SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2002
HIGHER SCHOOL CERTIFICATE
APRIL EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time — 5 minutes
- Working time — 90 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks — 60
- Attempt questions 1–3
- All questions are of equal value

Examiner: D.M.Hespe

Note: This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 60
Attempt Questions 1–3
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (20 marks) Use a SEPARATE writing booklet.

(a) (i) Calculate $|15-20i|$, **1**

(ii) Find $arg\left(-1-\frac{3}{\sqrt{2}}i\right)$. **1**

(b) Solve $x^2 = 4x-20$ over the complex field. **2**

(c) The polynomial $z^3-7z^2+25z-39$ has one zero equal to $2+3i$. Write down its three linear factors. **3**

(d) (i) Differentiate $x\ln x$. **1**

(ii) Hence show that a primitive of $\ln x$ is $x\ln x - x$. **2**

(e) Give the domain and range of $3\cos^{-1}3x$. **2**

(f) Find the square roots of $5+12i$. **2**

(g) Show that $z-i$ is a factor of z^3+2iz^2+3i. **1**

(h) Prove that the complex numbers $1+6i$, $3+10i$, and $4+12i$ are collinear on an Argand diagram. **3**

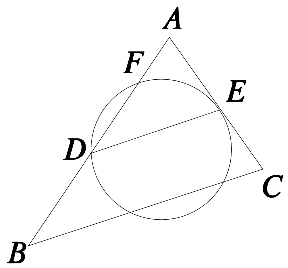
(i) The curves $y=\ln x$ and $y=x-1{\cdot}1$ intersect near $x=1{\cdot}5$. Use one iteration of Newton's method to get a better estimate. **2**

Marks

Question 2 (20 marks) Use a SEPARATE writing booklet.

(a) Prove that $\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$. **3**

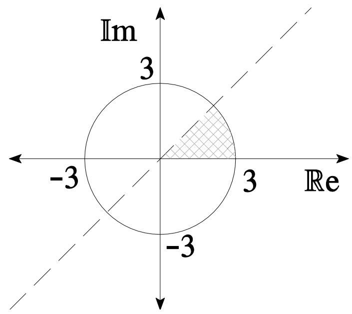
(b)



A line DE parallel to the base BC of a triangle ABC cuts AB, AC in D and E respectively. The circle which passes through D, and touches AC at E, meets AB at F. Prove that F, E, C, B lie on a circle. **3**

(c) (i) Sketch the locus of z on an Argand diagram such that $|z| = |z+1-\sqrt{3}i|$. **2**

(ii) Describe the locus of z shaded on the diagram. **3**



(d) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. **3**

(ii) Use the result to solve the equation $8x^3 - 6x + 1 = 0$. **2**

(iii) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$. **2**

(e) $\int_0^2 \frac{dx}{\sqrt{2x-x^2}}$. [Hint: try completing the square.] **2**

Marks

Question 3 (20 marks) Use a SEPARATE writing booklet.

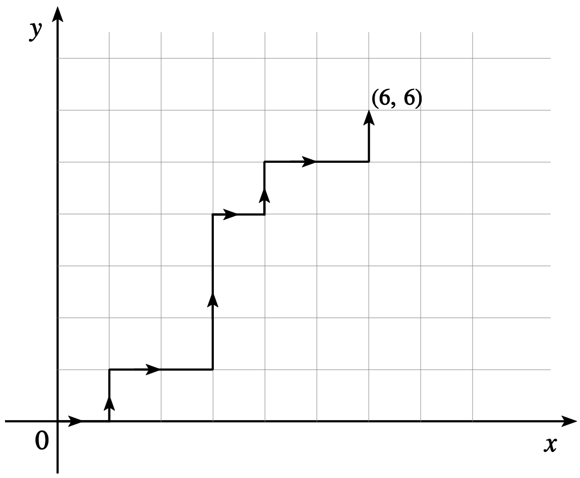
(a) (i) Show that $\dfrac{1}{1+u^2} = 1 - u^2 + u^4 - u^6 + u^8 - \ldots$ **2**

(ii) By integrating both sides of the above expression from 0 to x, show by a suitable choice of x, that $\dfrac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ **3**

(b) (i) If $z = x + iy$ and $z' = 1 + \dfrac{1}{z}$, obtain an expression for z' in the form $x' + iy'$, and hence express each of x' and y' in terms of x and y. **3**

(ii) Find an algebraic relation between x' and y' when z has constant argument θ. **3**

(c)



The lines in the above figure represent the streets in a city. A messenger wants to get from (0, 0) to (x, y), where x, y are non-negative integers, moving in the positive directions of x and y only.

