STUDENT NUMBER

St Aloysius' College
Year 12 Mid-Year Examinations
2012

EXTENSION 2 MATHEMATICS
(Additional paper)

General Instructions

Reading time – 5 minutes
Working time – 2 hours

- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Examination papers must NOT be removed from the examination room.

Total marks - 66

- Attempt all questions

Section I – 6 Marks

- All questions are of equal value
- These are objective response questions.

Section II – 60 Marks

- Questions 7–10 are of equal value.
- Marks for each part are shown in the margin
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Section I **Multiple Choice**

Select the best response from options A, B, C and D. Give your answers on the **multiple choice answer sheet** provided.

Question 1

The eccentricity of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=4$ is:

(A) $\frac{4}{9}$ (B) $\frac{5}{9}$ (C) $\frac{\sqrt{5}}{3}$ (D) $\frac{9}{4}$

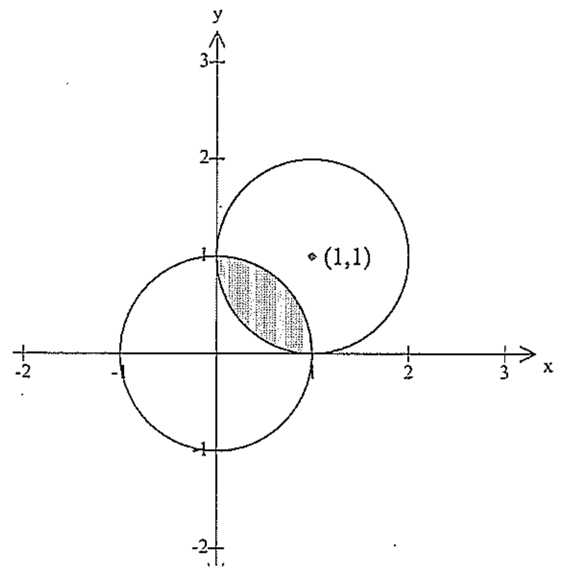
Question 2

Let $z=3-i$. What is the value of $\overline{iz}$?

(A) $-i-3i$ (B) $-1+3i$ (C) $1-3i$ (D) $1+3i$

Question 3

Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z|\le 1$ and $|z-(1-i)|\ge 1$

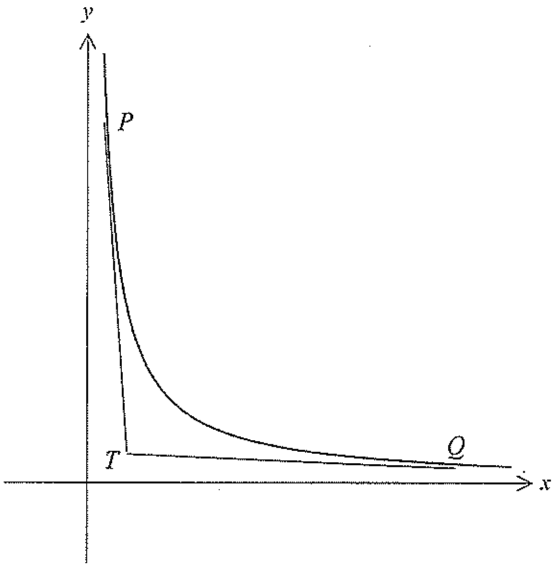
(B) $|z|\le 1$ and $|z-(1+i)|\ge 1$

(C) $|z|\le 1$ and $|z-(1-i)|\le 1$

(D) $|z|\le 1$ and $|z-(1+i)|\le 1$

Question 4

The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, $p \neq q$, lie on the same branch of the hyperbola $xy = c^2$.
The tangents at P and Q meet at the point T.



Which of the following expressions is the equation of the tangent to the hyperbola at Q?

(A) $x + q^2 y = 2cq$

(B) $x + q^2 y = 2c^2$

(C) $x + p^2 y = 2cp$

(D) $x + p^2 y = 2c^2$

Question 5

What are the solutions of the equation $z^2 = i\bar{z}$?

