St Aloysius' College
Year 12 Mid-Year Examinations
2008

EXTENSION 2 MATHEMATICS (Additional paper)

Total marks - 80

Reading time - 5 minutes
Working time – 2 hours

Examination papers must NOT be removed from the examination room.

General Instructions

- Start each question in a new booklet
- Board approved calculators may be used
- Marks may be deducted for careless or badly arranged work
- Show all necessary work

SUPERVISOR'S INSTRUCTIONS:
Please issue five 4 page answer booklets.
Please collect the examination paper with the answer booklets.

Question 1 *(16 marks)*

(a) Let $\mathscr{H}$ be the hyperbola $\dfrac{x^2}{5^2} - \dfrac{y^2}{3^2} = 1$

(i) Calculate the eccentricity of $\mathscr{H}$ **1**

(ii) Find the coordinates of the foci and the directrices of $\mathscr{H}$ **3**

(iii) Find the equations of the asymptotes of $\mathscr{H}$. **1**

(b) The point $P(x_1, y_1)$ lies on the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{7} = 1$.

The tangent at P cuts the x axis at T, and has equation $\dfrac{x_1 x}{9} + \dfrac{y_1 y}{7} = 1$

(i) Find the coordinates of T. **1**

(ii) Using the focus-directrix definition, or otherwise, show that $\dfrac{PS}{PS'} = \dfrac{TS}{TS'}$ where S and S' are the foci of the ellipse. **3**

(c) (i) Write down the equation of the chord of contact from $T(x_0, y_0)$ to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$. **1**

(ii) Can the chord of contact ever be a diameter? **1**

(d) (i) Let x be a fixed number satisfying $0 < x < 1$. Use the method of mathematical induction to prove that **3**

$$(1-x)^n > 1 - nx \qquad \text{for} \qquad n = 2, 3, \ldots$$

(ii) Deduce that **2**

$$\left(1 - \frac{1}{3n}\right)^n > \frac{2}{3} \qquad \text{for} \qquad n = 2, 3, \ldots$$

Question 2 *(16 marks)* START A NEW BOOKLET

(a) (i) Express $-1-i\sqrt{3}$ in modulus-argument form. **2**

(ii) Hence evaluate $(-1-i\sqrt{3})^5$ in the form $a+ib$ **2**

(b) Sketch the region where the inequalities **3**

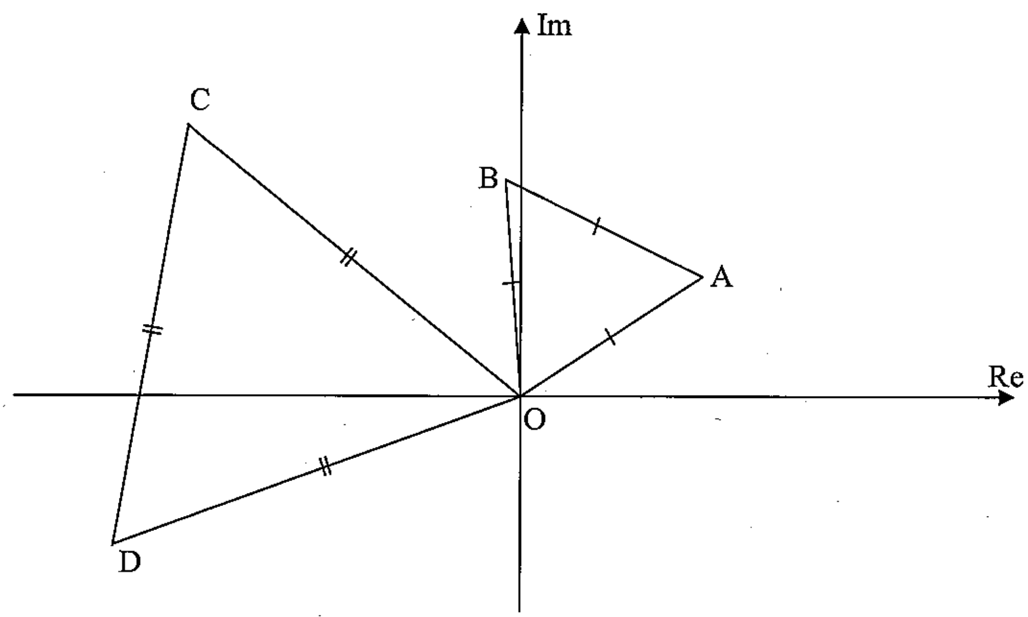
$|z-4-i| \le 5$ and $|z+2| \le |z-2|$ both hold.

