SYDNEY BOYS HIGH SCHOOL

4 UNIT MATHEMATICS

1997

YEAR 12 ASSESSMENT

[redacted]

TERM 2

TIME ALLOWED: 90 minutes

TOTAL MARKS: 95

EXAMINER: C. KOURTESIS

INSTRUCTIONS:

* Attempt all questions.
* Marks allocated to each question are indicated on the exam paper.
* All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
* Each question is to be returned in a separate Writing Booklet clearly marked Question 1, Question 2, etc on the cover. Each booklet must show your name.

Question 1 [17 marks]

[4] [a] If w is a complex number defined by $w = \frac{1 + 2i}{3 + 4i}$

determine:

[i] $|w|$ [ii] $\overline{w}$ [iii] Re (w)

[3] [b] If z represents the complex number $1 + 2i$ indicate on the Argand Diagram the points:

$$z,\ z^2,\ \frac{1}{z}$$

[3] [c] Find all pairs of integers x and y such that $(x + iy)^2 = -3 - 4i$.

[4] [d] On separate Argand Diagrams indicate the locus of points which satisfy:

(i) $|z| = 2$ (ii) $\text{Re}(z) > 2$ (iii) $\arg\left(\frac{z - 1}{z - i}\right) = \frac{\Pi}{4}$

[3] [e] Find the equation of the locus of the point P representing $z = x + iy$ where

$$|z^2 - (\overline{z})^2| = 4$$

Question 2 [18 marks]

[a] If α, β, γ are the roots of
$2x^3 - 4x^2 - 6x + 5 = 0$
find the values of:

[2] [i] $(\alpha - 1)(\beta - 1)(\gamma - 1)$

[2] [ii] $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

[2] [iii] $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

[2] [b] [i] Show that $1 + i$ is a root of

$$P(x) = 3x^3 - 7x^2 + 8x - 2 = 0.$$

[2] [ii] Find all roots of $P(x) = 0$

[3] [c] The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ.
Find an equation with numerical coefficients having roots $-\alpha, -\beta, -\gamma$.

[2] [d] [i] If α is a double zero of the polynomial $P(x)$, show that α is a zero of $P^1(x)$.

[3] [ii] Given that $(x - 2)^2$ is factor of

$x^5 + 2x^4 + ax^3 + bx^2$

find the values of a and b.

Question 3 [16 marks]

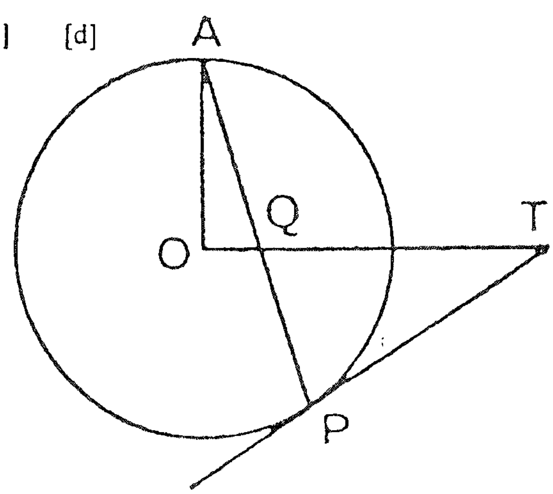
[4] [a] Prove by mathematical induction for integers $n \geq 1$ that $3^{4n} - 1$ is divisible by 80.

[4] [b] On an Argand Diagram shade in the region containing all points representing complex numbers z such that

$$0 \leq \arg(z+1) \leq \frac{\Pi}{4} \quad \text{and} \quad |z+i| < 2$$

[4] [c] In an Argand Diagram OABC is a rhombus where O is the origin and A is the point (1,2). If $B\hat{O}C = 30^0$ and B is in the second quadrant find the complex numbers representing the points B and C.

[4] [d]



In the diagram O is the centre of the circle, $A\hat{O}T$ is a right angle and TP is a tangent.

Prove TP = TQ. (Join O to P).

<u>Question 4</u> [15 marks]

[2] [a] [i] Express $\sqrt{3} + i$ and $\sqrt{3} - i$ in modulus - argument form.

[4] [ii] Hence find the exact values of

(α) $(\sqrt{3} + i)^{10} - (\sqrt{3} - i)^{10}$

(β) $\dfrac{(\sqrt{3} + i)^{10}}{(\sqrt{3} - i)^{10}}$

[b] Consider the equation $x^4 + x^2 + 1 = 0$ which is known to have four complex roots.

[2] [i] If w is a root of this equation, show that $w^6 = 1$.

[2] [ii] Deduce that w^2 is also a root of $x^4 + x^2 + 1 = 0$

[2] [iii] Write down the other complex roots of

$x^4 + x^2 + 1 = 0$ in terms of w.

[3] [c] Show that $x^3 + ax + b = o$ has only one real root if $a > o$.

Question 5 [14 marks]

[4] [a] Prove by mathematical induction that for all positive integers $n \geq 1$

$$3 \times 1! + 7 \times 2! + 13 \times 3! + \dots\dots\dots + (n^2 + n + 1)n! = (n + 1)^2 n! - 1$$

[2] [b] [i] Solve the equation $x^3 - 3x^2 - x + 3 = 0$.

[4] [ii] Determine the range of values of m for which the equation

$$x^4 - 4x^3 - 2x^2 + 12x + m = 0$$

has four distinct real roots.

[4] [c] Find the complex number with least positive argument satisfying the condition

$$|z - 5i| \leq 3$$