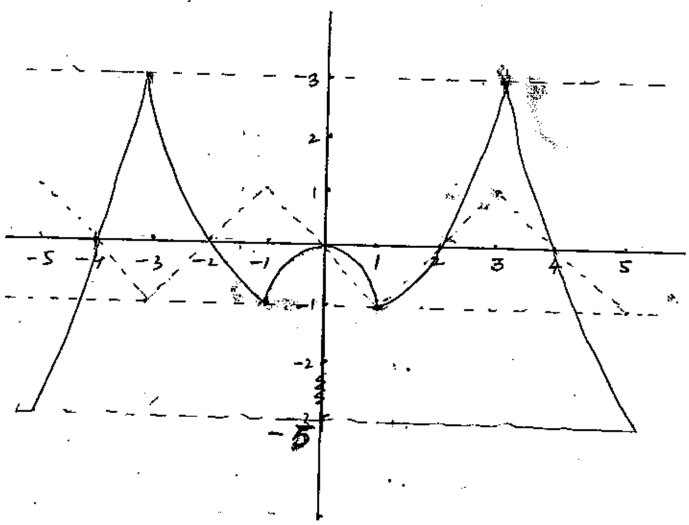
(ii) $y = \frac{1}{f(x)}$



(iii) $y = \sqrt{f(x)}$



(iv) $y = x f(x)$



(b) $y = \frac{9x-27}{x^2-x-2} = \frac{9(x-3)}{(x-2)(x+1)}$ D: all real x, $x \neq 2, -1$

x-intercept $x = 3$

$x = 0$, $y = \frac{13}{2}$

Neither odd nor even

(ii) $y' = \frac{(x^2-x-2)(9) - (9x-27)(2x-1)}{(x^2-x-2)^2}$

$= \frac{9x^2-9x-18 - 18x^2 + 54x + 9x - 27}{(x^2-x-2)^2}$

$= \frac{-9x^2 + 54x - 45}{(x^2-x-2)^2} = -\frac{9(x-5)(x-1)}{[(x-2)(x+1)]^2}$

Stationary pt. $y' = 0$

$-9(x^2-6x+5) = 0$

$(x-5)(x-1) = 0$

$x = 5$ or 1

$x = 5$, $y = \frac{9(2)}{(3)(6)} = 1$

$x = 1$, $y = \frac{9(-2)}{(-1)(2)} = 9$

x	5^-	5	5^+
y'	>0		<0

$(5, 1)$ is a max pt.

x	1^-	1	1^+
y'	<0		>0

$(1, 9)$ is a min pt.

(i) Asymptotes

Vertical $x = 2$, $x = -1$

Horizontal $y = 0$

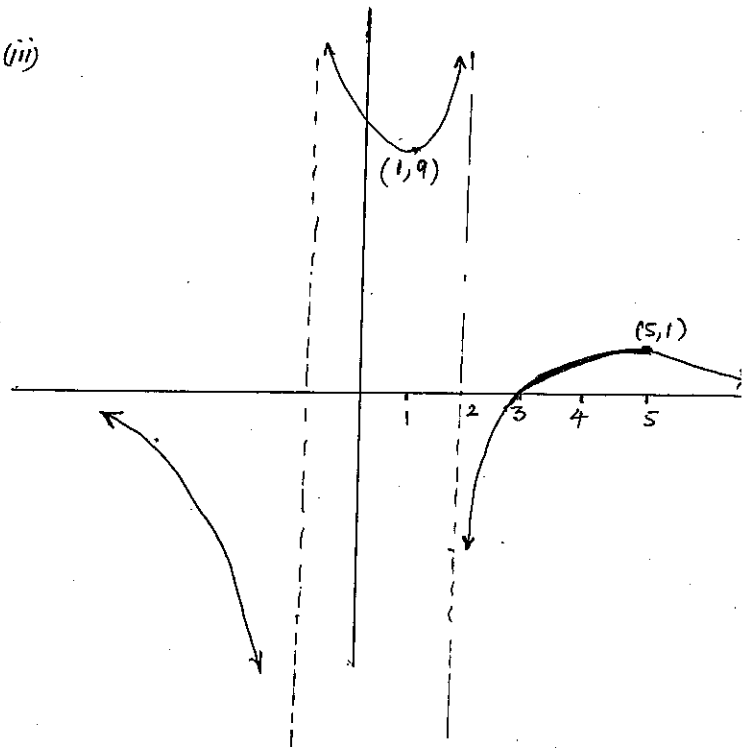
When $x = 2^+$; $y \to -\infty$

$x = 2^-$; $y \to \infty$

when $x \to -1^+$ $y \to \infty$

$x \to -1^-$, $y \to -\infty$

(iii)



