THE SCOTS COLLEGE

2012

HSC Pre-Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 8–11

Total marks – 67

Section I - Multi-choice
7 marks

- Attempt Questions 1–7
- Allow about 10 minutes for this section

Section II
60 marks

- Attempt Questions 8–11
- Allow about 1 hour 50 minutes for this section

Section 1 – Multiple choice
Total marks – 7
Attempt Questions 1-7
All questions are of equal value
Answer on the given multiple choice answer sheet

QUESTION 1 - The reciprocal of the number i is

A) 1

B) i

C) -1

D) $-i$

QUESTION 2 - The derivative of $x^2 + y^2 = xy + 5$ is

A) $\dfrac{dy}{dx} = \dfrac{2x - y}{2y - x}$

B) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

C) $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$

D) $\dfrac{dy}{dx} = \dfrac{y + x - 2x}{2y}$

QUESTION 3 - After using an appropriate substitution, the integral $\int_0^4 (2x + 3)\sqrt{2x + 1}\,dx$ is equivalent to,

A) $\frac{1}{2}\int_0^4 (u + 2)\sqrt{u}\,du$

B) $2\int_1^9 (u + 2)\sqrt{u}\,du$

C) $2\int_0^4 (u + 2)\sqrt{u}\,du$

D) $\frac{1}{2}\int_1^9 (u + 2)\sqrt{u}\,du$

QUESTION 4 - If $arg(1 + ai) = -\frac{\pi}{3}$, then the real number a is

A) $-\dfrac{\sqrt{3}}{2}$

B) $-\sqrt{3}$

C) $\dfrac{1}{\sqrt{3}}$

D) $\sqrt{3}$

QUESTION 5 - The length of the major axis of a hyperbola is 12. The equations of the asymptotes for the hyperbola are $y = \pm\frac{2}{3}x$. If the vertices of the hyperbola are on the *y-axis,* determine its equation.

A) $\frac{x^2}{16} - \frac{y^2}{36} = -1$

B) $\frac{x^2}{9} - \frac{y^2}{4} = -1$

C) $\frac{x^2}{36} - \frac{y^2}{16} = -1$

D) $\frac{x^2}{81} - \frac{y^2}{36} = -1$

QUESTION 6 – The range of the conic defined by $16x^2 + 25y^2 - 400 = 0$ is

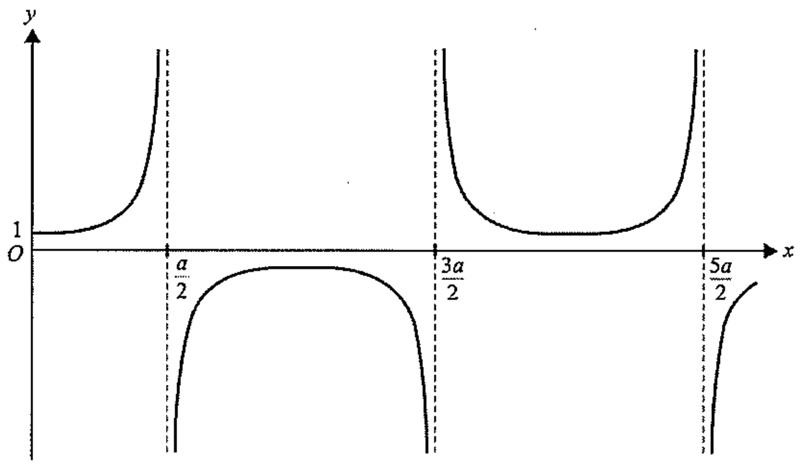
A) $|y| \leq 4$

B) $|y| \leq 5$

C) $|y| < 4$

D) $|y| < 5$

QUESTION 7



A rule for the above function is

A) $y = cosec\left(\frac{2\pi}{a}\left(x - \frac{a}{2}\right)\right)$

B) $y = cosec\left(\frac{2\pi}{a}\left(x + \frac{a}{4}\right)\right)$

C) $y = cosec\left(\frac{\pi}{a}\left(x - \frac{a}{2}\right)\right)$

D) $y = cosec\left(\frac{\pi}{a}\left(x + \frac{a}{2}\right)\right)$

Section 2 - Total marks – 60
All questions are of equal value
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION EIGHT (15 MARKS) Use a SEPARATE writing booklet.

a) Find $\int \frac{1}{x\log_e x}\,dx$ *(2)*

b) Using the substitution $u = e^x + 1$ or otherwise, *(3)*

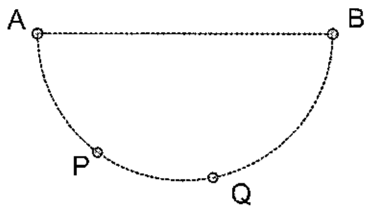
Evaluate $\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx$

c) *i)* Find a, b, and c, such that *(2)*

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}$$

ii) Hence, or otherwise, find $\int \frac{16}{(x^2+4)(2-x)}\,dx$ *(3)*

d)



Points P and Q are on the circumference of a semicircle with diameter AB.

The chords AP and BQ produced intersect at F. The chord AQ and BP intersect at G. FG produced meets AB at H.

Redraw the above diagram into your solution booklet

i) Prove that $PFQG$ is a cyclic quadrilateral. *(2)*
ii) Prove that $\angle PFG = \angle PQG = \angle PBA$. *(2)*
iii) Prove that FH *is perpendicular to* AB. *(1)*

End of Question 8

QUESTION NINE (15 MARKS) Use a SEPARATE writing booklet.

a) If $z = \sqrt{3} + i$, plot on the argand diagram

i) iz *(1)*

ii) $z(1+i)$ *(1)*

b) *i)* If $z = cos\frac{\pi}{3} + isin\frac{\pi}{3}$, find z^5 in the domain $-\pi < \theta \leq \pi$. *(2)*

ii) Plot on an argand diagram, all complex numbers that are the solutions to $z^5 = 1$. *(2)*

c) Draw on an argand diagram, the region which satisfies both $|z - 2 + 3i| < 2$ and $Re(z) \geq 3$. *(2)*

d) The complex numbers z_1 and z_2 are such that

$$z_1 = 5 + 2pi \quad \text{and} \quad \frac{z_1}{z_2} = 1 - i \qquad \text{where } p \text{ is a real constant.}$$

i) Find z_2 in the form $a + bi$, giving the real numbers a and b in terms of p. *(3)*

