THE SCOTS COLLEGE

MATHEMATICS EXTENSION II

YEAR 12 PRETRIAL

26TH MARCH 2014

GENERAL INSTRUCTIONS

- Reading time – 5 minutes
- Working time - 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working in Section II

WEIGHTING

30%

TOTAL MARKS

70

SECTION I (5 MARKS)

- Answers to be recorded on the multiple choice answer sheet provided
- Clearly label your answer sheet with your student number
- Allow about 10 minutes for this section

SECTION II (65 MARKS)

- Questions 6 - 9
- Answers to be recorded in the answer booklets provided
- *Each question must be completed in a new answer booklet.*
- Label each answer booklet with your student number and the question number attempted. Clearly indicate the booklet order if more than one booklet is used for a question. E.g. Book 1 of 2 and 2 of 2.

SECTION I

QUESTION 1

Consider a polynomial $P(x)$ of degree 3 and the two real numbers, a and b, such that the following three conditions hold true:

1) $a < b$
2) $P(a) > P(b) > 0$
3) $P'(a) = P'(b) = 0$

The polynomial $P(x)$ has:

(A) 3 real zeros

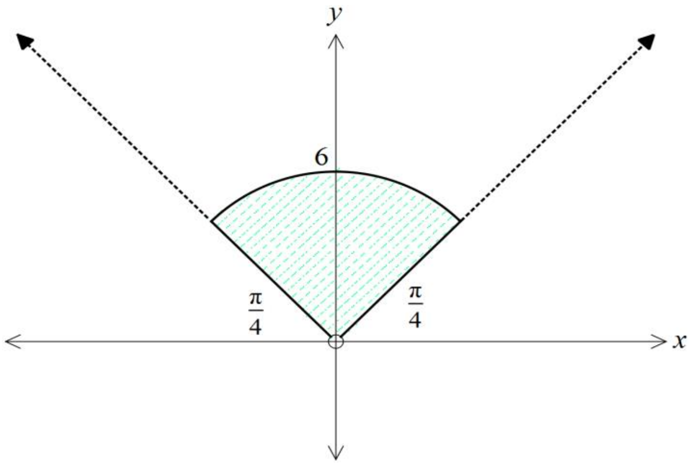
(B) 1 real zero γ such that $\gamma < a$

(C) 1 real zero γ such that $a < \gamma < b$

(D) 1 real zero γ such that $\gamma > b$

QUESTION 2

The shaded region of the Argand plane below represents the points where which of the following inequalities hold simultaneously?



(A) $|z| \leq 6$ and $-\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}$

(B) $|z| \leq 6$ and $\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$

(C) $|z| \geq \sqrt{6}$ and $-\frac{\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}$

(D) $|z| \geq \sqrt{6}$ and $\frac{\pi}{4} \leq \arg(z) \leq \frac{3\pi}{4}$

QUESTION 3

The graph $f(x) = \frac{1}{x^2 + mx - n}$, where m and n are real constants, has no vertical asymptotes if:

(A) $m^2 < -4n$

(B) $m^2 > -4n$

(C) $m^2 < 4n$

(D) $m^2 > 4n$

QUESTION 4

The complex number ω is a root of the equation $z^3 + 1 = 0$. Which of the following is FALSE?

(A) $\overline{w}$ is also a root.

(B) $w^2 + 1 - w = 0$

(C) $\frac{1}{w}$ is also a root.

(D) $(w - 1)^2 = -1$

QUESTION 5

Given the equation of the hyperbola, $\mathcal{H}$, is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has eccentricity *e*, then the ellipse, $\mathcal{E}$, with equation $\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1$ has which of the following as its eccentricity?

(A) $-e$

(B) $\frac{1}{e}$

(C) $\sqrt{e}$

(D) e^2

END OF SECTION I

SECTION II

QUESTION 6 (START A NEW ANSWER BOOKLET) 21 MARKS

a) Consider the complex number $z = x + iy$, where *x* and *y* are real, and the complex number $w = iz + \bar{z}$.

i) Find *w* in terms of *x* and *y* and hence show that Re(*w*) = Im(*w*). [2]

ii) Find all possible values of the principal argument of *w*. [1]

iii) Given that *x* and *y* are both positive and *x* < *y*, draw vectors representing z, iz, $\bar{z}$ and *w* on an Argand diagram. [2]

b) Let $z = 12cis\dfrac{13\pi}{15}$ and $w = 3cis\dfrac{\pi}{5}$

i) Evaluate $\dfrac{z}{w}$ and express your answer in the form $a + ib$. [2]

ii) Explain why $\dfrac{z^3}{w^3}$ is a real number. [2]

c) The locus of *z* is the region simultaneously satisfied by the following conditions:

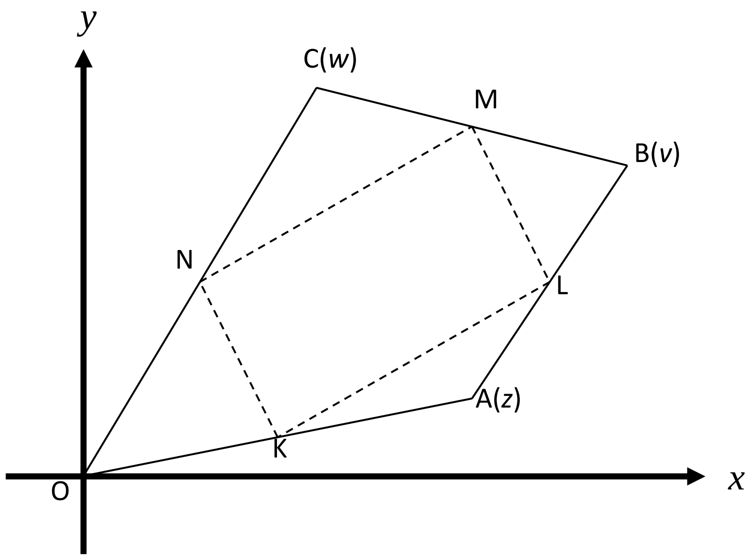
$|z - i| \leq 2$ & $\text{Im}(z)$ £ 1

i) Draw a neat sketch of the locus on an Argand diagram. [2]

ii) Find the values of z for which $|z|$ is a maximum and justify your answer. [2]

QUESTION 6 CONTINUES ON THE FOLLOWING PAGE…

d) OABC is a quadrilateral in the complex plane. The complex numbers, 0, z, v and w are represented by O, A, B and C respectively. K, L, M and N are the midpoints of OA, AB, BC and CO respectively.