Q2(c)

(i) $f(x) = \ln(\ln x)$

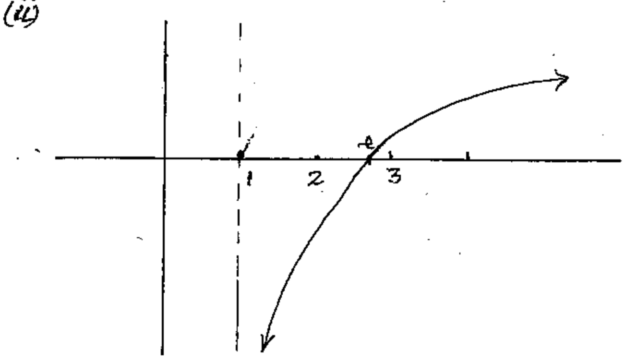
$\ln x > 0 \Rightarrow x > 1$ is the domain ✓

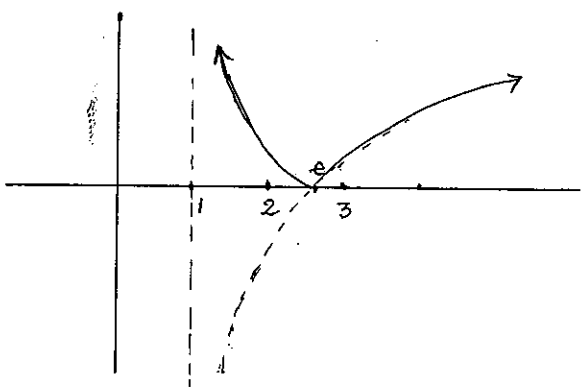
$f'(x) = \frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{1}{x}$

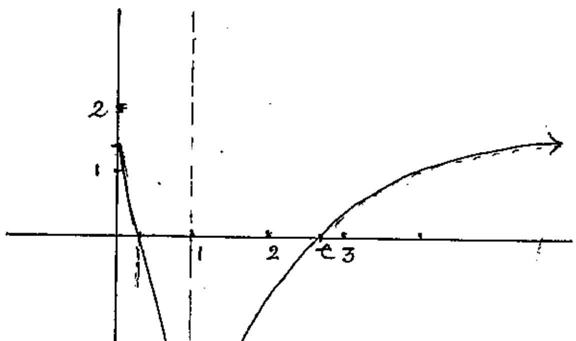
> 0 for all $x > 1$

$\therefore f(x)$ is increasing for all $x > 1$.

(ii)







Q3(a) $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(i) $a = 3, \; b = 4$

$e = \sqrt{(1 - \frac{9}{16})} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$

Foci $(0, \pm be) \Rightarrow (0, \pm\sqrt{7})$ ✓

(ii) $y = \pm\frac{b}{e} \Rightarrow y = \pm\frac{4}{\sqrt{7}/4}$

$y = \pm\frac{16\sqrt{7}}{7}$ ✓

(b) (i) $4x^2 + 9y^2 = 36$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$

$8x + 18y\frac{dy}{dx} = -\frac{4x}{9y}$ ✓

at $(1, \frac{4\sqrt{2}}{3})$ $\frac{dy}{dx} = \frac{-4 \times 1}{9 \times 4\sqrt{2}} \times 3$

$= -\frac{1}{3\sqrt{2}}$

Equation of tangent is

$y - \frac{4\sqrt{2}}{3} = -\frac{1}{3\sqrt{2}}(x - 1)$

$3\sqrt{2}y - \frac{4\sqrt{2}}{\cancel{3}} \times \cancel{3}\sqrt{2} = -x + 1$

$3\sqrt{2}y + x - 9 = 0$

(ii) $S(\pm 6, 0)$ asymptotes $3y = \pm 4x$.

$\Rightarrow y = \pm\frac{4}{3}x$.

$ae = 6$, $\frac{b}{a} = \frac{4}{3}$

$b = \frac{4a}{3}$.