Given that $arg(z_2) = tan^{-1}(4)$,

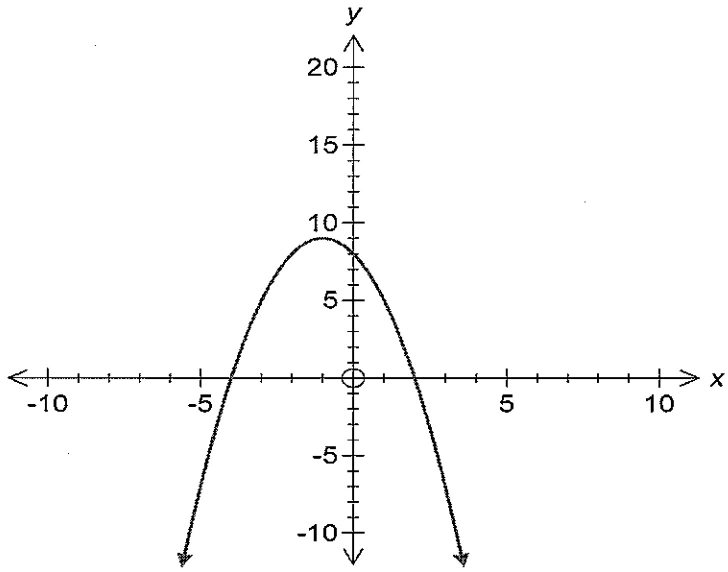
ii) Show that the value of $p = \frac{3}{2}$. *(2)*

iii) Find the value of $|z_1|$. *(2)*

End of Question 9

QUESTION TEN (15 MARKS) Use a SEPARATE writing booklet.

a)



Let $f(x) = -(x-2)(x+4)$, given above.

Without the use of calculus, on separate diagrams, sketch the following graphs.

i) $y^2 = f(x)$ *(2)*

ii) $y = \frac{1}{f(x)}$ *(2)*

iii) $y = e^{f(x)}$ *(2)*

iv) $y = log_e(f(x))$ *(2)*

b) Given $y = cos(x)$ *for* $0 \leq x \leq 2\pi$

i) Graph $y = |f(x)|$ *(2)*

ii) Where is the graph of $y = |f(x)|$ not differentiable, giving reasons for your answer. *(2)*

iii) Hence or otherwise find the area bounded by $y = |f(x)|$ and the *x-axis*. *(1)*

c) Sketch the curve of

$$|x| - |y| = 2$$ *(2)*

End of Question 10

QUESTION ELEVEN (15 MARKS) Use a SEPARATE writing booklet.

a) The ellipse D has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ and the ellipse E has equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

i. Sketch D and E on the same diagram, showing the coordinates of the points where each curve crosses the axes. *(2)*

ii. Calculate the eccentricity of ellipse D *(1)*

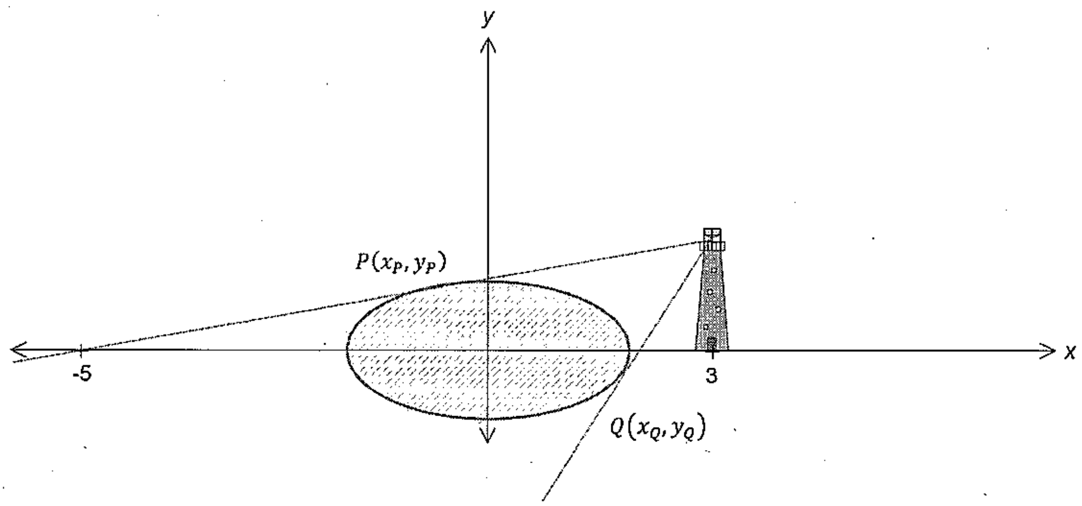
iii. The point S is a focus of D and the point T is a focus of E. Find the length of ST. *(3)*

b) Determine the real values of λ for which the equation

$$\frac{x^2}{5-\lambda}+\frac{y^2}{3-\lambda}=1 \quad \text{defines a hyperbola}$$ *(1)*

Question Eleven continues over page...

c)



The diagram above shows a lighthouse, with base at *(3,0)*, shining a light onto an elliptical shape with the equation $x^2 + 4y^2 = 5$. The rays shown above are tangents to the curve, at *P* and *Q*, and represent the boundaries of the shadow produced.

i) Show that the equation of a tangent to the general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ *(2)*

ii) Calculate the point of contact of the tangent at *P*. *(1)*

iii) Hence or otherwise, show that the top of the lighthouse is located at *(3,2)*. *(1)*

iv) Hence or otherwise calculate the equation of the chord of contact of the two tangents in the above diagram. *(2)*

v) Show that the angle between the two tangents is $\frac{\pi}{2} - tan^{-1}\left(\frac{1}{4}\right) - tan^{-1}\left(\frac{4}{11}\right)$ *(2)*

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Yr12 – Ext 12 – Pretrial – solutions – 2012