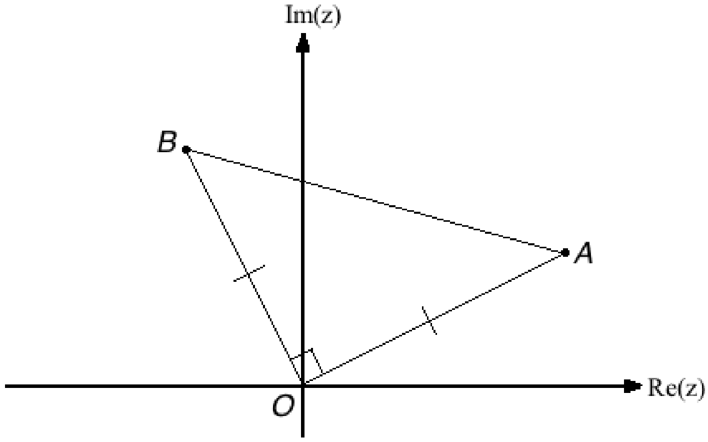


i) Show that the complex number represented by L is $\dfrac{v+z}{2}$. [2]

ii) Find similar results for K, M and N and hence calculate the complex numbers representing the vectors $\overrightarrow{NM}$ and $\overrightarrow{KL}$ and show that they are equal. [2]

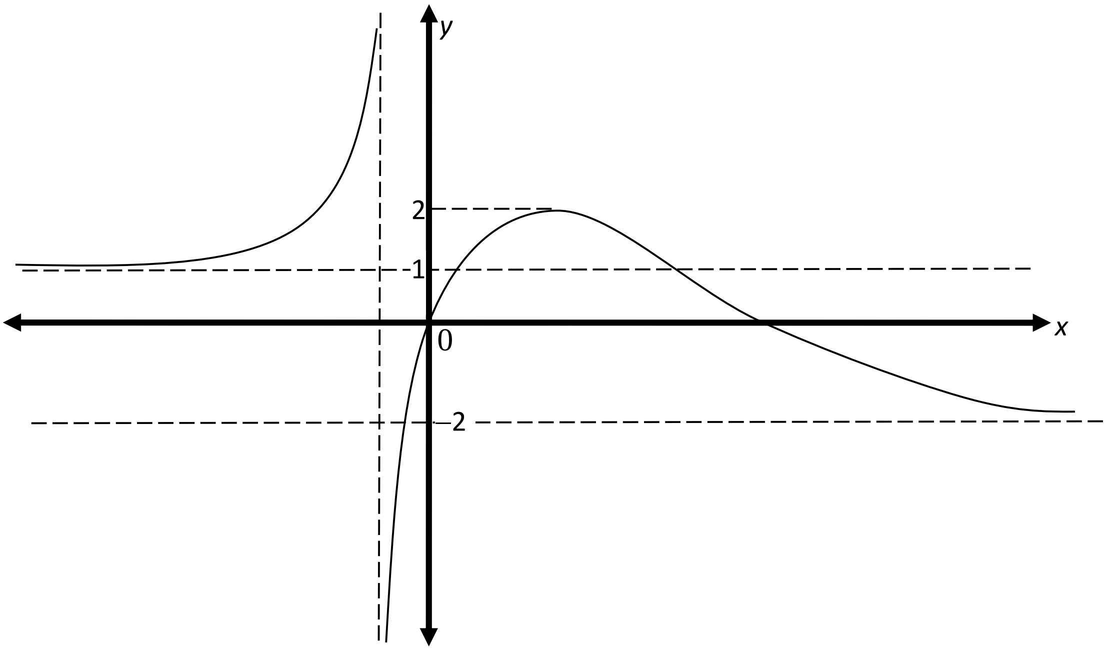
iii) What type of quadrilateral is KLMN? Justify your answer. [1]

e) The Argand diagram below shows the points A and B, which represent the complex numbers z_1 and z_2 respectively. Given that $\triangle BOA$ is a right-angled, isosceles triangle, prove that $(z_1 + z_2)^2 = 2z_1z_2$. [3]



END OF QUESTION 6

a) The diagram below shows the graph of the function $y = f(x)$.



Draw a one half-page sketch for each of the following. Make sure you indicate all significant features.

i) $y = f(-x)$ [2]

ii) $y = |f(x)|$ [2]

iii) $y = \dfrac{1}{f(x)}$ [2]

iv) $y = [f(x)]^2$ [2]

v) $y^2 = f(x)$ [2]

b) Consider the curve $x^3 + 3x^2y - 2y^3 = 16$

i) Show that $\dfrac{dy}{dx} = \dfrac{x^2 + 2xy}{2y^2 - x^2}$ [2]

ii) Find the coordinates of the stationary points on the curve. You do not need to determine the nature of the stationary points. [2]

END OF QUESTION 7

QUESTION 8 (START A NEW ANSWER BOOKLET) 9 MARKS

a) Consider the polynomial $P(x) = x^6 - 6x^4 + 5x^2 + 12$

i) Show that i is a zero of $P(x)$ and explain why $-i$ is also a zero of $P(x)$. [2]

ii) Explain why $(x^2 + 1)$ is a factor of $P(x)$. [1]

iii) Given that $x^4 - 7x^2 + 12 = (x^2 - 3)(x^2 - 4)$, factorise $P(x)$ over: [3]

α) the rational field **Q**

β) the real field **R**

γ) the complex field **C**

b) The polynomial $P(x) = x^4 - 3x^3 + 7x^2 + ax + b$ has real coefficients.

i) Given that 2 is a multiple zero of $P(x)$, find the values of a and b. [2]

ii) Explain why 2 is a double zero of $P(x)$ [1]

QUESTION 9 (START A NEW ANSWER BOOKLET) 21 MARKS

a) The equation of an ellipse is $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$.

i) Show that the eccentricity of the ellipse is $\dfrac{4}{5}$ and determine the foci and directrices of the ellipse. [3]

ii) Draw a neat sketch of the ellipse showing significant features. [2]

b) P$\left(cp, \dfrac{c}{p}\right)$and Q$\left(cq, \dfrac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. The tangents at the points P and Q meet at the point T.