(A) 0 and i

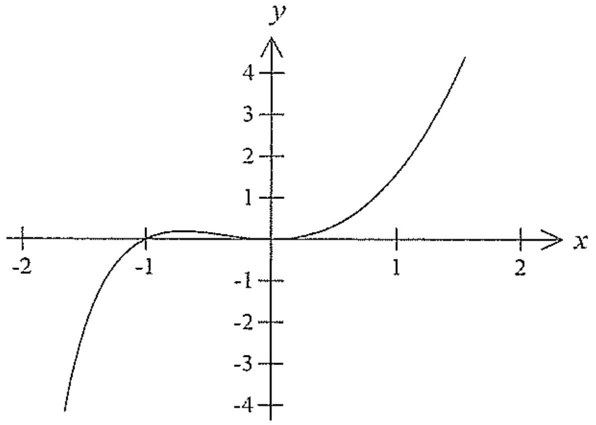
(B) 0 and $-i$

(C) 0, $-i$, $\frac{\sqrt{3}+i}{2}$ and $\frac{-\sqrt{3}+i}{2}$

(D) 0, i, $\frac{\sqrt{3}+i}{2}$ and $\frac{-\sqrt{3}+i}{2}$

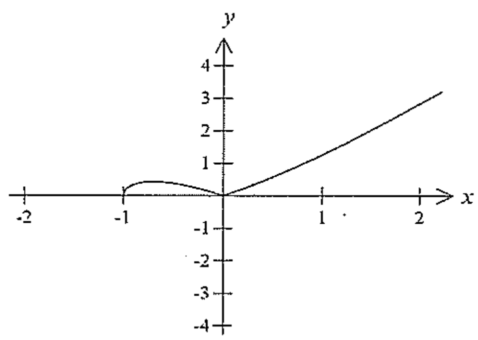
Question 6

The diagram shows the graph of the function $y = f(x)$.

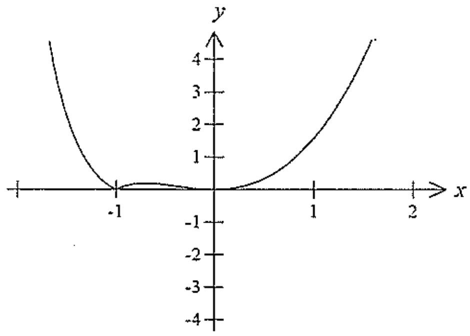


Which of the following is the graph of $y = f(x)^2$?

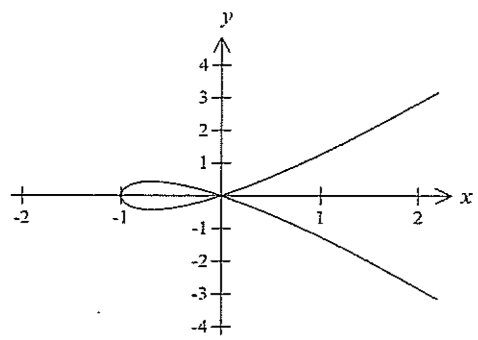
(A)



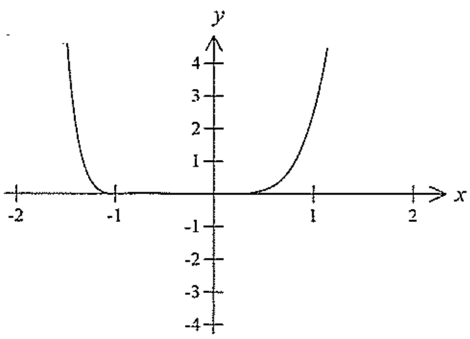
(B)



(C)



(D)



End of Section I

Section II **Begin each question in a new answer booklet.**
All necessary working is required.