Let the number of possible routes be denoted by $f(x,y)$.

(i) Evaluate $f(k,1)$. **2**

(ii) Show that $f(k+1,2) = f(k,2) + f(k+1,1)$. **3**

(iii) By means of mathematical induction or otherwise, prove that $f(m,2) = \frac{1}{2}(m^2 + 3m + 2)$ for all non-negative integral values of m. **4**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a1>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

QUESTION 1. 2002 Ext 2 Solutions Task #1

(a) (i) $|15-20i| = \sqrt{15^2+20^2}$
$= \sqrt{625}$
$= 25$

(ii) $\arg\left(-1-\frac{3}{\sqrt{2}}i\right) = -(\pi - 1.13029)$
$\doteqdot -2.0113$
or $(-115°14')$

(b) $x^2-4x = -20$
$(x-2)^2 = -16$
$x-2 = \pm 4i$
$x = 2\pm 4i$

(c) $P(z) = z^3-7z^2+25z-39$.

now one zero is $2+3i$

∴ another zero is $2-3i$ (by conjugate root theorem)

now if the other root is β.

$2+3i+2-3i+\beta = 7 \quad \left(\sum \alpha = -\frac{b}{a}\right)$
$\therefore \beta = 3$

$\therefore P(z) = (z-2+3i)(z-2-3i)(z-3)$

(d) $y = x\ln x$
$y' = x\times\frac{1}{x}+1.\ln x$
$= 1+\ln x$

$\therefore \int(1+\ln x)\,dx = x\ln x$

$\int \ln x\,dx = x\ln x - \int 1.\,dx$

$\therefore \int \ln x\,dx = x\ln x - x + c$.

(e) $D: |x| \le \frac{1}{3}$

$R: 0 \le y \le 3\pi$.

(f) $\sqrt{5+12i} = a+ib$
$5+12i = a^2-b^2+2abi$

$\therefore a^2-b^2 = 5$ — ①
$2ab = 12$ — ②

now $(a^2+b^2)^2 = (a^2-b^2)^2+(2ab)^2$
$= 25+144$
$= 169$

$\therefore a^2+b^2 = 13$ ③

① + ③

$2a^2 = 18$
$a^2 = 9$
$a = \pm 3$

$\therefore b = \pm 2$

Hence roots are $\pm(3+2i)$

(g) $P(i) = i^5 + 2i^3 + 3i$

$= -i - 2i + 3i$

$= 0$

$\therefore (z - i)$ is a factor.

(h) Let $z_1 = 1 + 6i$

$z_2 = 3 + 10i$

$z_3 = 4 + 12i$

now $z_2 - z_1 = 2 + 4i$

$\therefore \arg(z_2 - z_1) = \tan^{-1} 2.$

Also $\arg(z_3 - z_2) = \arg(1 + 2i)$

$= \tan^{-1} 2.$

$\therefore$ vectors $z_2 - z_1$ and $z_3 - z_2$

are parallel.

& z_2 is on both

$\therefore$ collinear!

(i) given $\ln x = x - 1.1$.