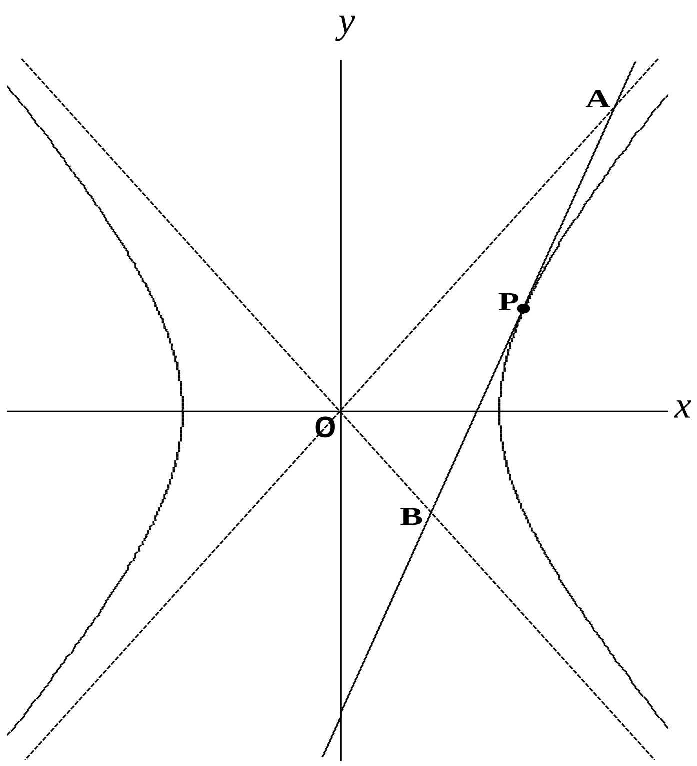
i) Show that the equation of the tangent at P is $x + p^2 y = 2cp$. [2]

ii) Show that the coordinates of T are $\left(\dfrac{2pqc}{p+q}, \dfrac{2c}{p+q}\right)$. [2]

iii) If $q = 3p$, show that the locus of T is a rectangular hyperbola and find its foci. [3]

QUESTION 9 CONTINUES ON THE FOLLOWING PAGE…

c) The equation of the hyperbola, $\mathcal{H}$, is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at the point P with coordinates $(a\sec\theta, b\tan\theta)$ on $\mathcal{H}$ cuts the asymptotes at A and B as shown in the diagram below.



i) Write down the equations of the asymptotes. [1]

ii) Show that the equation of AP is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ [3]

iii) Show that AP crosses the x-axis at the point $\left(\frac{a}{\sec\theta}, 0\right)$. [1]

iv) Show that A is the point $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$. [2]

v) Given that B is the point $\left(\frac{a}{\sec\theta + \tan\theta}, -\frac{b}{\sec\theta + \tan\theta}\right)$, find the exact area of ΔOAB. [2]

END OF SECTION II. END OF EXAM.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

THE SCOTS COLLEGE–2014–MATHEMATICS EXTENSION 2 MATHEMATICS PRE-TRIAL HSC

CANDIDATE NUMBER: ______________________

SECTION I – MULTIPLE CHOICE ANSWER SHEET (5 MARKS)

Mark the correct answer by filling in the circle. To make a correction, neatly place a cross over the circle and then fill in the correct circle.

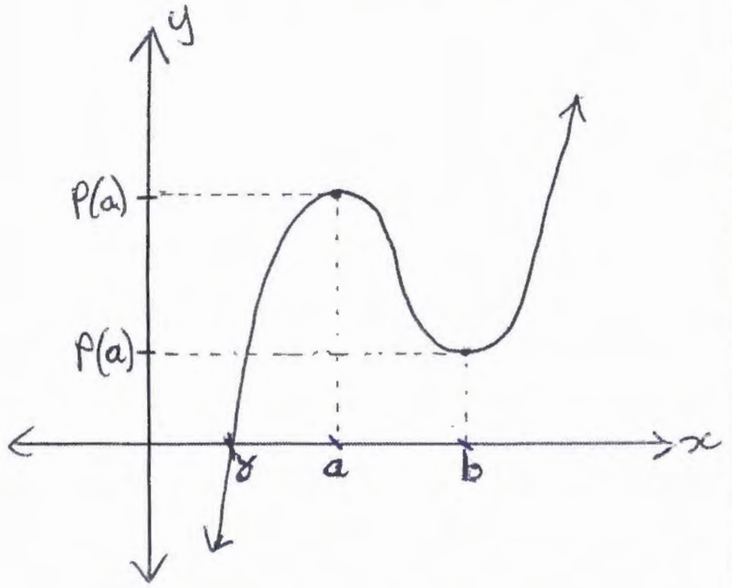
EXAMPLE:	A	B	C	D
	○	⊗	●	○

	A	**B**	**C**	**D**
Question 1	○	○	○	○
Question 2	○	○	○	○
Question 3	○	○	○	○
Question 4	○	○	○	○
Question 5	○	○	○	○

Mathematics Extension 2 - Pretrial 2014

Section 1:

① B: $P(x)$ has 1 real zero γ such that $\gamma < \alpha$



$a < b$, $P(a) > P(b) > 0$ and
$P'(a) = P'(b) = 0$

② B: $|z| \leq 6$ and $\frac{\pi}{4} \leq \arg z \leq \frac{3\pi}{4}$

③ A: $f(x) = \dfrac{1}{x^2 + mx - n}$

For $f(x)$ to have no vertical asymptotes, we need $\Delta < 0$ for $x^2 + mx - n$

$$\Delta < 0$$
$$m^2 - 4(1)(-n) < 0$$
$$m^2 + 4n < 0$$
$$m^2 < -4n$$

④ D: $z^3 + 1 = 0$

$$z^3 = -1$$

$$z^3 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$
for $k = 0, 1, 2$

$$z = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3},$$
$$\cos\pi + i\sin\pi \text{ or}$$
$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, -1 \text{ or } \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

$\therefore$ $\bar{w}$ is a root

$\frac{1}{w}$ is a root

Also, $w^3 + 1 = 0$
$$(w+1)(w^2 - w + 1) = 0$$
So $w^2 + 1 - w = 0$

But if $w = -1$, $(w-1)^2 = (-1-1)^2$
$$= (-2)^2$$
$$= 4$$
$$\neq -1$$

So D is false

(5) B: For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

we have

$$b^2 = a^2(e^2 - 1)$$

$$\frac{b^2}{a^2} = e^2 - 1$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$= \frac{a^2 + b^2}{a^2}$$

$$e = \frac{\sqrt{a^2 + b^2}}{a}$$

For the ellipse $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$

we have

$$b^2 = (a^2+b^2)(1 - E^2)$$

$$\frac{b^2}{a^2+b^2} = 1 - E^2$$

$$E^2 = 1 - \frac{b^2}{a^2+b^2}$$

$$= \frac{a^2+b^2-b^2}{a^2+b^2}$$

$$= \frac{a^2}{a^2+b^2}$$

$$E = \frac{a}{\sqrt{a^2+b^2}}$$

$$= \frac{1}{e}$$ ✓

Alternatively:

Hyperbolae have eccentricity e such that $e > 1$

So $-e < 0$

$$\frac{1}{e} < 1$$

$$\sqrt{e} > 1$$

and $e^2 > 1$

Ellipses have eccentricity e such that $0 < e < 1$

$\therefore$ $\frac{1}{e}$ is the ellipse eccentricity

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⑥ Complex Numbers

(a)(i) $w = iz + \bar{z}$

$= i(x+iy) + x - iy$

$= ix - y + x - iy$

$= x - y + i(x-y)$

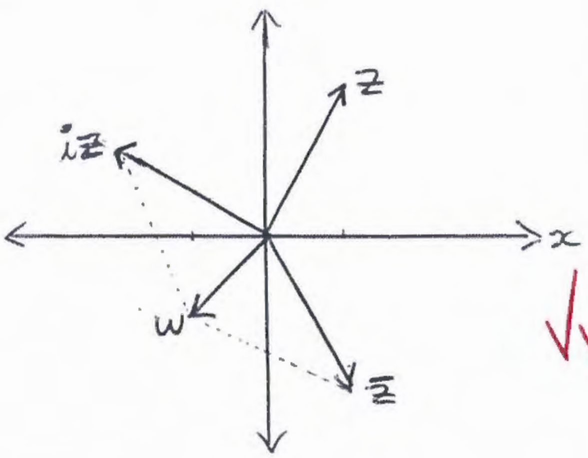
$\text{Re}(w) = x - y$

$= \text{Im}(w)$

(ii) If $\text{Re}(w) = \text{Im}(w)$, then

$\arg w = \frac{\pi}{4}$ or $-\frac{3\pi}{4}$

(iii)



(b)(i) $\frac{z}{w} = \frac{12 \text{ cis } \frac{13\pi}{15}}{3 \text{ cis } \frac{\pi}{5}}$

$= \frac{12}{3} \times \text{cis}\left(\frac{13\pi}{15} - \frac{\pi}{5}\right)$

$= 4 \times \text{cis}\left(\frac{13\pi}{15} - \frac{3\pi}{15}\right)$

$= 4 \text{ cis}\left(\frac{10\pi}{15}\right)$

$= 4 \text{ cis } \frac{2\pi}{3}$

$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -2 + 2\sqrt{3}i$

(ii) $\frac{z^3}{w^3} = \left(\frac{z}{w}\right)^3$

$= \left(4 \text{ cis } \frac{2\pi}{3}\right)^3$

$= 4^3 \times \text{cis}\left(\frac{3 \times 2\pi}{3}\right)$

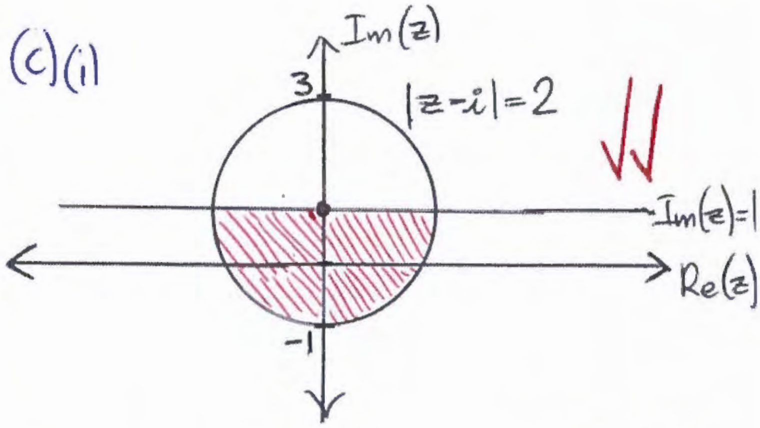
$= 64 \text{ cis } 2\pi$

$= 64(\cos 2\pi + i\sin 2\pi)$

$= 64(1 + 0i)$

$= 64$ which is a real number

(c)(i)



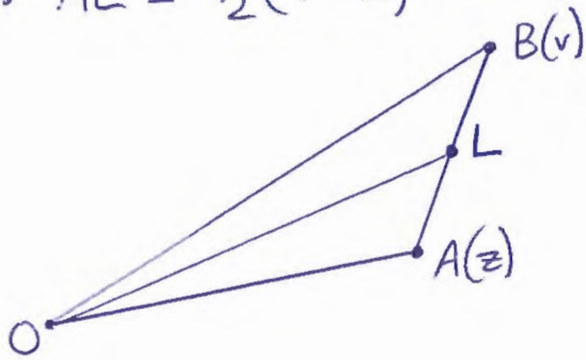
(ii) The points in the region furthest from the origin are where the line Im(z)=1 intersects the circle, so $|z|$ would be at a maximum value

$\therefore z = -2 + i$ or $2 + i$

(d)(i) $AL = \frac{1}{2}AB$

because L is the midpoint of AB

$\therefore AL = \frac{1}{2}(v - z)$



$OL = OA + AL$ (vector addition)

$= z + \frac{1}{2}(v - z)$

$= \frac{2z + v - z}{2}$

$= \frac{v + z}{2}$ ✓

So L is represented by the complex number $\frac{v+z}{2}$

(ii) Similarly, K, M and N are represented by the complex numbers

$\frac{z}{2}$, $\frac{v+w}{2}$ and $\frac{w}{2}$ respectively ✓

$\therefore \overrightarrow{NM} = \frac{v+w}{2} - \frac{w}{2}$

$= \frac{v}{2}$

$\overrightarrow{KL} = \frac{v+z}{2} - \frac{\;}{2}$

$= \frac{v}{2}$

So $\overrightarrow{NM} = \overrightarrow{KL}$ ✓

(iii) Because $\overrightarrow{NM} = \overrightarrow{KL}$, then $NM = KL$ (in length) and $NM // KL$ (parallel)

So KLMN is a parallelogram because it has an equal pair of opposite sides which are parallel ✓