Q1

$$\frac{1}{i} = \frac{i}{i^2} = -i$$

D

Q2

$$2x + 2y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

C

Q3

let $u = 2x+1$

$u - 1 = 2x$

$\frac{du}{dx} = 2$

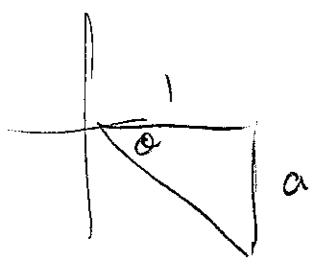
$\frac{1}{2}du = dx$

$\boxed{x=0 \quad u=1}$ $\boxed{x=4 \quad u=9}$

$$\frac{1}{2}\int_1^9 (u+2)\sqrt{u}\ du$$

D

Q4



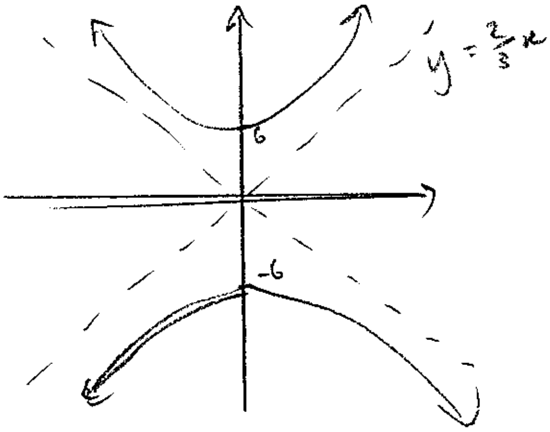
4th Quad

$$\frac{-\pi}{3} = \tan^{-1}\left(\frac{a}{1}\right)$$

$\therefore\ a = -\sqrt{3}$

B

Q5



$y = \pm \frac{b}{a} x$

Note: $b = 6$

$\frac{2}{3} = \frac{6}{a}$

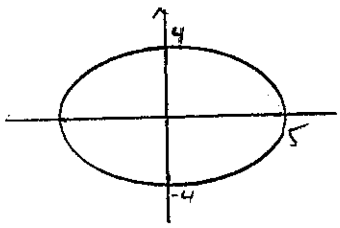
$a = 9$

$\frac{y^2}{36} - \frac{x^2}{81} = 1$ $\therefore$ D

Q6

$16x^2 + 25y^2 = 400$

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

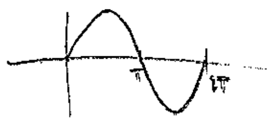


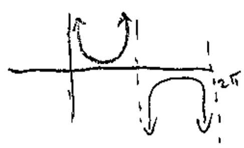
$\therefore$ A

Q7

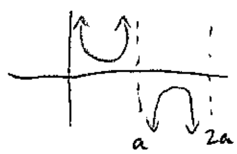
$y = \operatorname{cosec} x = \frac{1}{\sin x}$

D

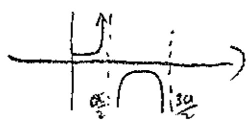




$y = \operatorname{cosec} x$



let $\pi = a$ $y = \operatorname{cosec}\left(\frac{\pi}{a} x\right)$



shift by $\frac{\pi}{2}$ $y = \operatorname{cosec}\left(\frac{\pi}{a}x + \frac{\pi}{2}\right)$

Q8

a) $\int \frac{1}{x} \cdot \frac{1}{\log_e x} \, dx$

let $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx$

$\int \frac{1}{u} \, du$

$\left[\log_e(u) + c \right]$

$\log_e\left[\log_e x\right] + c$

b) $\int_0^1 \frac{e^x}{(1+e^x)^2} \, dx$

$u = e^x + 1$

$\frac{du}{dx} = e^x \iff du = e^x \, dx$

when $\boxed{\begin{array}{l} x = 0 \\ u = e^0 + 1 = 2 \end{array}}$ $\boxed{\begin{array}{l} x = 1 \\ u = e + 1 \end{array}}$

$\int_2^{e+1} \frac{1}{u^2} \, du = \int_2^{e+1} u^{-2} \, du$

$\left[-u^{-1} \right]_2^{e+1}$

$\left[-\frac{1}{e+1} + \frac{1}{2} \right]$

Q8 b) i) $\dfrac{16}{(x^2+4)(2-x)} = \dfrac{(ax+b)(2-x) + c(x^2+4)}{(x^2+4)(2-x)}$

$\therefore$

$$2ax - ax^2 + 2b - bx + cx^2 + 4c = 16$$

$\therefore$

$c - a = 0 \quad \Rightarrow \quad c = a$

$2a - b = 0 \quad \Rightarrow \quad 2c - b = 0$

$\qquad\qquad\qquad\qquad 4c - 2b = 0$

$2b + 4c = 16$

Solving

$2b + 4c = 16$

$-2b + 4c = 0$

$4b \qquad = +16$

$$\boxed{\begin{array}{ll} b & = +4 \\ c & = 2 \\ a & = 2 \end{array}}$$

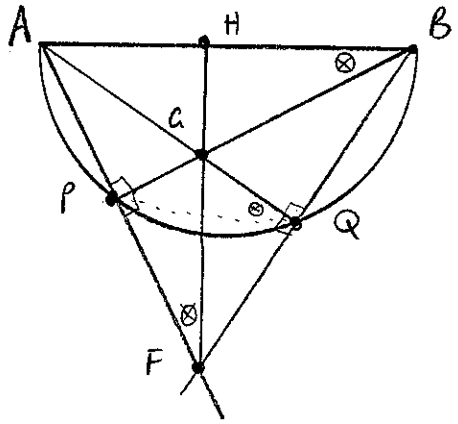
ii)

$$\int \frac{2x+4}{x^2+4} + \frac{2}{2-x} \, dx$$

$$\int \frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{2}{2-x} \, dx$$

$$\left[\ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) - 2\ln|2-x| \right] + C$$

Q8 d)



i) $\angle APB = \angle BQA = 90°$ (angles subtended from a diameter AB)