(c) Let $z_1 = \dfrac{4-3i}{5}$ and $z_2 = \dfrac{8+15i}{17}$ so that $|z_1| = |z_2| = 1$

(i) Find $z_1 z_2$ and $\overline{z}_1 z_2$ in the form $a+ib$ **2**

(ii) Hence find two distinct ways of writing 85^2 as the sum $x^2 + y^2$, where x and y are positive integers. **2**

(d)



Let $\omega = cis\dfrac{\pi}{3}$ and the points A, B, C and D correspond to the complex numbers α, β, γ and δ repectively. ΔOAB and ΔOCD are equilateral.

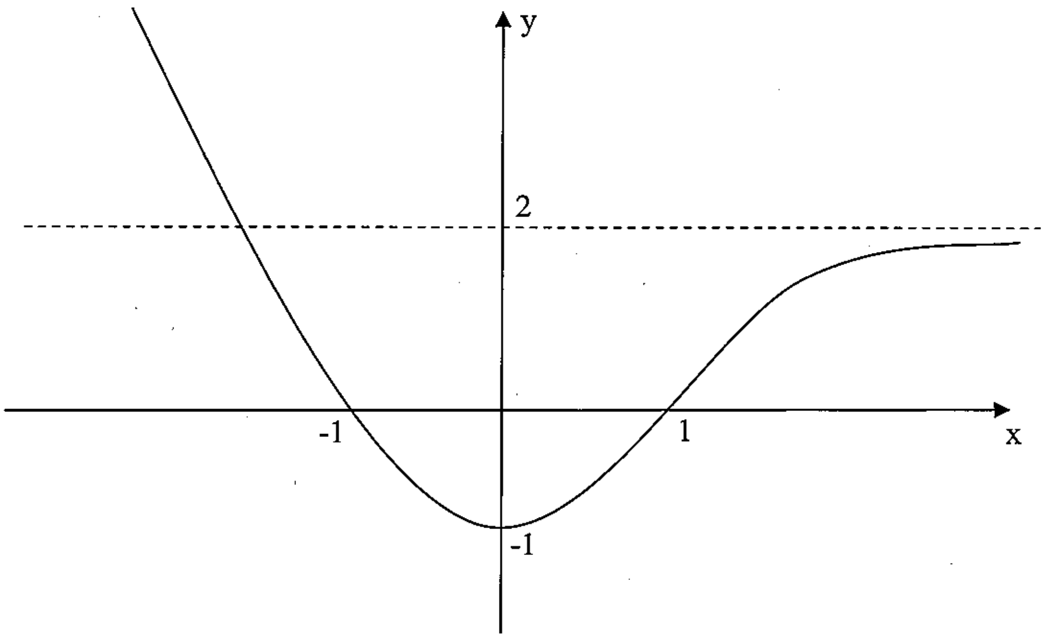
(i) Explain why $\delta = \omega\gamma$ **2**

(ii) Find the complex number α in terms of β **1**

(iii) Using complex numbers, show that the lengths AC and BD are equal. **2**

Question 3 *(16 marks)* START A NEW BOOKLET

The function $y = f(x)$ is defined by the following graph.