Consider $f(x) = \ln x - x + 1.1$.

now $f'(x) = \frac{1}{x} - 1$

If $x_1 = 1.5$

$x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$

$\doteqdot 1.516355$

$\doteqdot \underline{1.52}$ (3.sig.figs)

2

(a) If $u = \cot^{-1} x$

then $x = \cot u$

$\frac{dx}{du} = -\frac{1}{\sin^2 u}$

(3)

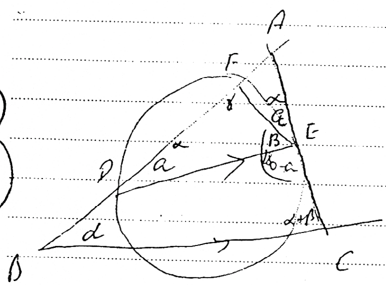
$\frac{dx}{du} = -\operatorname{cosec}^2 u$

$\frac{d}{dx} \cot^{-1} x = \frac{-1}{\operatorname{cosec}^2 u}$

$= \frac{-1}{1+x^2}$

(b)

(3)



$\hat{AEF} = \hat{ADE} = \hat{B}$

$= a^\circ$

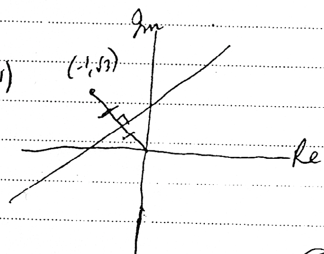
$\hat{FEC} = 180 - a^\circ$

opposite ∠'s supplementary

Cyclic.

(c) (i)

(2)



(3)

(ii) $0 \le \arg z < \pi/4$

$|z| \le 3.$

(d) (i) $\operatorname{cis} 3\theta = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

(3)

$\cos 3\theta = \cos^3\theta - 3\cos\theta\, i \sin^2\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

(d)

(ii) $8x^3 - 6x = -1$

$4x^3 - 3x = -1/2$

$\cos 3\theta = -1/2$

(2)

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\theta = 2\pi/9, 4\pi/9, 8\pi/9$

(2)

(iii) Sum of roots $= -b/a = 0$

$\cos 2\pi/9 + \cos 4\pi/9 + \cos 8\pi/9 = 0$

$\cos 2\pi/9 + \cos 4\pi/9 = -\cos 8\pi/9$

$\cos 2\pi/9 + \cos 4\pi/9 = \cos \pi/9$

(e) $\int_0^2 \frac{dx}{\sqrt{2x - x^2}}$

$\int_0^2 \frac{dx}{\sqrt{1 - (x^2 - 2x + 1)}}$

$\int_0^2 \frac{dx}{\sqrt{1 - (x-1)^2}}$

$= \left[\sin^{-1}(x-1)\right]_0^2$

$= \sin^{-1} 1 - \sin^{-1} -1$

$= \pi/2 + \pi/2$

$= \pi.$

Question 3

(a) (i) NTP $\frac{1}{1+u^2}=1-u^2+u^4-\cdots$ for $|u|<1$ 2

RHS = $1-u^2+u^4-\cdots$ an infinite geometric series with $a=1$, $r=-u^2$

$\therefore S_\infty$ exists since $|u|<1 \Rightarrow u^2<1 \therefore |r|<1$

$S_\infty=\frac{a}{1-r}=\frac{1}{1-(-u^2)}=\frac{1}{1+u^2}$ QED

(ii) $\int_0^x \frac{1}{1+u^2}\,du=\int_0^x \left(1-u^2+u^4-\cdots\right)du$ 3

$\therefore \left.\tan^{-1} u\right]_0^x=\left.u-\frac{u^3}{3}+\frac{u^5}{5}-\cdots\right]_0^x$

$\therefore \tan^{-1} x=x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots$

Let $x=1$

$\therefore \tan^{-1}1=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots$ QED

(b) (i) $z=x+iy, z'=x'+iy'=1+\frac{1}{z}$ 3

$\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{x-iy}{x^2+y^2}$

$1+\frac{1}{z}=1+\frac{x-iy}{x^2+y^2}=\left(1+\frac{x}{x^2+y^2}\right)-i\left(\frac{y}{x^2+y^2}\right)$