(e) Since $\triangle BOA$ is right-angled and isosceles, we have ✓

$z_2 = iz_1$

$\therefore (z_1 + z_2)^2 = (z_1 + iz_1)^2$

$= z_1^2 + 2iz_1^2 + i^2 z_1^2$

$= z_1^2 + 2iz_1^2 - z_1^2$ ✓

$= 2iz_1^2$

$= 2 \times z_1 \times iz_1$

$= 2 \times z_1 \times z_2$

$= 2z_1z_2$ ✓

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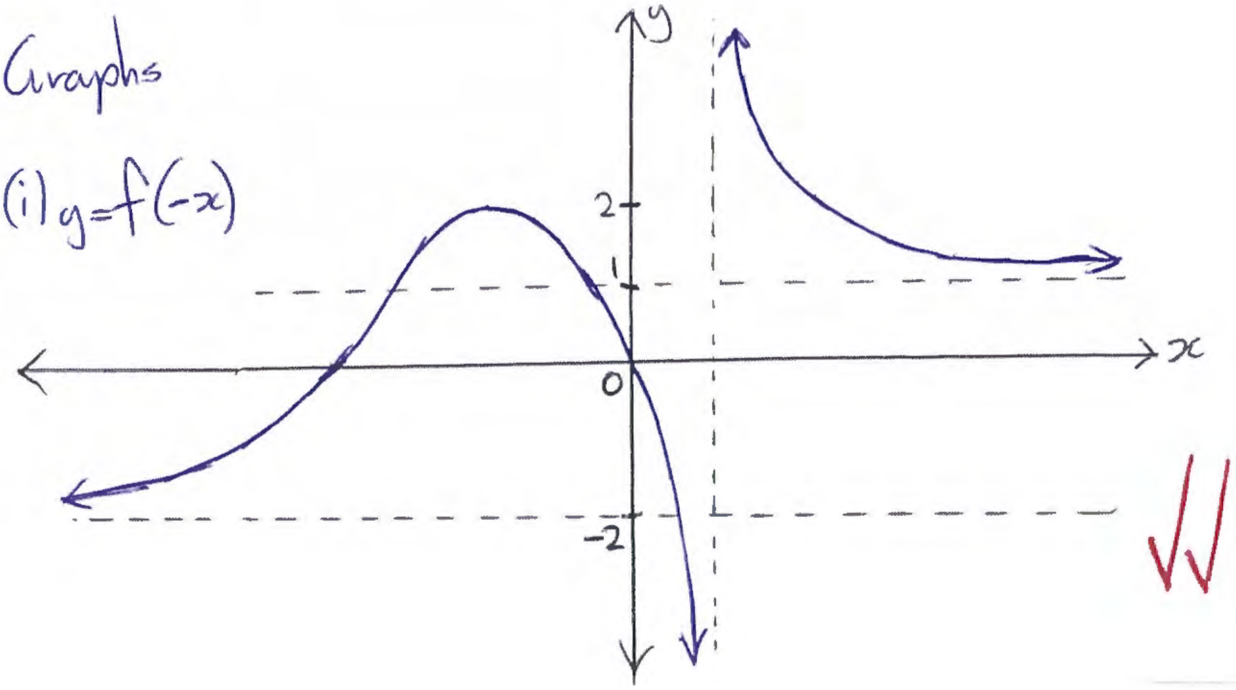
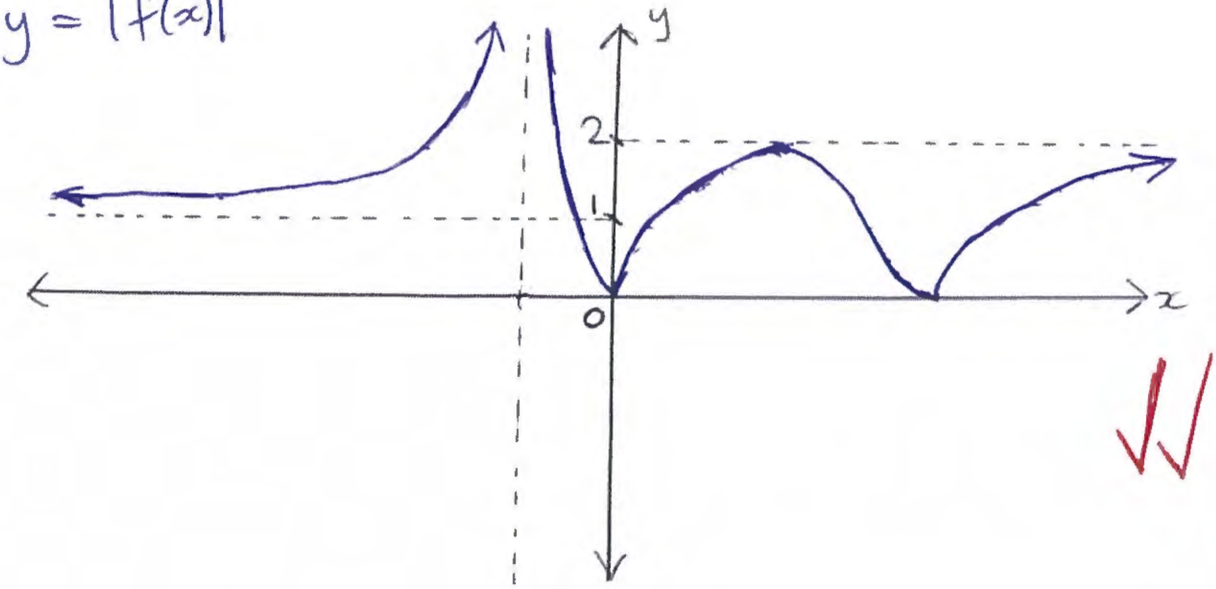
⑦ Graphs