On separate diagrams, provide sketches of the following:

(i) $y = \dfrac{1}{f(x)}$ **3**

(ii) $y = f(|x|)$ **2**

(iii) $y = |f(x)|$ **2**

(iv) $y^2 = f(x)$ **3**

(v) $y = \ln f(x)$ **3**

(vi) $y = f(\ln x)$ **3**

Question 4 *(16 marks)* START A NEW BOOKLET

(a) (i) Derive the equation of the tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$. **2**

(ii) Derive the equation of the tangent to the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$. **2**

(iii) Show that the point of intersection of these tangents is $\left(a, b\tan\frac{\theta}{2}\right)$. **3**

(b) Find in mod-arg form the values of z which satisfy $z^5=\dfrac{1+i}{\sqrt{2}}$ **4**

(c) The sequence of numbers 1, 3, 7, 17, ... is defined by **5**
$T_1=1,\quad T_2=3,\quad T_{n+2}=2T_{n+1}+T_n\quad$ for $\quad n\geq 1$
Use mathematical induction to prove that $T_n=\frac{1}{2}\left(1+\sqrt{2}\right)^n+\frac{1}{2}\left(1-\sqrt{2}\right)^n$ for $n\geq 1$

Question 5 *(16 marks)* START A NEW BOOKLET

(a) Find real numbers a, b and c such that $\dfrac{x^2-x}{2(x+1)}=ax+b+\dfrac{c}{x+1}$ **3**

(b) Consider the function $y=\dfrac{x^2-x}{2(x+1)}$

(i) Using part (a), or otherwise, write down the equations of the oblique asymptote and the vertical asymptote. **2**

(ii) Find the intersection of the asymptotes, **1**

(iii) Find the coordinates and the nature of any stationary points. **5**

(iv) Sketch the function taking care with the relative positions of the critical features. **4**

(v) For what two values of k does the equation $kx=\dfrac{x^2-x}{2(x+1)}$ have one distinct solution only. **1**

END OF EXAM

Ext 2 Mid-Year. S.A.C. 2008.

Q1

a) $\frac{x^2}{5^2} - \frac{y^2}{3^2} = 1$

(i) $b^2 = a^2(e^2-1)$

$\therefore 9 = 25(e^2-1)$

$\therefore e^2 = \frac{9}{25} + 1$

$e = \frac{\sqrt{34}}{5}$ (1)

(ii) $ae = \sqrt{34}$

$\therefore S(\sqrt{34},0)$ $S'(-\sqrt{34},0)$ (1)

and $\frac{a}{e} = \frac{25}{\sqrt{34}}$

$\therefore x = \pm\frac{25}{\sqrt{34}}$ (1)

and (1) for twins.

(iii) $\frac{y^2}{9} = \frac{x^2}{25} - 1$

$y = \pm\frac{3}{5}\sqrt{x^2-25}$

$\therefore$ Asymptotes $y = \frac{\pm 3x}{5}$ (1)

b) (i) $\frac{x_1 x}{9} + \frac{y_1 y}{7} = 1$ Let $y=0 \Rightarrow x = \frac{9}{x_1}$ $\therefore T\left(\frac{9}{x_1}, 0\right)$ (1)

(ii) $\frac{PS}{PS'} = \frac{\cancel{e}PN}{\cancel{e}PN'}$ where N, N' are feet of perpendicular from P to the directrices. (1)

$= \frac{\frac{3}{e} - x_1}{\frac{3}{e} + x_1}$

$= \frac{3 - ex_1}{3 + ex_1}$ (1)

But $\frac{TS}{TS'} = \frac{\frac{9}{x_1} - 3e}{\frac{9}{x_1} + 3e}$

$= \frac{3 - ex_1}{3 + ex_1}$ (1)

$\therefore \frac{PS}{PS'} = \frac{TS}{TS'}$

c) (i) $\frac{xx_0}{16} + \frac{yy_0}{4} = 1$ (1)

(ii) No because $(0,0)$ does not satisfy the above equation. (1)

d) (i) Let $S(n)$ be the statement that $(1-x)^n > 1-nx$ for $n=2,3\ldots$ and $0<x<1$

For $n=2$ LHS $= 1-2x+x^2$

$> 1-2x$

$=$ RHS

$\therefore S(2)$ is true. (1)

Assume $S(k)$ ie $(1-x)^k > 1-kx$ for $k=2,3\ldots$

Consider $S(k+1)$ LHS $= (1-x)^{k+1}$

$= (1-x)^k(1-x)$

$> (1-kx)(1-x)$ by assumption (1)

$= 1-(k+1)x + kx^2$

$> 1-(k+1)x$ (1)

$=$ RHS

Hence $S(n)$ is true for all integers $n \geq 2$ by the Principle of M.I.