Question 7

(a) Calculate $|5-2i|$ **(1)**

(b) For $z=2-i$ and $w=-3+2i$ find $\frac{w}{\bar{z}}$ in the form $a+ib$ **(2)**

(c) (i) Find $-\sqrt{3}+i$ in mod-arg form **(2)**

(ii) Hence, or otherwise, find $\left(-\sqrt{3}+i\right)^4$ in the from $a+ib$ **(2)**

(d) Find the locus of z which satisfies $\left|\frac{z+1}{z-1}\right|=\sqrt{2}$ and sketch the locus on an Argand diagram. **(3)**

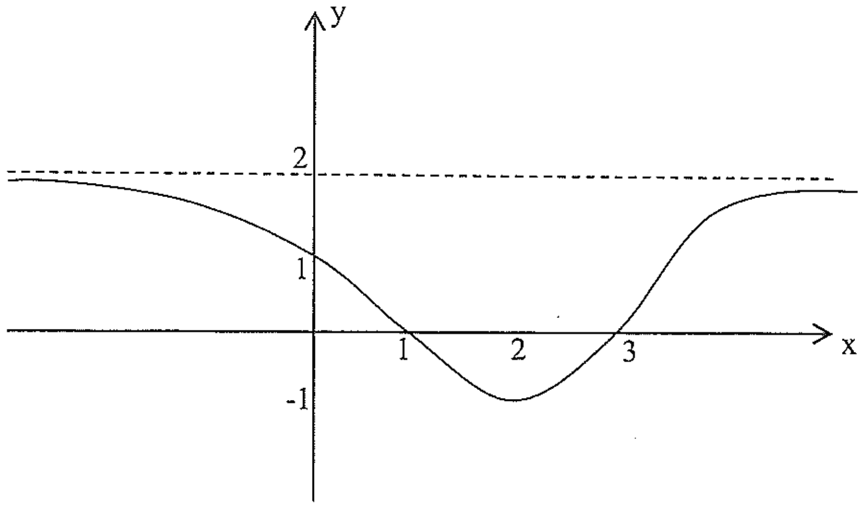
(e) Find the roots of $z^3=\frac{-1+i\sqrt{3}}{2}$ in mod-arg form. **(3)**

(f) If a complex number z has modulus greater than one and a positive acute argument, show on an Argand diagram vectors representing z, $\frac{1}{z}$ and $z-\frac{1}{z}$ **(2)**

Question 8   **Begin a new booklet**

(a) Using the domain $-2 \le x \le 4$ and a diagram at least half a page in height, sketch $y = x(x-2)$ and $y^3 = x(x-2)$ on the same diagram. **(3)**

(b)



Use the above sketch of $y = f(x)$ to draw separate sketches of:

(i) $y = [f(x)]^2$ **(2)**

(ii) $y = \dfrac{1}{f(x)}$ **(2)**

(iii) $y = f(x^2)$ **(2)**

(iv) $y = e^{f(x)}$ **(2)**

(c) For the relation $x^2y + xy^2 = 16$

(i) Differentiate implicitly to find y' in terms of x and y. **(2)**

(ii) Find the coordinates of any stationary point(s) **(2)**

Question 9 **Begin a new booklet**

(a) $P(a\sec\theta, \sqrt{3}a\tan\theta)$ and $Q(a\sec\varphi, \sqrt{3}a\tan\varphi)$ are distinct points on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{3a^2}=1$ such that $PS+QS'=a$ where S and S' are the two foci.

Show that $\sec\theta+\sec\varphi=\frac{1}{2}$ **(4)**

(b) For the ellipse E: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the hyperbola H: $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

(i) State the equations of the asymptotes of H. **(1)**

(ii) Show that the point of intersection, P, of an asymptote and E in the first quadrant is $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ **(2)**

(iii) Show that the tangent at $Q(a\sec\theta, b\tan\theta)$ on H has the equation **(2)**

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$

(iv) The tangent at Q passes through P, find the coordinates of Q in exact simplified form. **(4)**

(c) For the ellipse and the hyperbola in part (b), find a relationship for the eccentricity of H, e_H, in terms of the eccentricity of E, e_E, when $a^2 \ge b^2$. **(2)**

Question 10 **Begin a new booklet**

(a) (i) For the cubic function $y = x^3 - 3px$ where $p > 0$, find the coordinates of the stationary points in terms of p **(2)**

(ii) Deduce that the cubic function $y = x^3 - 3px + q$ where q is real intersects the x axis three times if and only if $q^2 < 4p^3$ **(2)**

(b) Suppose that the cubic equation $x^3 - 3px + q = 0$ has the real root α.