$\therefore x'=1+\frac{x}{x^2+y^2}, y'=-\frac{y}{x^2+y^2}$

(ii) $\arg z=\theta=\text{constant}$ 3

$\tan\theta=\frac{y}{x}$

$x'-1=\frac{x}{x^2+y^2}$

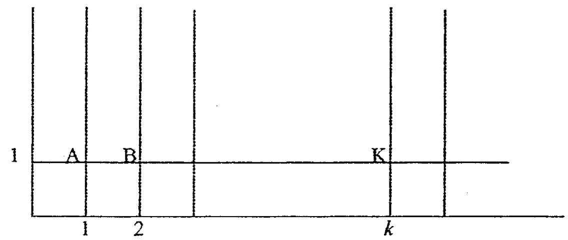
$-y'=\frac{y}{x^2+y^2}$

$\therefore \frac{-y'}{x'-1}=\frac{\frac{y}{x^2+y^2}}{\frac{x}{x^2+y^2}}=\frac{y}{x}=\tan\theta$

$\therefore \tan\theta=\frac{y'}{1-x'}$

(c) (i) 2

Method 1: *Addition Principle*



A corresponds to (1,1)
B corresponds to (2,1)
K corresponds to $(k,1)$

There is 1 way to get to (0,1) and 1 way to get to (1,0). So there is 1 + 1 = 2 ways to get to A.
There is 1 way to get to (2,0) and 2 ways to get to (1,1). So there is 1 + 2 = 3 ways to get to B.

So there are $k + 1$ ways to get to K.

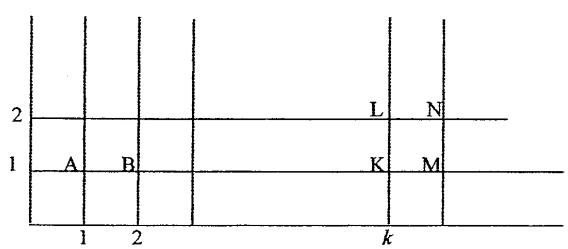
Method 2:

Use R to denote every time one move to the **RIGHT** is taken, and U to denote one move **UP**. To get to K, there must be k R's and only 1 U. So that 1 possible route is represented by the word $\underbrace{RRR\cdots RRR}_{k}U$. So all possible words would give all possible routes.

This means the number of permutations of $k + 1$ letters, with k letters exactly the same

ie $\dfrac{(k+1)!}{k!} = k+1 = \dbinom{k+1}{1}$

(ii) **Method 1:** *Addition Principle*



The number of ways to get to N is equal to the number of ways of getting to L + the number of ways of getting to M. 3

$f(k,2)$ = number ways to L.
$f(k+1,1)$ = number ways to M.
$f(k+2,2)$ = number ways to N.

ie $f(k,2) + f(k+1,1) = f(k+2,2)$
QED

(c) (ii) **Method 2**

$$f(k,2)=\binom{k+2}{2}, f(k+1,1)=\binom{k+2}{1}, f(k+1,2)=\binom{k+3}{2}$$

$$\therefore \binom{k+2}{2}+\binom{k+2}{1}=\binom{k+3}{2}$$

because of Pascal's relationship ie $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$

(c) (iii) **Note that $m \geq 0$**

$$f(m,2)=\binom{m+2}{2}=\frac{(m+2)!}{m!2!}=\frac{(m+2)(m+1)}{2}=\tfrac{1}{2}\left(m^2+3m+2\right)$$ 4

OR with mathematical induction

When $m=0$, LHS$=f(0,2)=1$
RHS$=\tfrac{1}{2}\left(0^2+3\times 0+2\right)=1$
So true for $m=0$

Assume true for $m=k$ ie $f(k,2)=\tfrac{1}{2}\left(k^2+3k+2\right)$
NTP that $f(k+1,2)=\tfrac{1}{2}\left((k+1)^2+3(k+1)+2\right)=\tfrac{1}{2}\left(k^2+5k+6\right)$
LHS$=f(k+1,2)$

$=f(k,2)+f(k+1,1)$ from (ii)
$=\tfrac{1}{2}\left(k^2+3k+2\right)+(k+2)$ from (i)
$=\tfrac{1}{2}\left(k^2+3k+2\right)+\tfrac{1}{2}(2k+4)$
$=\tfrac{1}{2}\left(k^2+5k+6\right)$
$=$ RHS

So by the principle of mathematical induction, $f(m,2)=\tfrac{1}{2}\left(m^2+3m+2\right), m \geq 0$