(ii) Let $x = \frac{+1}{3n}$ (1)

$\therefore 0<x<1$ for $n=2,3\ldots$

$\therefore$ Part(i) $\Rightarrow \left(1-\frac{1}{3n}\right)^n > 1-\frac{1}{3}$

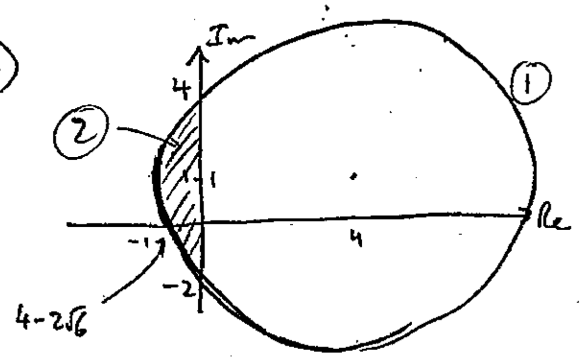
$= \frac{2}{3}$ (1)

Q2 (a) (i) $-1-i\sqrt{3} = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ 2

(ii) $(-1-i\sqrt{3})^5 = 2^5\left[\cos\frac{-10\pi}{3} + i\sin\frac{-10\pi}{3}\right]$ [De Moivre] 1

$= -16 + 16\sqrt{3}\,i$ 1

(b)



(c) (i) $z_1 z_2 = \frac{4-3i}{5}\cdot\frac{8+15i}{17}$

$= \frac{77+36i}{85}$ 1

$\bar{z}_1 z_2 = \frac{4+3i}{5}\cdot\frac{8+15i}{17}$

$= \frac{-13+84i}{85}$ 1

(ii) $|z_1| = |z_2| = 1 \quad \therefore |z_1 z_2| = 1$ and $|\bar{z}_1 z_2| = 1$

$\therefore 77^2 + 36^2 = 85^2 \Rightarrow x = 77,\ y = 36$ 1

and $13^2 + 84^2 = 85^2 \Rightarrow x = 13,\ y = 84$ 1

(d) (i) $|\omega| = 1 \quad \therefore |\delta| = |\omega\gamma|$ since $OD = OC$ 1

and $\angle COD = \frac{\pi}{3} \quad \therefore \arg\delta = \arg\gamma + \frac{\pi}{3}$ [Not principle]

$= \arg\gamma + \arg\omega$

$= \arg\omega\gamma$ 1

$\therefore \delta = \omega\gamma$

(ii) $\beta = \omega\alpha \Rightarrow \alpha = \frac{\beta}{\omega}$

$= \bar{\omega}\beta$ 1

(iii) $AC = |\gamma - \alpha|$ and $BD = |\delta - \beta|$

$= |\gamma - \bar{\omega}\beta|$ $\qquad = |\omega\gamma - \beta|$ 1

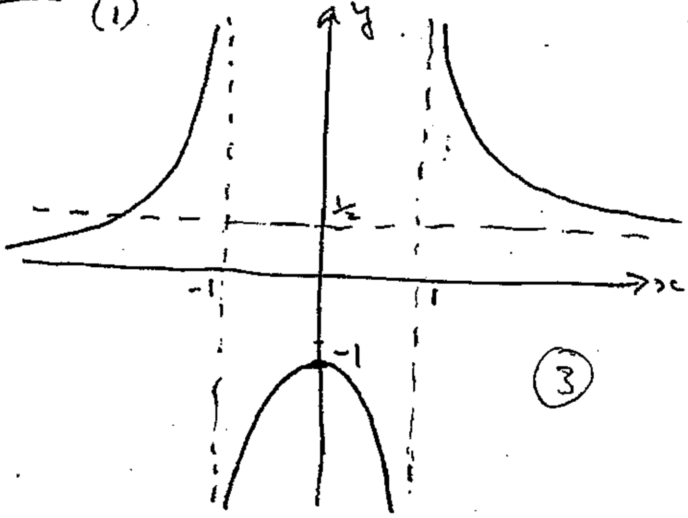
$= |\bar{\omega}|\,|\omega\gamma - \beta|$