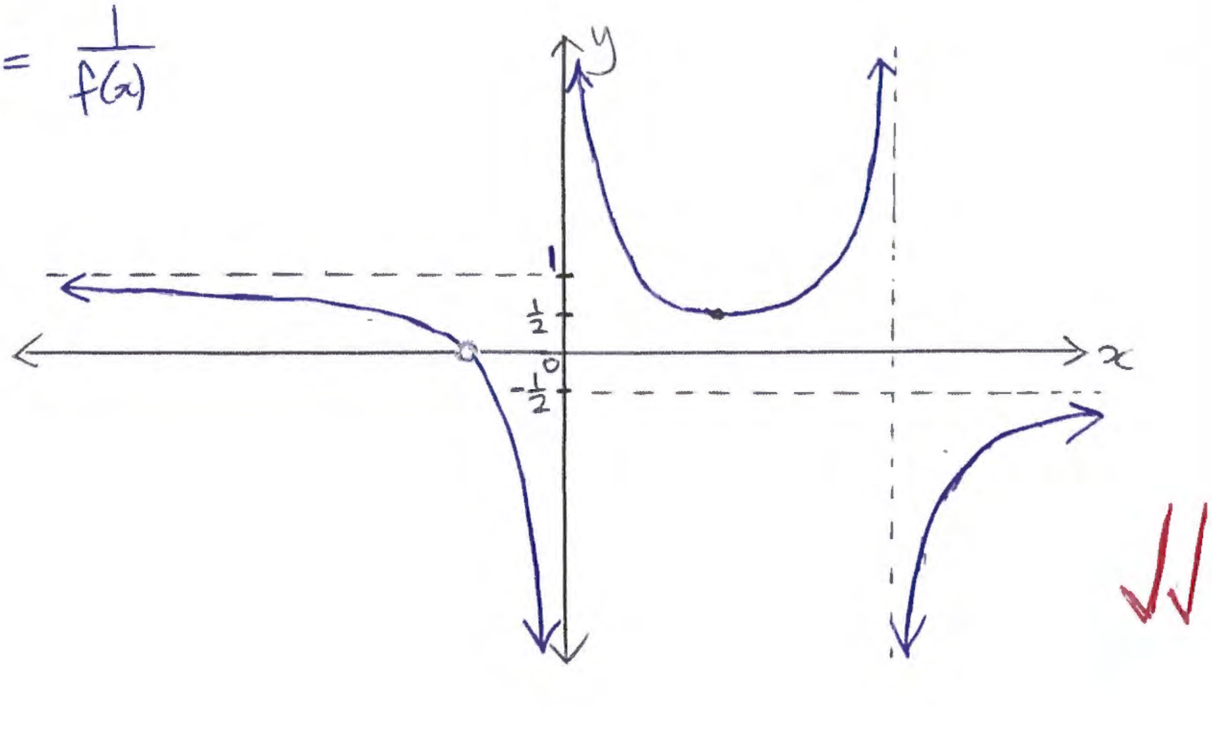
(a) (i) $y = f(-x)$



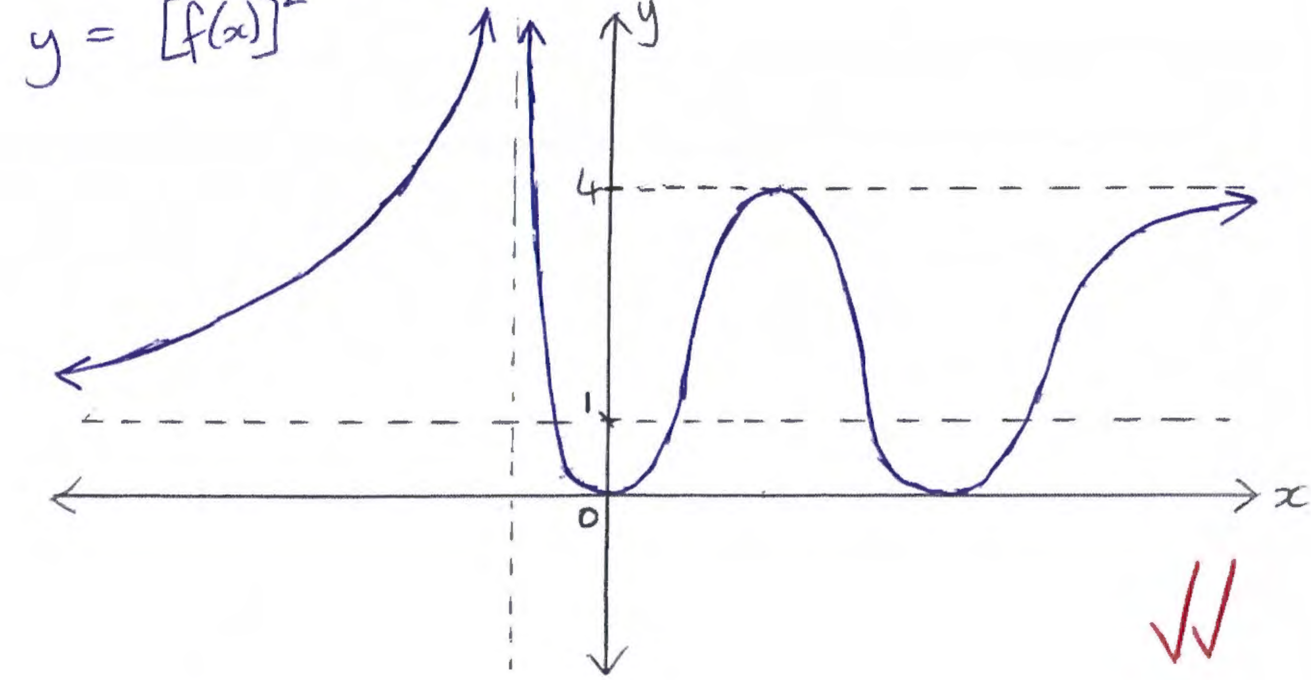
(ii) $y = |f(x)|$



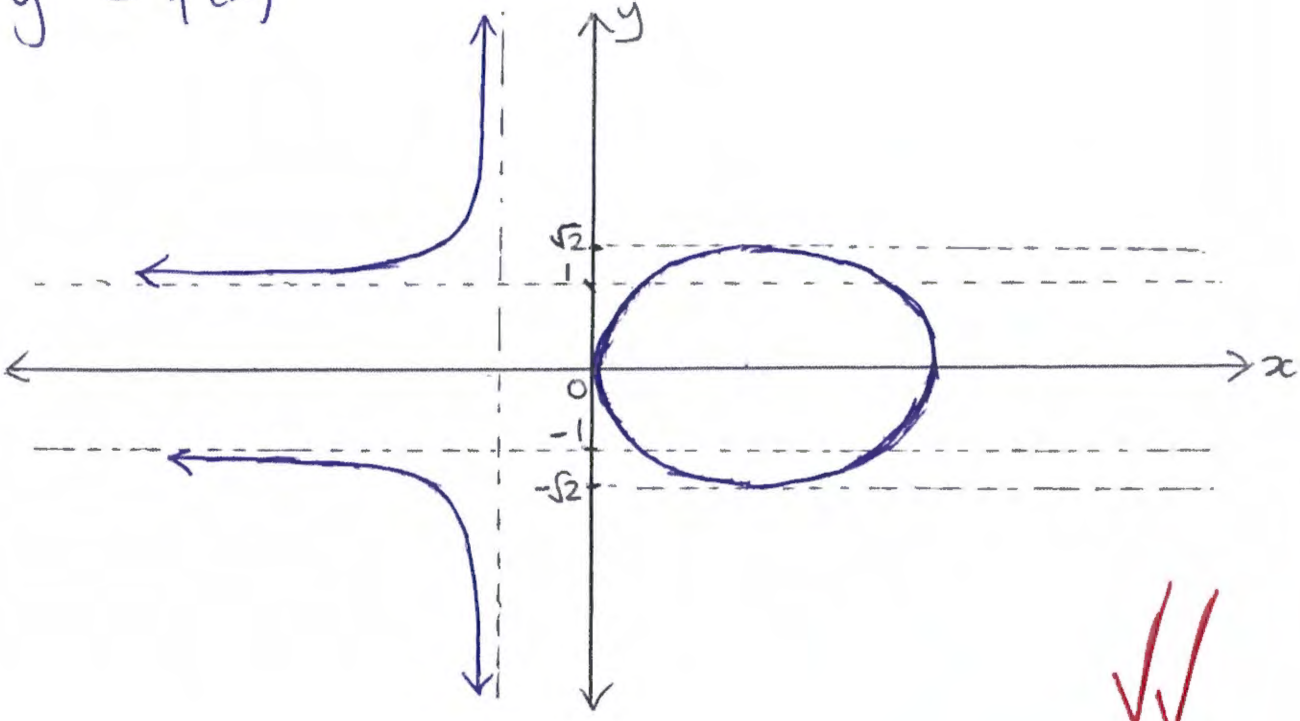
(iii) $y = \frac{1}{f(x)}$



(iv) $y = [f(x)]^2$



(v) $y^2 = f(x)$



(b)(i) $x^3 + 3x^2y - 2y^3 = 16$

$$\frac{d}{dx}\left(x^3 + 3x^2y - 2y^3\right) = \frac{d}{dx}(16)$$

$$3x^2 + 6xy + 3x^2\frac{dy}{dx} - 6y^2\frac{dy}{dx} = 0$$

$$x^2 + 2xy + x^2\frac{dy}{dx} - 2y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(x^2-2y^2\right) = -x^2-2xy$$

$$\frac{dy}{dx} = \frac{-(x^2+2xy)}{x^2-2y^2}$$

$$= \frac{x^2+2xy}{2y^2-x^2}$$

✓

(ii) $\frac{dy}{dx} = 0$ for stationary points

$$x^2+2xy = 0$$

$$x(x+2y) = 0$$

$$x = 0 \text{ or } -2y$$

when $x=0$, $(0)^3+3(0)^2y-2y^3 = 16$

$$-2y^3 = 16$$

$$y^3 = -8$$

$$y = -2$$

$\therefore$ $(0,-2)$ is a stationary point ✓

Also, when $x=-2y$, $(-2y)^3+3(-2y)^2y-2y^3 = 16$

$$-8y^3+12y^3-2y^3 = 16$$

$$2y^3 = 16$$

$$y^3 = 8$$

$$y = 2$$

when $y=2$, $x = -2(2)$

$$= -4$$

$\therefore$ $(-4,2)$ is a stationary point ✓

So there are stationary points at $(0,-2)$ and $(-4,2)$

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POLYNOMIALS

8 (a) $P(i) = (i)^6 - 6(i)^4 + 5(i)^2 + 12$
$= -1 - 6 - 5 + 12$