Further, suppose that there is a real number u such that $\alpha = u + \frac{p}{u}$

(i) Show by substitution that $u^3 + \frac{p^3}{u^3} + q = 0$ **(2)**

(ii) Show that u is real if and only if $q^2 \geq 4p^3$ **(2)**

(iii) Using part (a), or otherwise, deduce that if u is real then the cubic equation cannot have three distinct real roots. **(2)**

(c) (i) Make a sketch of $\alpha = u + \frac{p}{u}$ in which the u axis is horizontal and the α axis is the vertical. Find and show the turning points. **(2)**

(ii) Deduce that, if α is the only distinct real root of the cubic equation in part (b) then $|\alpha| \geq 2\sqrt{p}$ **(1)**

(d) Consider again the cubic equation of part (b) with the root $\alpha = u + \frac{p}{u}$ where u is real. Let ω be a complex cube root of unity. **(2)**

Show by substitution that $\beta = u\omega + \frac{p}{u}\overline{\omega}$ is also a root of the cubic equation

$x^3 - 3px + q = 0$

End of Examination

2012 Mid Year Extension 2 Maths — Section II

Multiple Choice					
1	2	3	4	5	6
C	C	D	A	C	D

Q7

(a) $|5-2i| = \sqrt{25+4}$
$= \sqrt{29}$ (1)

(b) $z = 2-i$; $w = -3+2i$ $\therefore \frac{w}{\bar{z}} = \frac{-3+2i}{2+i}$

$= \frac{(-3+2i)(2-i)}{4+1}$ (1)

$= \frac{-6+2+i(3+4)}{5}$

$= \frac{-4}{5} + i.\frac{7}{5}$ (1)

(c) (i) $|-\sqrt{3}+i| = \sqrt{3+1}$
$= 2$

$\arg(-\sqrt{3}+i) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ 2nd Quadrant
$= \frac{5\pi}{6}$ or $150°$

$\therefore -\sqrt{3}+i = 2 \text{ cis } \frac{5\pi}{6}$ or $2 \text{ cis } 150°$ (2)

(ii) $(-\sqrt{3}+i)^4 = 2^4 \text{ cis } \frac{10\pi}{3}$ (1)

$= 16 \text{ cis}\left(-\frac{2\pi}{3}\right)$ [de Moivre]

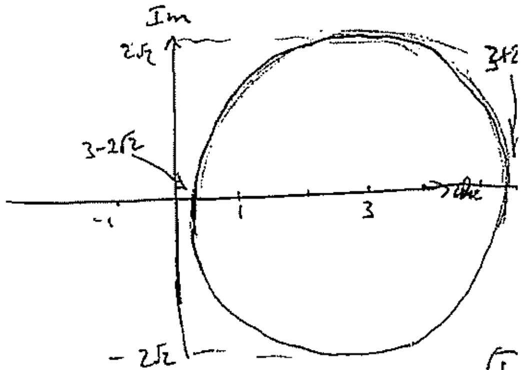
$= 16\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$ (1)

$= -8 - 8i\sqrt{3}$

(d) $(x+1)^2 + y^2 = 2\left[(x-1)^2 + y^2\right]$ (1)

$0 = x^2 - 6x + 1 + y^2$

$8 = (x-3)^2 + y^2$ (1)



(1)

(e) $\left|\frac{-1+i\sqrt{3}}{2}\right| = 1$ $\therefore$ Let $z = \text{cis}\,\theta$