$= |\omega\gamma - \beta|$ 1

$\therefore AC = BD$

Q3

(i)



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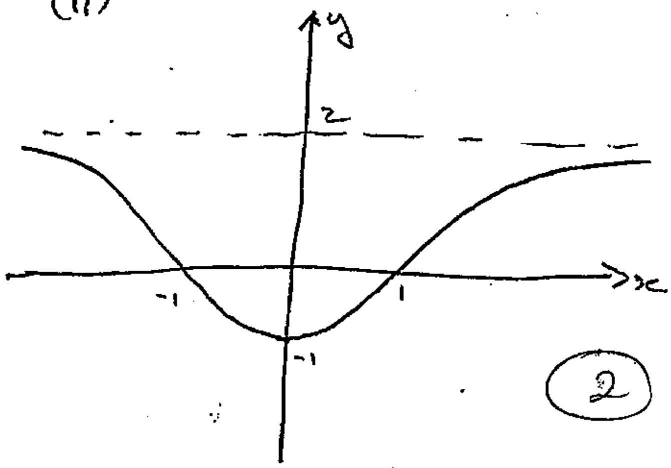
(ii)



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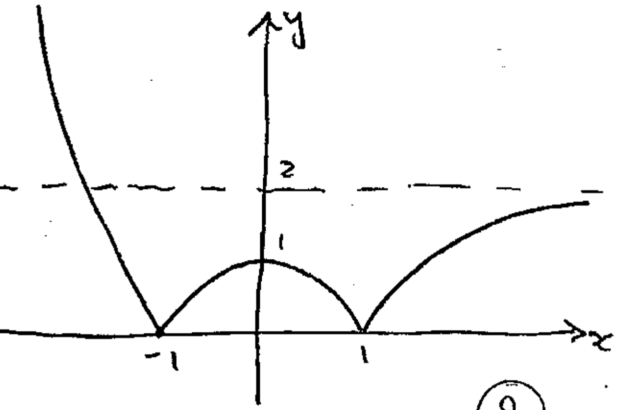
(iii)



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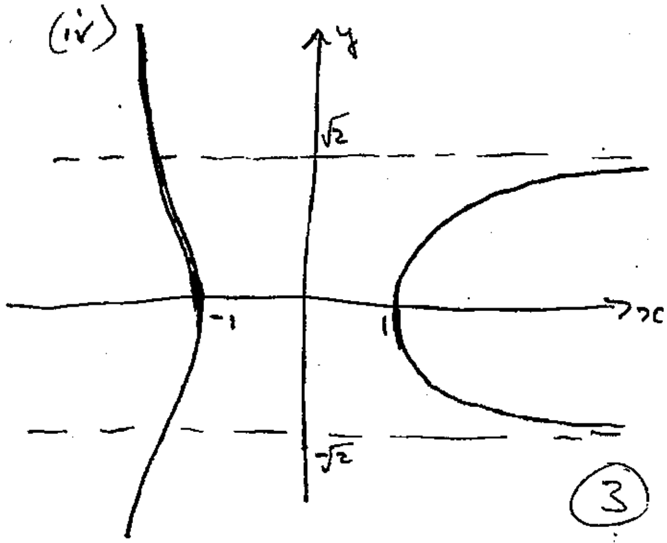
(iv)



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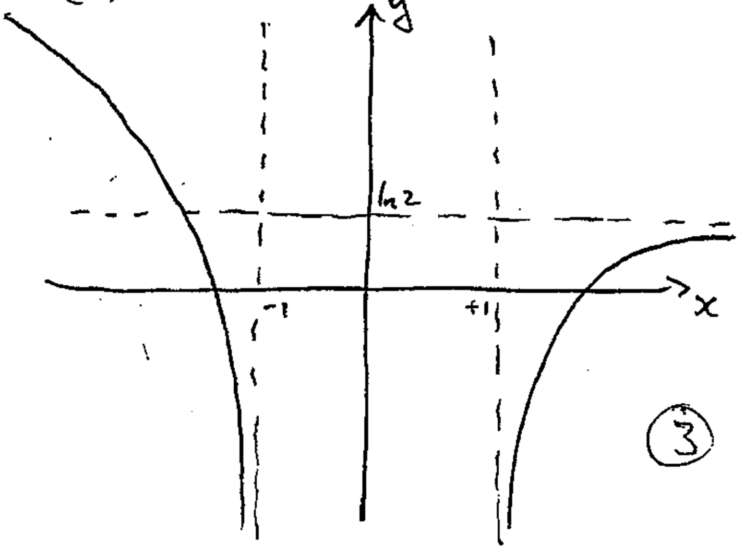
(v)