$\therefore$

$\angle GPF = \angle GQF = 90°$ (angle sum of straight line)

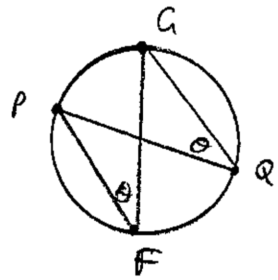
$\therefore$

GF is a diameter with P & Q lying on circumference

as $\angle GPF$ & $\angle GQF$ are subtended from GF and equal $90°$

$\therefore$ PFQG is cyclic quad.

ii) $\angle PFG = \angle PQG$ (angle from same arc PG of circle to circumference are equal)

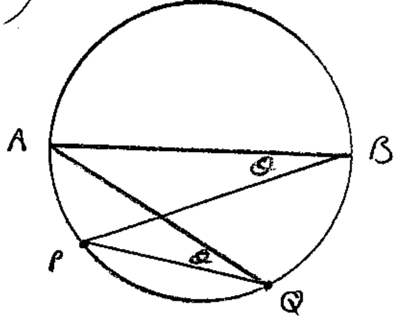


$\angle PQG = \angle PQA$ (A, G, Q are collinear, given)

$\angle PQA = \angle PBA$ (angles subtended from same arc AP of circle to circumference are equal)

$\therefore$

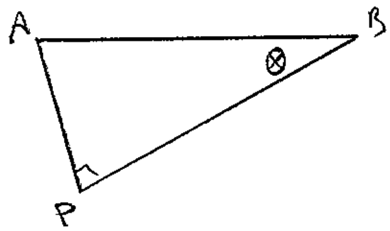
$\angle PFG = \angle PQG = \angle PBA$



Q8 d) iii)

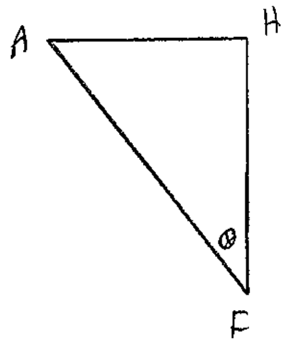
Consider

$\triangle ABP$



and

$\triangle AFH$



$\angle ABP = \angle AFH$ (proven in part ii)

$\angle PAB = \angle HAF$ (common)

$\therefore$

By angle sum of $\triangle$

$\angle APB = \angle AHF = 90^\circ$

$\therefore$

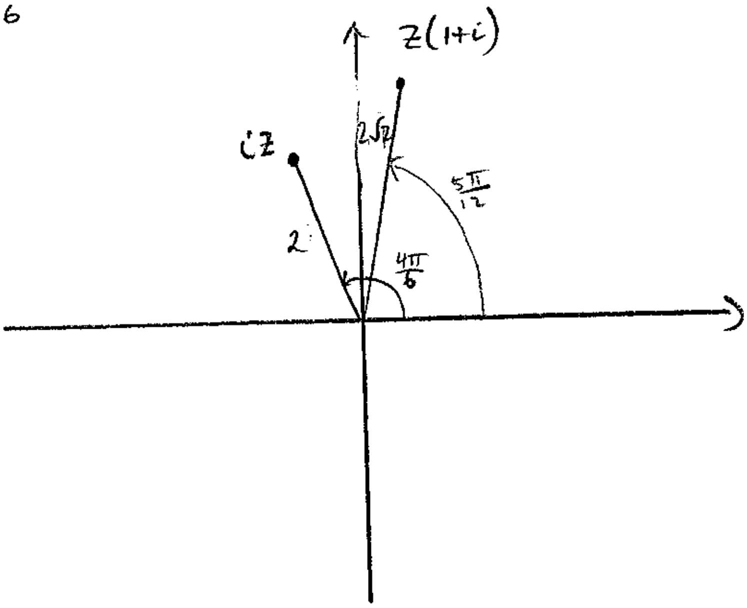
$FH \perp AB$

9 a)

$z = \sqrt{3} + i$

$\arg(z) = \frac{\pi}{6}$

$|z| = 2$



i) $iz = 2\,\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{2}\right)$

$= 2\,\text{cis}\left(\frac{4\pi}{6}\right)$

ii) $z(1+i) = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12}\right)$

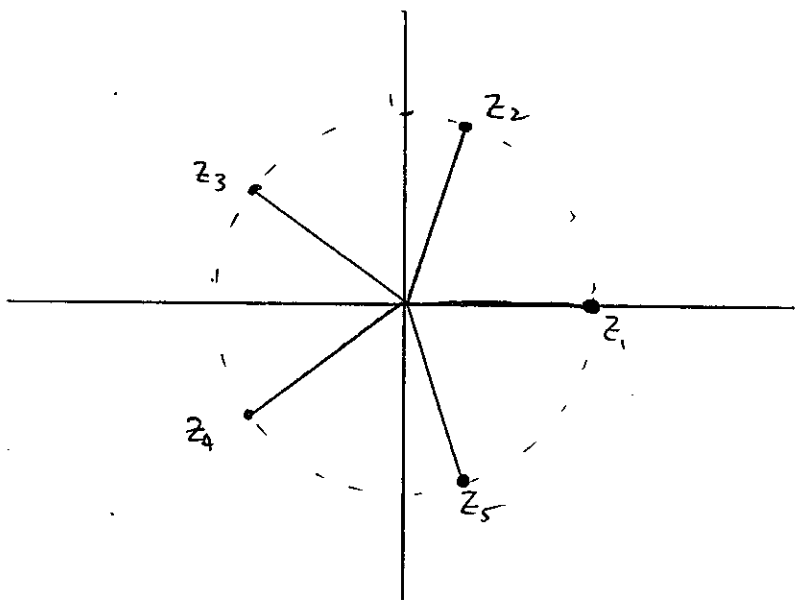
b)

$z = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$z^5 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

$z^5 = \cos\left(\frac{-\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$

ii)