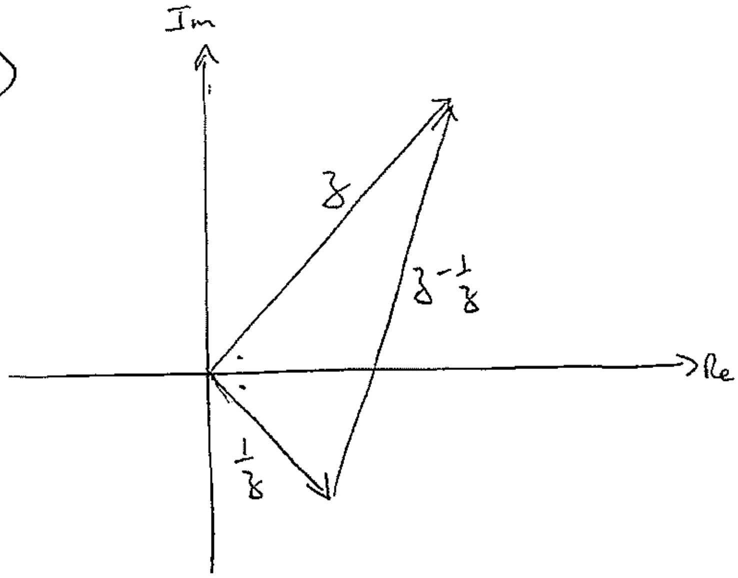
$\therefore \text{cis}\,3\theta = \frac{-1+i\sqrt{3}}{2}$ [de Moivre] (1)

$\cos 3\theta = -\frac{1}{2}$ and $\sin 3\theta = \frac{\sqrt{3}}{2}$ $\therefore 3\theta = \frac{2\pi}{3}, -\frac{4\pi}{3}, \frac{8\pi}{3}$ (1)

$\theta = \frac{2\pi}{9}, -\frac{4\pi}{9}, \frac{8\pi}{9}$

$\therefore z = \text{cis}\,\frac{2\pi}{9}, \text{cis}\,-\frac{4\pi}{9}, \text{cis}\,\frac{8\pi}{9}$ [or cis 40°, cis −80°, cis 160°] (1)

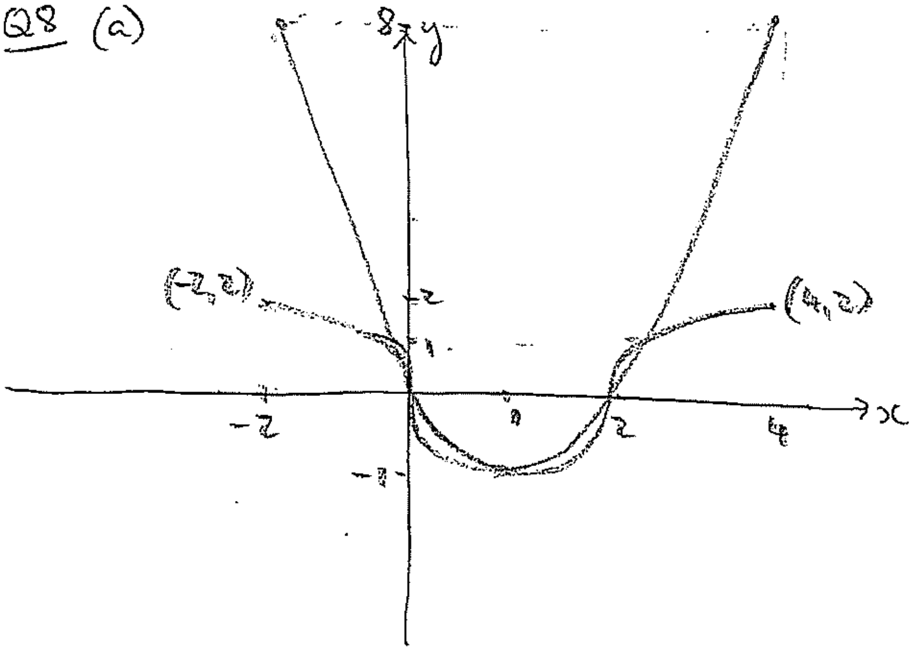
(f)



(1)

(1)

Q8 (a)