$a^2 + b^2 = a^2e^2$

$a^2 + \frac{16a^2}{9} = 36$

$9a^2 + 16a^2 = 9 \times 36$

$25a^2 = 9 \times 36$

$a^2 = \frac{9 \times 36}{25} = \frac{324}{25}$

$\therefore b^2 = \frac{16}{9} \times \frac{324}{25}$

$= \frac{576}{25}$

$\therefore$ Equation is $\frac{x^2}{\frac{324}{25}} - \frac{y^2}{\frac{576}{25}} = 1$ ✓

$400x^2 - 225y^2 = 5184$ ✓

Q 3

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y=mx+c$

(i)(α) $bx^2+a^2(m^2x^2+2mcx+c^2)=a^2b^2$

$(b^2+a^2m^2)x^2+2a^2cmx+a^2(c^2-b^2)=0$

$\Delta=0$

$\Rightarrow \cancel{4}a^4c^2m^2-\cancel{4}a^2(c^2-b^2)(b^2+a^2m^2)=0$

$\Rightarrow a^4c^2m^2-a^2(c^2b^2-b^4+a^2m^2c^2-a^2b^2m^2)=0$

$\Rightarrow a^4\cancel{c^2m^2}+a^2b^4+a^4b^2m^2-a^2b^2c^2-a^4\cancel{c^2m^2}=0$

$\Rightarrow a^2b^2[b^2+a^2m^2-c^2]=0$

$\Rightarrow c^2=a^2m^2+b^2$

(β) $\frac{x^2}{16}+\frac{y^2}{9}=1$ (3, 4)

$y=mx+c$

$4=3m+c$

$c^2=16m^2+9$

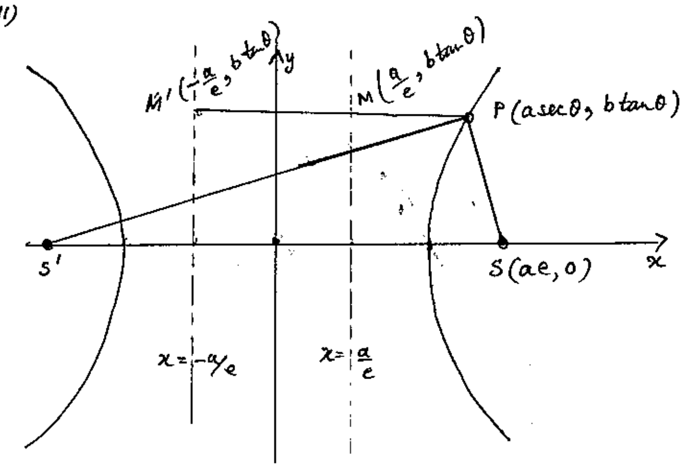
$(4-3m)^2=16m^2+9$

$16-24m+9m^2=16m^2+9$

$7m^2-24m-7=0$

$\Rightarrow m_1m_2=\frac{c}{a}=-\frac{7}{7}=-1$

(ii)



$\frac{PS}{PM}=e \Rightarrow PS=ePM$ ✓

$\frac{PS'}{PM'}=e \Rightarrow PS'=ePM'$

$|PS-PS'|=e|PM-PM'|$ ✓

$=e\left|a\sec\theta-\frac{a}{e}-\frac{a}{e}-a\sec\theta\right|$

$=e\left|-\frac{2a}{e}\right|$

$=2a$ ✓

Q.3 (d)

(i) $xy = c^2$

$y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2} = \frac{-c^2}{c^2q^2}$

$= -\frac{1}{q^2}$ at $Q(cq, \frac{c}{q})$ ✓

Normal at $Q(cq, \frac{c}{q})$ has slope q^2

Equation at Q

$y - \frac{c}{q} = q^2(x - cq)$

i.e. $qy - c = q^3(x - cq)$ ✓

(ii) The normal at Q intersects the hyperbola at $\cancel{xy} = c^2$ at P

From (i) $y = q^2x - cq^3 + \frac{c}{q}$ ...

at Point P

$x(q^2x - cq^3 + \frac{c}{q}) = c^2$

$q^2x^2 - (cq^3 - \frac{c}{q})x - c^2 = 0$

$x^2 - \frac{c}{q^2}(q^3 - \frac{1}{q})x - \frac{c^2}{q^2} = 0$

$x^2 - c(q - \frac{1}{q^3})x - \frac{c^2}{q^2} = 0$ ✓

The roots of this quadratic equation in x are the x-coordinates of Q and P.

(iii) Let the x-coordinate of P be X

then product of roots.

$cq.X = -\frac{c^2}{q^2}$

$= X = \frac{-c}{q^3}$

substitute this in

$xy = c^2$

$y = c^2 / -\frac{c}{q^3} = -cq^3$

The coordinates of P are. ✓

$(-\frac{c}{q^3}, -cq^3)$

(iv)

Equation of OQ is $y = mx$

$m = \frac{\frac{c}{q} - 0}{cq - 0} = \frac{1}{q^2}$

the line $y = \frac{1}{q^2}x$ cuts $xy = c^2$ at L

where $x \cdot \frac{1}{q^2}x = c^2$

$x^2 = q^2c^2$

$\therefore x = \pm cq$.