$z_1 = 1$

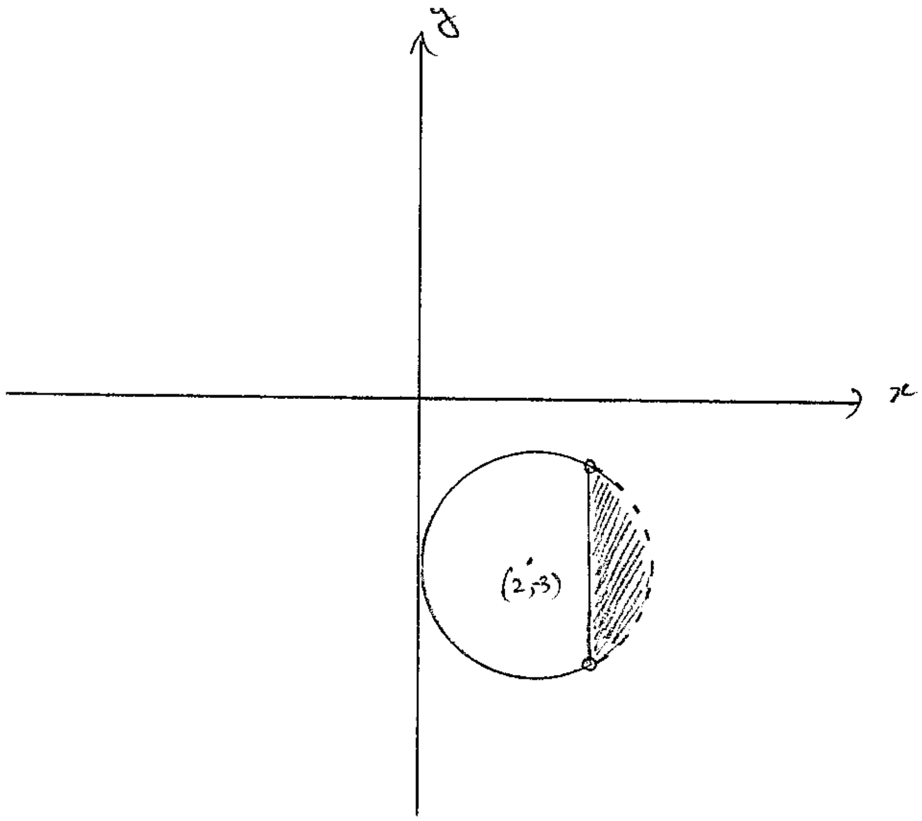
$z_2 = \text{cis}\left(\frac{2\pi}{5}\right)$

$z_3 = \text{cis}\left(\frac{4\pi}{5}\right)$

$z_4 = \text{cis}\left(\frac{-4\pi}{5}\right)$

$z_5 = \text{cis}\left(\frac{-2\pi}{5}\right)$

Q9 c)



Q9 a)

i) $z_1 = 5 + 2pi$

$\frac{z_1}{z_2} = 1 - i$

$\therefore$

$z_2 = \frac{z_1}{1-i}$

$z_2 = \frac{5+2pi}{1-i} \times \frac{1+i}{1+i}$

$= \frac{5+5i+2pi-2p}{2} = \frac{5-2p}{2} + \frac{5+2p}{2}i$

ii) $\arg(z_2) = \tan^{-1}(4)$

$\arg(z_2) = \tan^{-1}\left(\frac{\frac{5+2p}{2}}{\frac{5-2p}{2}}\right)$

$\therefore$

$4 = \frac{5+2p}{5-2p}$

$20-8p = 5+2p$

$15 = 10p$

$\frac{15}{10} = p$

$\frac{3}{2} = p$ as required

iii) $z_1 = 5+3i$

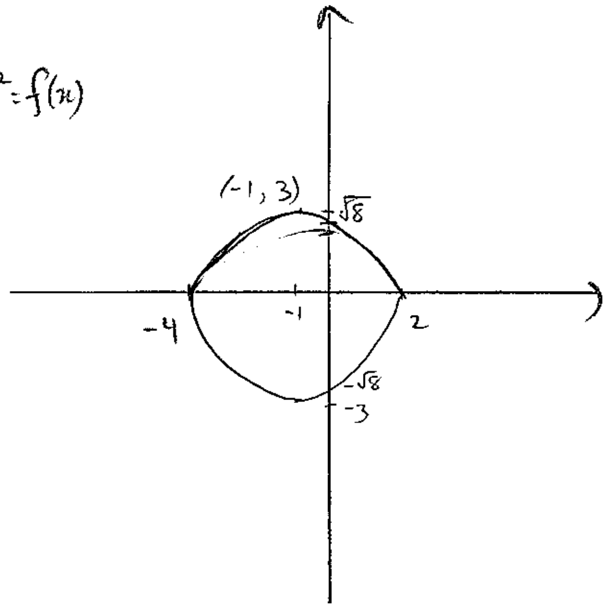
$|z_1| = \sqrt{5^2+3^2}$

$= \sqrt{34}$

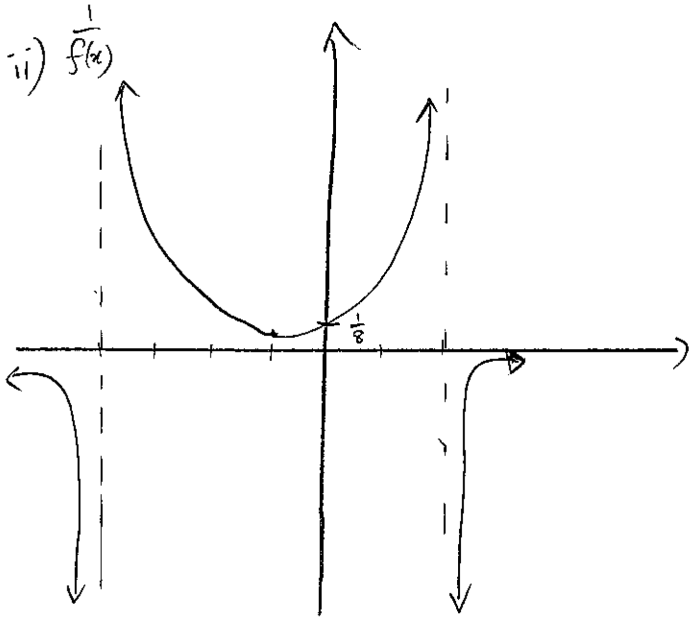
Q10

a)