①

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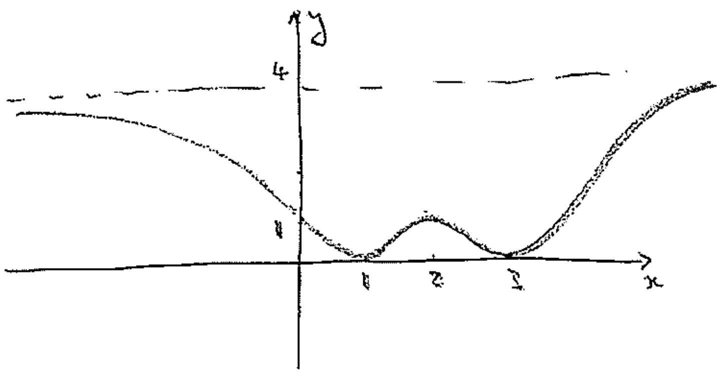
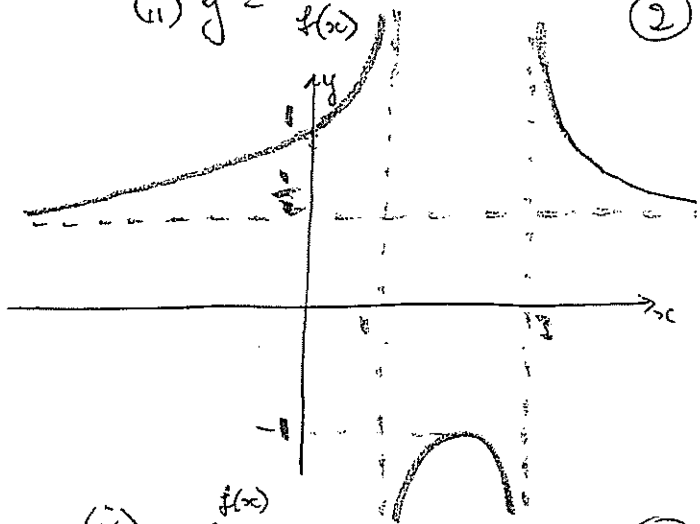
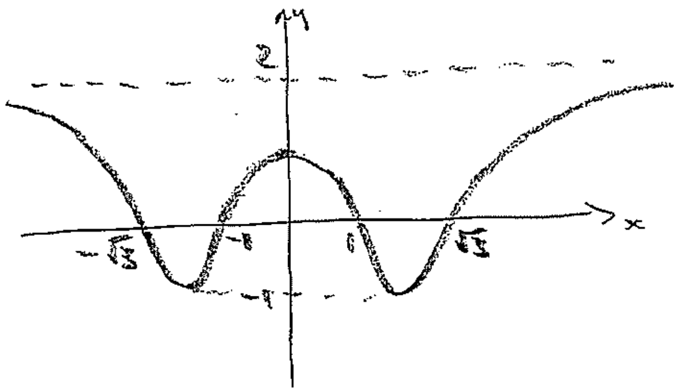
(b) (i) $y = [f(x)]^2$ ②



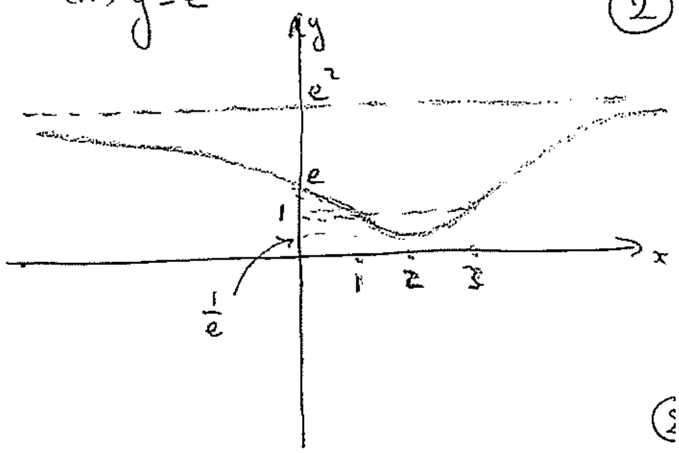
(ii) $y = \frac{1}{f(x)}$ ②



(iii) $y = f(x^2)$ ②



(iv) $y = e^{f(x)}$ ②



(c)(i) $x^2y + xy^2 = 16 \Rightarrow 2xy + x^2y' + y^2 + 2xyy' = 0 \Rightarrow y' = \frac{-y(2x+y)}{x(x+2y)}$

(ii) $y' = 0 \Rightarrow y = -2x$ only as $y = 0$ is not in the ~~domain~~ range.