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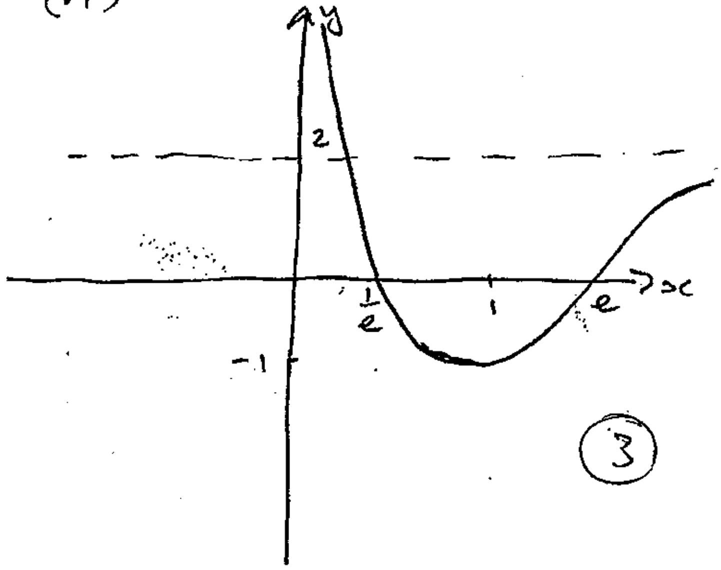
(vi)



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Q4 (a) (i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= -\dfrac{b\cos\theta}{a\sin\theta}$ (1)

$\therefore y - b\sin\theta = \dfrac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$\therefore \dfrac{y\sin\theta}{b} - \sin^2\theta = -\dfrac{x\cos\theta}{a} + \cos^2\theta$

$\therefore \dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ (A) (1)

(ii) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$= \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta}$

$= \dfrac{b\sec\theta}{a\tan\theta}$ (1)

$\therefore y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$\dfrac{y\tan\theta}{b} - \tan^2\theta = \dfrac{x\sec\theta}{a} - \sec^2\theta$

$\therefore \dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ (B) (1)

(iii) (A) + (B)$\times\cos\theta \Rightarrow \dfrac{x\cos\theta}{a} + \dfrac{x}{a} = 1 + \cos\theta$

$x\left(\dfrac{\cos\theta + 1}{a}\right) = 1 + \cos\theta$ (1)

$\therefore x = a$

$\rightarrow$ (A) $\Rightarrow \dfrac{y\sin\theta}{b} = 1 - \cos\theta$

$y = \dfrac{b(1-\cos\theta)}{\sin\theta}$ (1)

$= \dfrac{b \times 2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= b\tan\frac{\theta}{2}$ (1)

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b) Since $\left|\dfrac{1+i}{\sqrt{2}}\right| = 1 \quad \therefore$ Let $z = \text{cis}\,\theta$ (1)

$\therefore \cos 5\theta + i\sin 5\theta = \dfrac{1+i}{\sqrt{2}}$ using De Moivre

$\therefore \cos 5\theta = \frac{1}{\sqrt{2}}$ and $\sin 5\theta = \frac{1}{\sqrt{2}}$ (1)

$\therefore 5\theta = \frac{\pi}{4}, -\frac{7\pi}{4}, \frac{9\pi}{4}, -\frac{15\pi}{4}, \frac{17\pi}{4}$ (or $45^\circ, -315^\circ, 405^\circ, -675^\circ, 765^\circ$)

$\theta = \frac{\pi}{20}, -\frac{7\pi}{20}, \frac{9\pi}{20}, -\frac{3\pi}{4}, \frac{17\pi}{20}$ (1) ~~(or 2.25°, -15.75°, 20.25°, -35.75°, 38.25°)~~ (or $9^\circ, -63^\circ, 81^\circ, -135^\circ, 153^\circ$)

$z = \text{cis}\frac{\pi}{20}, \text{cis}-\frac{7\pi}{20}, \text{cis}\frac{9\pi}{20}, \text{cis}-\frac{3\pi}{4}, \text{cis}\frac{17\pi}{20}$ (1)

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c) See over.