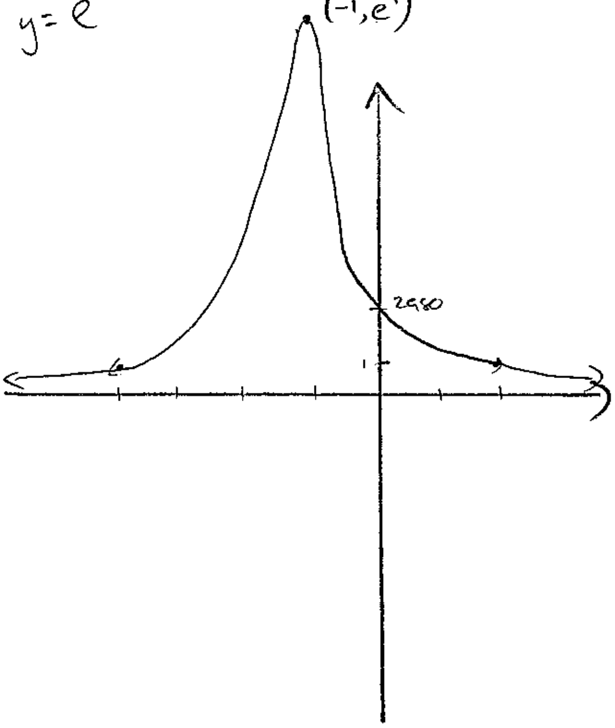
i) $y^2 = f(x)$



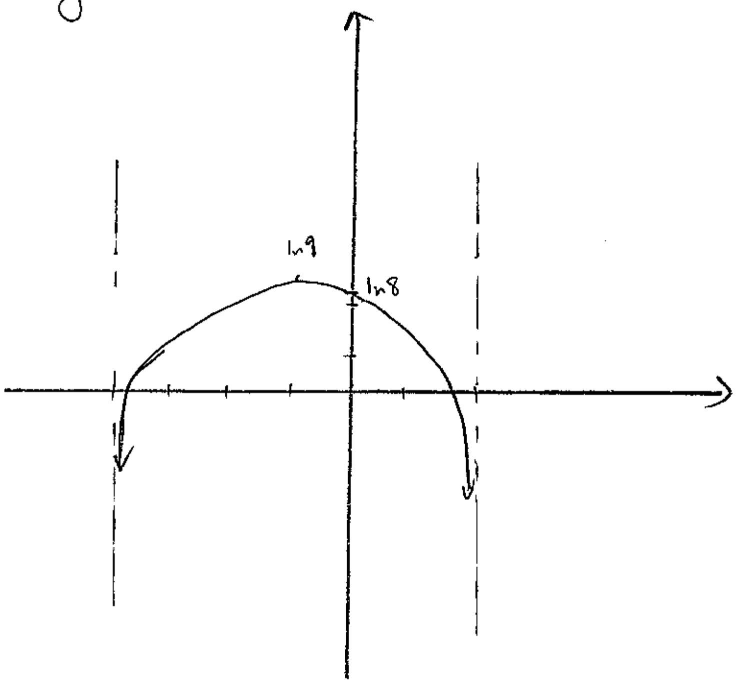
ii) $\frac{1}{f(x)}$



iii) $y = e^{f(x)}$



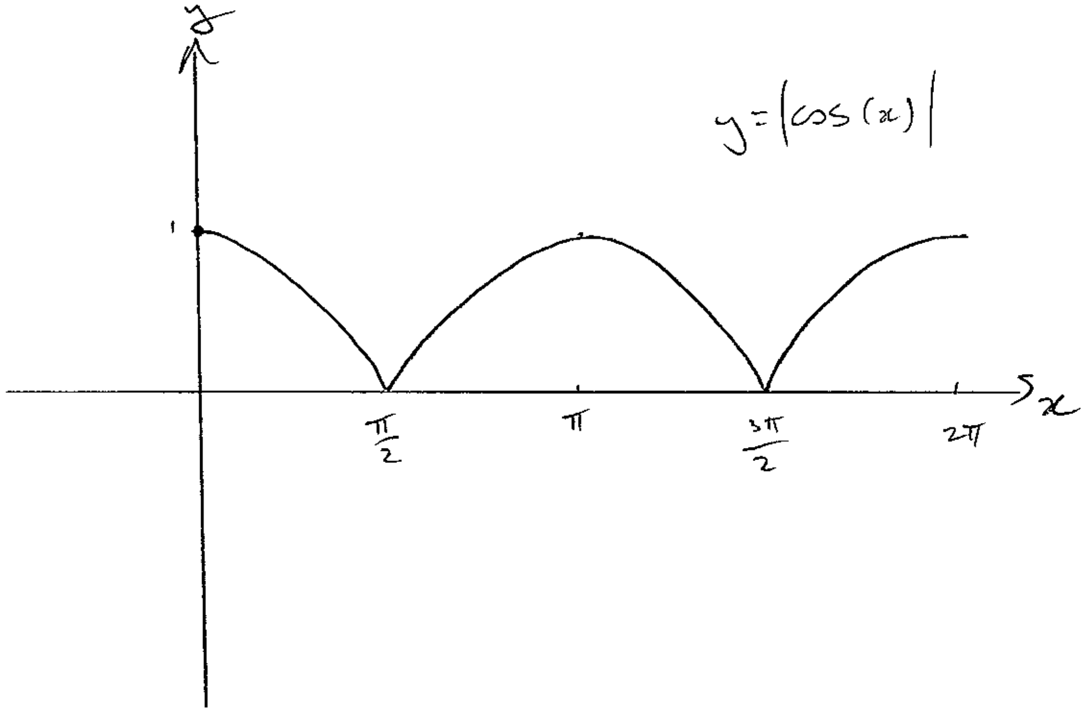
iv) $\log_e f(x)$



Q10

b)

i)



ii) Not differetiable at

$x = \frac{\pi}{2}$ $\qquad \lim_{x^+ \to \frac{\pi}{2}} f'(x) \neq \lim_{x^- \to \frac{\pi}{2}} f'(x)$