L has x-coordinate $-cq$

Sub. in $xy = c^2$

$-cq \cdot y = c^2$

$y = -\frac{c}{q}$ ✓

i.e. L is $(-cq, -\frac{c}{q})$ [can also be made by observation]

$m_{PL} = \frac{-\frac{c}{q} + cq^3}{-cq + \frac{c}{q^3}}$

$= \frac{-\frac{1}{q} + q^3}{-q + \frac{1}{q^3}}$

$= \frac{-1 + q^4}{q} \times \frac{q^3}{-q^4 + 1}$

$= -q^2$ ✓

$m_{QL} = \frac{c/q}{cq} = \frac{1}{q^2}$

$m_{PL} \times m_{QL} = -q^2 \times \frac{1}{q^2}$

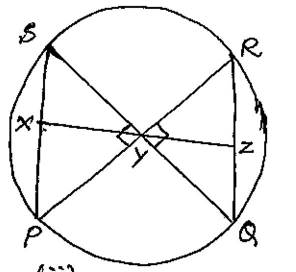
$= -1$ ✓

Therefore $\angle PLQ = 90°$.

Q4

(a) (i) $\angle SYP = 90^\circ$ (Given)

$\therefore$ SP is the diameter of circle passing through Y.



X is the mid pt. of SP (Given)

$\therefore$ X is the centre

$\therefore$ $SX = XP = XY$ (equal radii)

(ii) Let $\angle XSY = x$, $\angle XPY = y$

$\angle XSY = x$ (equal angles opp. to equal sides SX & XY of isosceles $\triangle SXY$)

Similarly $\angle PYX = y$

$\therefore x + y = 90^\circ$ ($\because SYP = 90^\circ$ given)

$\angle RYZ = y$ (vertically opp. to $\angle XYP$)

$\angle YRZ = x$ (angles in the same segment standing on arc PQ)

IN $\triangle RYZ$

$\angle RZY = 180 - (x+Y)$ ($\angle$ sum of $\triangle$)

$= 180^\circ - 90^\circ$ ($\because X + Y = 90^\circ$)

$= 90^\circ$

$\therefore XZ \perp RQ$

(b) (i) $2x^3 - x^2 - 6x + 3 = 0$

$\alpha + \beta + \gamma = \frac{1}{2}$

$\alpha\beta + \beta\gamma + \gamma\alpha = -3$

$\alpha\beta\gamma = -\frac{3}{2}$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= (-3)^2 - 2(-\frac{3}{2})(\frac{1}{2})$

$= 9 + \frac{3}{2}$

$= 21/2$

(ii) $x^3 + 3x^2 + 7x + q = 0$

Let the roots by $\alpha - d, \alpha, \alpha + d$

$\alpha - d + \alpha + \alpha + d = -3$

$3\alpha = -3$

$\therefore \alpha = -1$

$(-1-d)(-1) + (-1)(-1+d) + (-1-d)(-1+d) = 7$

$1 + \cancel{d} + 1 - \cancel{d} + 1 - d^2 = 7$

$1 - d^2 = 7 - 2$

$= 5$

$(1-d)(-1)(-1+d) = -q$

$-(1-d^2) = -q$

$-5 = -q$

$q = 5$ ✓

(c) $2^{2^n} \geq 5^{2n}$ for $n \geq 5$

Step 1. prove true for $n = 5$

L.H.S. $= 2^{32}$ TRUE

R.H.S $= 5^{10}$

Step 2. Assume true for $n = k$.

i.e. $2^{2^k} \geq 5^{2k}$

or $2^{2^k} - 5^{2k} \geq 0$

Step 3. Prove true for $n = k+1$

i.e. $2^{2^{k+1}} \geq 5^{2(k+1)}$

i.e. $2^{2^{k+1}} - 5^{2(k+1)} \geq 0$

L.H.S. $2^{2^k \cdot 2^1} = (2^{2^k})^2$

$\geq (5^{2k})^2$

$= 5^{4k}$

$= 5^{2k+2k}$

$\geq 5^{2k+10}$ ($\because k \geq 5$)

$\geq 5^{2k+2}$

$\geq 5^{2(k+1)}$

∗. Hence true for $n = k+1$.

Hence by the principles of mathematical induction the statement is true $\forall$ $n \geq 5$.