(i) or $\dfrac{2x}{a^2} + \dfrac{2y\cdot y'}{b^2} = 0$

$y' = -\dfrac{b^2x}{a^2y}$

(ii) or $\dfrac{2x}{a^2} - \dfrac{2y\cdot y'}{b^2} = 0$

$y' = \dfrac{b^2x}{a^2y}$

Q4 c) For $T_1 = 1$; $T_2 = 3$; $T_{n+2} = 2T_{n+1} + T_n$ $n \geq 1$

RTP $T_n = \frac{1}{2}(1+\sqrt{2})^n + \frac{1}{2}(1-\sqrt{2})^n$ $n \geq 1$

For $n=1$ $T_1 = \frac{1}{2}(1+\sqrt{2}) + \frac{1}{2}(1-\sqrt{2})$

$= 1$

For $n=2$ $T_2 = \frac{1}{2}(3+2\sqrt{2}) + \frac{1}{2}(3-2\sqrt{2})$

$= 3$

$\therefore$ True for $n=1$ and $n=2$

Assume that

$T_k = \frac{1}{2}(1+\sqrt{2})^k + \frac{1}{2}(1-\sqrt{2})^k$

and $T_{k+1} = \frac{1}{2}(1+\sqrt{2})^{k+1} + \frac{1}{2}(1-\sqrt{2})^{k+1}$

for any integer $k \geq 1$

Now $T_{k+2} = 2T_{k+1} + T_k$

$= (1+\sqrt{2})^{k+1} + (1-\sqrt{2})^{k+1} + \frac{1}{2}(1+\sqrt{2})^k + \frac{1}{2}(1-\sqrt{2})^k$ by assumption

$= \frac{1}{2}(1+\sqrt{2})^k(2+2\sqrt{2}+1) + \frac{1}{2}(1-\sqrt{2})^k(2-2\sqrt{2}+1)$

$= \frac{1}{2}(1+\sqrt{2})^k(1+\sqrt{2})^2 + \frac{1}{2}(1-\sqrt{2})^k(1-\sqrt{2})^2$

$= \frac{1}{2}(1+\sqrt{2})^{k+2} + \frac{1}{2}(1-\sqrt{2})^{k+2}$ as required.

Hence if the proposition is true for any two ~~succe~~ consecutive integers then it is true for the next integer.

Since it is true for $n=1$ and $n=2$, therefore it is true for all $n \geq 1$ by the Principle of Mathematical Induction.

Q5 a) $ax + b + \frac{c}{x+1} = \frac{ax^2 + (a+b)x + b + c}{x+1}$ (3)

$\therefore$ For the required equality $a = \frac{1}{2}$, $b = -1$, $c = 1$

b) (i) $y = \frac{x}{2} - 1$ and $x = -1$ (2)

(ii) $\left(-1, -\frac{3}{2}\right)$ (1)

(iii) $y' = \frac{\cancel{2}(2x-1)(x+1) - \cancel{2}(x^2 - x)}{\cancel{4}2(x+1)^2}$

$= \frac{x^2 + 2x - 1}{2(x+1)^2}$ (1)

(2)

$y' = 0 \Rightarrow x = -1 \pm \sqrt{2}$

For $x = \sqrt{2} - 1$, $y = \sqrt{2} - \frac{3}{2}$ (1)

x	0	$\sqrt{2}-1$	1
y'	$-\frac{1}{2}$	0	$+\frac{1}{4}$

(1)