$= 0$ ✓

$\therefore$ i is a zero of $P(x)$

Since the coefficients of $P(x)$ are all real numbers, the conjugate of i, which is $-i$, is also a zero because the complex roots of $P(x)$ must occur in conjugate pairs ✓

Check: $P(-i) = (-i)^6 - 6(-i)^4 + 5(-i)^2 + 12$
$= -1 - 6 - 5 + 12$
$= 0$

(ii) Both $(x-i)$ and $(x+i)$ are factors of $P(x)$ so we have
$(x-i)(x+i) = x^2+1$
which is a factor of $P(x)$ ✓

(iii) (α) $P(x) = (x^2+1)(x^2-3)(x^2-4)$
$= (x^2+1)(x^2-3)(x+2)(x-2)$
over the rational field $\mathbb{Q}$ ✓

(β) $P(x) = (x^2+1)(x+\sqrt{3})(x-\sqrt{3})(x+2)(x-2)$
over the real field $\mathbb{R}$ ✓

(γ) $P(x) = (x+i)(x-i)(x+\sqrt{3})(x-\sqrt{3})(x+2)(x-2)$
over the complex field $\mathbb{C}$ ✓

(b)(i) $P(x) = x^4 - 3x^3 + 7x^2 + ax + b$
$P'(x) = 4x^3 - 9x^2 + 14x + a$
$P'(2) = 4(2)^3 - 9(2)^2 + 14(2) + a$
$0 = 32 - 36 + 28 + a$
$a = -24$ ✓

Also, $P(2) = 0$
$0 = 2^4 - 3(2)^3 + 7(2)^2 - 24(2) + b$
$0 = 16 - 24 + 28 - 48 + b$
$b = 28$ ✓

$\therefore P(x) = x^4 - 3x^3 + 7x^2 - 24x + 28$

(ii) $P'(x) = 4x^3 - 9x^2 + 14x$
$P''(x) = 12x^2 - 18x + 14$
However, $P''(2) = 12(2)^2 - 18(2) + 14$
$= 48 - 36 + 14$
$= 26$
$\neq 0$

Because $P(2) = 0$ and $P'(2) = 0$ but $P''(2) \neq 0$, 2 is a double root of $P(x)$ ✓