$\&$ $\qquad$ because

$x = \frac{3\pi}{2}$ $\qquad \lim_{x^+ \to \frac{3\pi}{2}} f'(x) \neq \lim_{x^- \to \frac{3\pi}{2}} f'(x)$

Not differetiable at

$x = 0$ & $x = 2\pi$ (end points in a finite domain)

Q 10 b) iii)

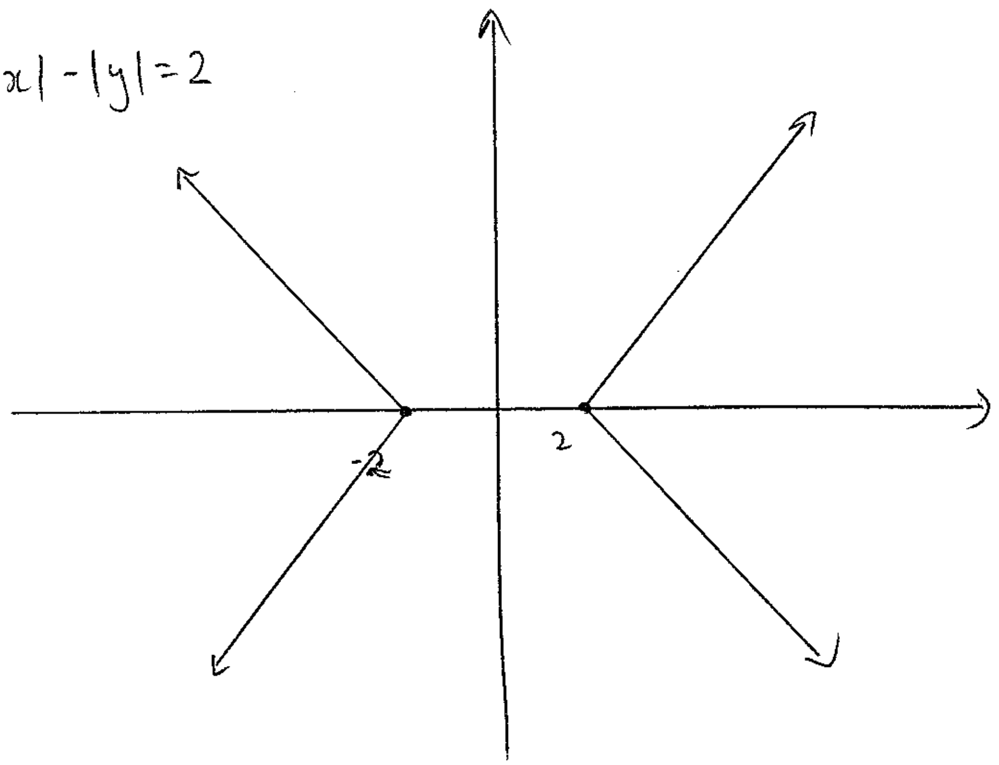
$$2\left|\int_0^{\frac{\pi}{2}} \cos x\right| \, dx$$

$$2\left|\left[-\sin x\right]_0^{\frac{\pi}{2}}\right| = 2\left(0 - - \sin\left(\tfrac{\pi}{2}\right)\right)$$

$$= 2$$

c)

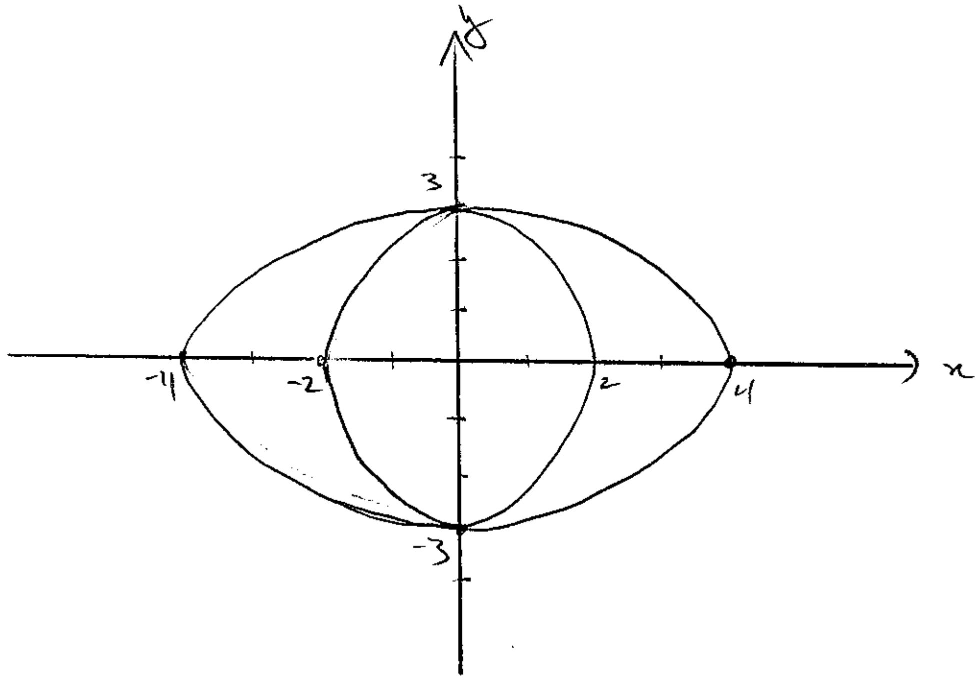
$|x| - |y| = 2$



Q11

a)

i)



ii) $b^2 = a^2(1-e^2)$

$e^2 = 1 - \frac{b^2}{a^2}$

$e^2 = 1 - \frac{9}{16}$

$e = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$

iii)

$S = ae = \sqrt{7}$

$T = be = \sqrt{5}$

$ST = \sqrt{7 + 5}$

$= \sqrt{12} = 2\sqrt{3}$

Q11 b)

$$5 - \lambda > 0 \quad \text{when} \quad 3 - \lambda < 0$$
$$\lambda < 5 \qquad\qquad \lambda > 3$$

$$3 < \lambda < 5$$

OR

$$5 - \lambda < 0 \quad \text{when} \quad 3 - \lambda > 0$$
$$5 < \lambda \qquad\qquad \lambda \not< 3$$