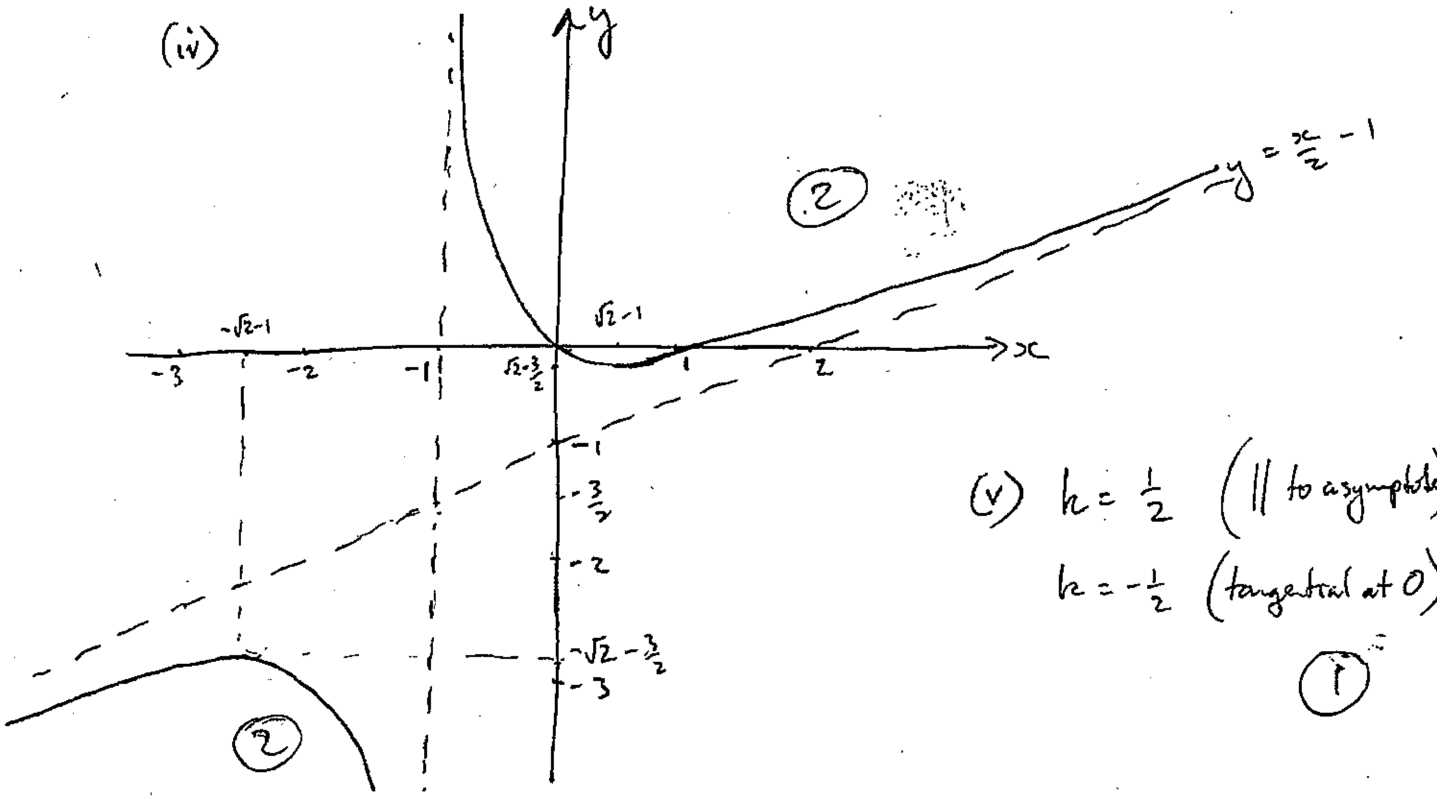
Minimum turning point.

For $x = -\sqrt{2} - 1$, $y = -\sqrt{2} - \frac{3}{2}$

x	-3	$-\sqrt{2}-1$	-2
y'	$+\frac{1}{4}$	0	$-\frac{1}{2}$

Maximum turning point.

(iv)



(v) $k = \frac{1}{2}$ (|| to asymptote)

$k = -\frac{1}{2}$ (tangential at O)

(1)