$\therefore$ Using $x^2y + xy^2 = 16 \Rightarrow -2x^3 + 4x^3 = 16 \Rightarrow x^3 = 8$ ②

$\therefore$ The only stationary point is $(2, -4)$

Q9 (a) $b^2 = a^2(e^2-1) \Rightarrow 3a^2 = a^2(e^2-1) \Rightarrow e = 2$ 1

$PS + QS' = a$ where $S(2a, 0)$ and $S'(-2a, 0)$

$\therefore 2PN + 2QN' = a$ using $\frac{PS}{PN} = e$; $\frac{QS'}{QN'} = e$ 1

$\therefore \left(a\sec\theta - \frac{a}{e}\right) + \left(a\sec\phi + \frac{a}{e}\right) = \frac{a}{2}$ 1

1

$\therefore \sec\theta + \sec\phi = \frac{1}{2}$

(b) (i) $y = \pm\frac{bx}{a}$ 1

(ii) For P, $\frac{x^2}{a^2} + \frac{b^2x^2}{a^2b^2} = 1 \Rightarrow x = \pm\frac{a}{\sqrt{2}}$ 1

$\therefore y = \pm\frac{b}{\sqrt{2}}$

P is in the 1st Quadrant $\therefore P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ 1

(iii) $y' = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

$= \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ Tangent at Q

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$ 1

$\sec^2\theta - \tan^2\theta = \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b}$ 1

$\therefore 1 = \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b}$

(iv) Substituting P $\therefore 1 = \frac{\sec\theta}{\sqrt{2}} - \frac{\tan\theta}{\sqrt{2}}$ 1

$\therefore \sqrt{2} + \tan\theta = \sec\theta$

$2 + 2\sqrt{2}\tan\theta + \tan^2\theta = \tan^2\theta + 1$ 1

$\therefore \tan\theta = \frac{-1}{2\sqrt{2}}$

$\therefore \sec\theta = \sqrt{2} - \frac{1}{2\sqrt{2}}$ 1

$= \frac{3}{2\sqrt{2}}$

$\therefore Q\left(\frac{3a}{2\sqrt{2}}, \frac{-b}{2\sqrt{2}}\right)$ 1

(c) $b^2 = a^2(1 - e_E^2)$ and $b^2 = a^2(e_H^2 - 1)$ 1

$\therefore e_H^2 - 1 = 1 - e_E^2$

$e_H = \sqrt{2 - e_E^2}$ 1

Q10 (a) (i) $y = x^3 - 3px \quad ; \; p > 0$

$y' = 3x^2 - 3p \quad \therefore y' = 0 \Rightarrow x = \pm\sqrt{p}$ 1

$\therefore (\sqrt{p}, -2p\sqrt{p})$ and $(-\sqrt{p}, 2p\sqrt{p})$ are stationary points 1

(ii) $y = x^3 - 3px + q$ is the function $y = x^3 - 3px$ shifted vertically by the distance $|q|$. The function will continue to have 3 distinct x intercepts if and only if that shift is less than the height of the stationary points i.e. $-2p\sqrt{p} < q < 2p\sqrt{p} \Rightarrow q^2 < 4p^3$ 1 1

~~(ii)~~ OR The stationary points of $y = x^3 - 3px + q$ are at $(\sqrt{p}, -2p\sqrt{p} + q)$ and $(-\sqrt{p}, 2p\sqrt{p} + q)$

There will be 3 distinct x intercepts iff these points are on opposite sides of the x axis –

$\Rightarrow q < 2p\sqrt{p}$ if $q > 0$ or $q > -2p\sqrt{p}$ if $q < 0$ –

$\therefore q^2 < 4p^3$ for all q.