∴ No solution

∴

$$3 < \lambda < 5$$

Q11 c)

i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{xb^2}{ya^2}$$

tangent at (x_1, y_1)

$$y - y_1 = -\frac{x_1 b^2}{y_2 a^2}(x - x_1)$$

$$2yy_1a^2 - 2y_1^2a^2 = -2x_1xb^2 + 2x_1^2b^2$$

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = -\frac{x_1x}{a^2} + \frac{x_1^2}{a^2}$$

$$\frac{x_1x}{a^2} + \frac{yy_1}{b^2} = \frac{a_1^2}{a^2} + \frac{y_1^2}{b^2}$$

but $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

∴

$$\frac{x_1x}{a^2} + \frac{yy_1}{b^2} = 1$$

as required

Q11 c) ii) tangent at (x_p, y_p)

$$\frac{x_p x}{a^2} + \frac{y_p y}{b^2} = 1 \quad \text{satisfies } (-5, 0) \text{ when } \begin{array}{l} a^2 = 5 \\ b^2 = 5/4 \end{array}$$

$$\frac{-5x_p}{5} + 0 = 1$$

$$-x_p = 1$$

$$x_p = -1$$

$\therefore$

$$(-1)^2 + 4(y_p)^2 = 5$$

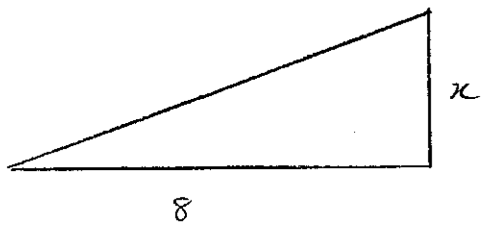
$$y_p = 1$$

point of contact is $(-1, 1)$

c) iii)

gradient of tangent is $m = \dfrac{-x_p}{4y_p}$ at $(-1, 1)$

$\therefore \quad m = \frac{1}{4}$



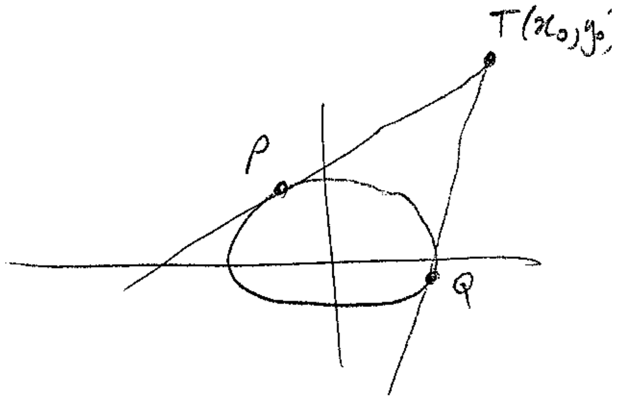
$$\frac{1}{4} = \frac{x}{8}$$

$x = 2$

$\therefore$ top of lighthouse is located at $(3, 2)$

Q11 c

iv) Tangent at $P(x_1, y_1)$



$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

tangent at $Q(x_2, y_2)$

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$$

$T(x_0, y_0)$ satisfies both

such that

$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$ is the chord of contact

at $(3, 2)$ on $\frac{x^2}{5} + \frac{y^2}{5/4} = 1$

$$\frac{3x}{5} + \frac{2y}{5/4} = 1$$

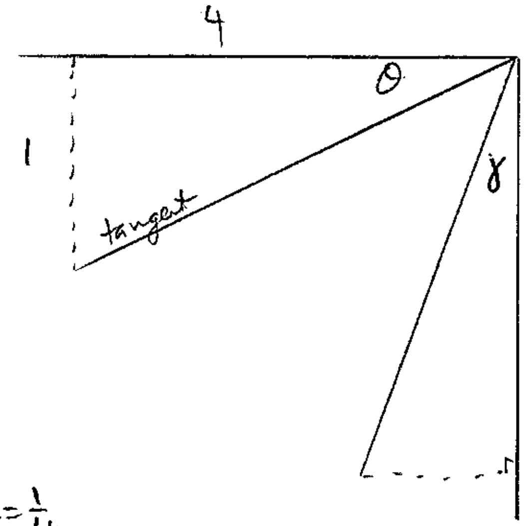
$$\frac{3x}{5} + \frac{8y}{5} = 1$$

$$3x + 8y = 5$$

$$3x + 8y - 5 = 0$$

Q11 c)

v)



Tangent at P has $m = \frac{1}{4}$

$\therefore \tan^{-1}\left(\frac{1}{4}\right) = \theta$

Find

$Q(x_2, y_2)$

$3x + 8y = 5$

$y = \frac{5}{8} - \frac{3}{8}x$

$x^2 + 4y^2 = 5$

$x^2 + 4\left(\frac{5}{8} - \frac{3}{8}x\right)^2 = 5$

$x^2 + 4\left(\frac{25}{64} - \frac{30}{64}x + \frac{9}{64}x^2\right) = 5$

$25x^2 - 30x - 55 = 0$

$5x^2 - 6x - 11 = 0$

$x = \frac{6 \pm \sqrt{256}}{10}$

$x = -\frac{10}{10}$ or $\frac{22}{10}$

$= -1 \qquad = 2.2$

$Q(2.2, y_2)$

$3(2.2) + 8y = 5$

$8y = 5 - 6.6$

$y = -\frac{3}{16}$

$Q(2.2, -0.2)$

$M_Q = \frac{-x}{4y} = \frac{-2.2}{4(-0.2)} = \frac{11}{4}$

$\therefore$ tangent makes angle with vertical of $\tan^{-1}\left(\frac{4}{11}\right)$

$\therefore$ Angle formed by tangents is $\frac{\pi}{2} - \theta - \gamma$

$\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{4}\right) - \tan^{-1}\left(\frac{4}{11}\right)$

OR

$\tan^{-1}\left(\frac{40}{27}\right)$