9

CONICS

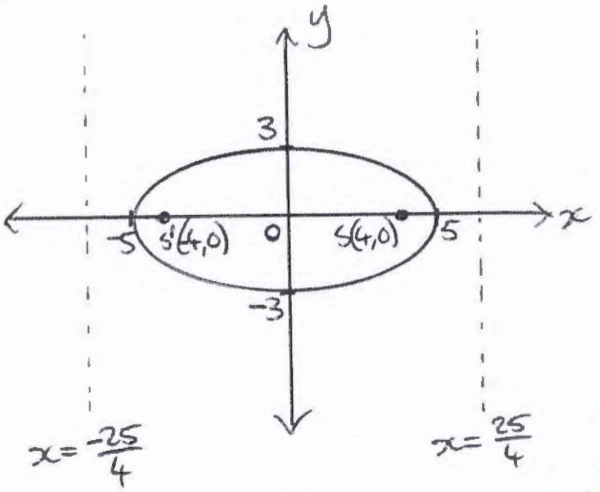
9 (a) (i)

$$b^2 = a^2(1-e^2)$$
$$9 = 25(1-e^2)$$
$$\frac{9}{25} = 1-e^2$$
$$e^2 = \frac{16}{25}$$
$$e = \sqrt{\frac{16}{25}}$$
$$= \frac{4}{5}$$

foci are $(\pm ae, 0)$
$$= (\pm 5 \times \tfrac{4}{5}, 0)$$
$$= (-4,0) \text{ and } (4,0)$$

directrices are $x = \pm \frac{a}{e}$
$$= \pm \frac{5}{\frac{4}{5}}$$
$$= \pm 5 \times \frac{5}{4}$$
$$= -\frac{25}{4} \text{ and } \frac{25}{4}$$

(ii)



$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

(b) (i)

$$xy = c^2$$
$$y = \frac{c^2}{x}$$
$$= c^2 x^{-1}$$
$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

At $P(cp, \frac{c}{p})$, $\frac{dy}{dx} = -\frac{c^2}{(cp)^2}$
$$= -\frac{c^2}{c^2p^2}$$
$$= -\frac{1}{p^2}$$

$$\therefore y - y_1 = m(x - x_1)$$
$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$
$$p^2y - cp = -x + cp$$
$$x + p^2y = 2cp$$

is the equation of the tangent at P

(ii) The tangent at Q has equation

$$x + q^2y = 2cq$$

So we have

$$x + p^2y = 2cp \quad (1)$$
$$x + q^2y = 2cq \quad (2)$$

(1) − (2):
$$(p^2-q^2)y = 2c(p-q)$$
$$(p+q)(p-q)y = 2c(p-q)$$
$$y = \frac{2c}{p+q}$$

when $y = \frac{2c}{p+q}$

$$x + p^2\left(\frac{2c}{p+q}\right) = 2cp$$

$$x = 2cp - \frac{2cp^2}{p+q}$$

$$= \frac{2cp(p+q) - 2cp^2}{p+q}$$

$$= \frac{2cp^2 + 2pqc - 2cp^2}{p+q}$$

$$= \frac{2pqc}{p+q}$$ ✓

So T is the point $\left(\frac{2pqc}{p+q}, \frac{2c}{p+q}\right)$

(iii) If $q = 3p$

$$T = \left(\frac{2p(3p)c}{p+}, \frac{2c}{p+3p}\right)$$

$$= \left(\frac{6p^2c}{4p}, \frac{2c}{4p}\right)$$

$$= \left(\frac{3pc}{2}, \frac{c}{2p}\right)$$ ✓

If $y = \frac{c}{2p}$

$$p = \frac{c}{2y}$$

and $x = \frac{3pc}{2}$

$$= \frac{3c}{2} \times \frac{c}{2y}$$

$$= \frac{3c^2}{4y}$$

$$\therefore xy = \frac{3c^2}{4}$$ ✓

This is a rectangular hyperbola

$$xy = \left(\frac{\sqrt{3}}{2}c\right)^2$$

$$= d^2 \text{ for } d = \frac{\sqrt{3}c}{2}$$

Foci of a rectangular hyperbola are at the points $(\pm a, \pm a)$

$$= (\pm\sqrt{2}d, \pm\sqrt{2}d)$$

$$= \left(\pm\sqrt{2}\times\frac{\sqrt{3}}{2}c, \pm\sqrt{2}\times\frac{\sqrt{3}}{2}c\right)$$

$$= \left(-\frac{\sqrt{6}}{2}c, -\frac{\sqrt{6}}{2}c\right) \text{ and}$$

$$\left(\frac{\sqrt{6}}{2}c, \frac{\sqrt{6}}{2}c\right)$$ ✓

(c) (i) Asymptotes: $y = -\frac{bx}{a}$ and $y = \frac{bx}{a}$ ✓

(ii) $$\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

At $P(a\sec\theta, b\tan\theta)$;

$$\frac{dy}{dx} = \frac{b^2}{a^2} \times \frac{a\sec\theta}{b\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$ ✓

$$\therefore \quad y - y_1 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$ab\sec^2\theta - ab\tan^2\theta = bx\sec\theta - ay\tan\theta$$

$$ab(\sec^2\theta - \tan^2\theta) = bx\sec\theta - ay\tan\theta$$

$$ab = bx\sec\theta - ay\tan\theta$$

$$1 = \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b}$$

(iii) When $y = 0$,

$$1 = \frac{x\sec\theta}{a}$$

$$a = x\sec\theta$$

$$x = \frac{a}{\sec\theta}$$

So AP crosses the x-axis at the point $\left(\frac{a}{\sec\theta}, 0\right)$

(iv) At A, we have $y = \frac{bx}{a}$

$$\therefore \quad \frac{x\sec\theta}{a} - \left(\frac{bx}{a}\right)\frac{\tan\theta}{b} = 1$$

$$\frac{x\sec\theta}{a} - \frac{x\tan\theta}{a} = 1$$

$$x(\sec\theta - \tan\theta) = a$$

$$x = \frac{a}{\sec\theta - \tan\theta}$$

$$y = \frac{b}{a}x$$

$$= \frac{b}{a} \times \frac{a}{\sec\theta - \tan\theta}$$

$$= \frac{b}{\sec\theta - \tan\theta}$$

$\therefore$ A is the point $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$.

(v) Area ΔOAB = Area ΔOAX + Area ΔOBX

Taking OX as the base length of each triangle, we have

$$\text{Area} = \frac{1}{2} \times \text{base} \times (\text{height}_1 + \text{height}_2)$$

$$= \frac{1}{2} \times \frac{a}{\sec\theta} \times \left(\frac{b}{\sec\theta - \tan\theta} + \frac{b}{\sec\theta + \tan\theta}\right)$$

$$= \frac{a}{2\sec\theta}\left(\frac{b(\sec\theta + \tan\theta) + b(\sec\theta - \tan\theta)}{\sec^2\theta - \tan^2\theta}\right)$$

$$= \frac{a}{2\sec\theta} \times \frac{2b\sec\theta}{1}$$

$$= \frac{2ab\sec\theta}{2\sec\theta}$$

$$= ab \quad \text{units}^2$$

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