(b)(i) Since α is a root $\alpha^3 - 3p\alpha + q = 0$

If $\alpha = u + \frac{p}{u}$ then $\left(u + \frac{p}{u}\right)^3 - 3p\left(u + \frac{p}{u}\right) + q = 0$ 1

$\therefore u^3 + 3pu + \frac{3p^2}{u} + \frac{p^3}{u^3} - 3pu - \frac{3p^2}{u} + q = 0$ 1

$\therefore u^3 + \frac{p^3}{u^3} + q = 0$ ①

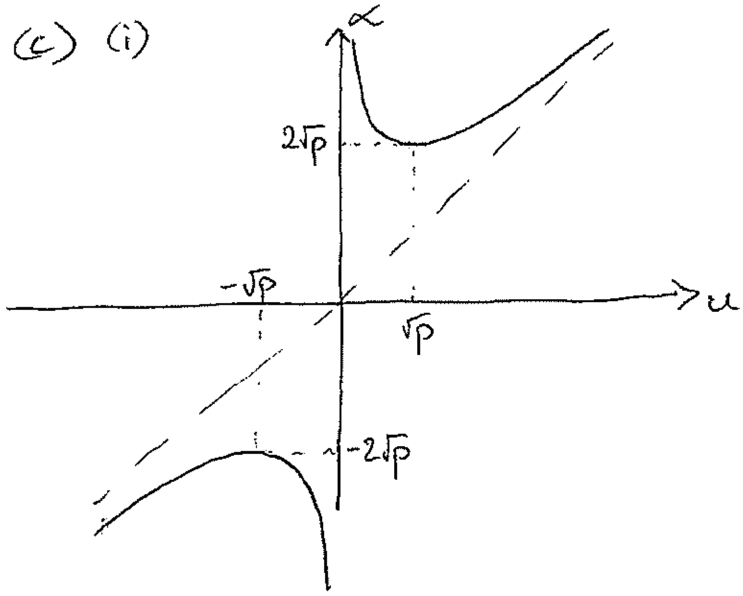
(ii) ① $\Rightarrow u^6 + qu^3 + p^3 = 0 \Rightarrow \Delta = q^2 - 4p^3$ 1

u^3, and hence u, is real iff $\Delta \geq 0$ 1

$\therefore q^2 \geq 4p^3$

(iii) This set of values for q for which u is real is the exact opposite of the set of values in part (a) (ii) for which there are 3 distinct real ~~roots~~ x intercepts, and hence roots. Hence u being real and there being 3 distinct real roots are mutually exclusive. 1 1

(c) (i)



$\alpha = u + \frac{p}{u}$

$\frac{d\alpha}{du} = 0 \Rightarrow u^2 = p$

$u = \pm\sqrt{p}$ 2

(ii) Since all cubic functions have at least one real root, if u is real then $\alpha = u + \frac{p}{u}$ is the only real root. The above graph shows the values of α for all real $u \neq 0$. 1

$\therefore \quad \alpha \geqslant 2\sqrt{p} \quad$ or $\quad \alpha \leqslant -2\sqrt{p}$

$\Rightarrow \quad |\alpha| \geqslant 2\sqrt{p}$

(d) Substituting $x = \beta$ into $x^3 - 3px + q$

$$\beta^3 - 3p\beta + q = \left(u\omega + \frac{p}{u}\bar{\omega}\right)^3 - 3p\left(u\omega + \frac{p}{u}\bar{\omega}\right) + q$$

$$= u^3\omega^3 + 3pu\,\omega^2\bar{\omega} + \frac{3p^2}{u}\omega\bar{\omega}^2 + \frac{p^3}{u^3}\bar{\omega}^3$$

$$- 3pu\omega - \frac{3p^2}{u}\bar{\omega} + q$$

Now $\omega^3 = \bar{\omega}^3 = 1$; $\omega\bar{\omega} = 1 \therefore \omega^2\bar{\omega} = \omega$ and $\omega\bar{\omega}^2 = \bar{\omega}$ 1

$$\therefore \beta^3 - 3p\beta + q = u^3 + 3pu\omega + \frac{3p^2}{u}\bar{\omega} + \frac{p^3}{u^3} - 3pu\omega - \frac{3p^2}{u}\bar{\omega} + q$$

$$= u^3 + \frac{p^3}{u^3} + q$$ 1

$$= 0 \quad \text{from part (b)(i)}$$

$\therefore \beta$ is also a root of the equation.