SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2004

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK # 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Hand in your answer booklets in **3** sections.
  Section A (Questions 1 - 5), Section B (Questions 6 - 9) and Section C (Questions 10 - 13).
- Start each **NEW** section in a separate answer booklet.

Total Marks - 90 Marks

- Attempt Sections A - C
- All questions are NOT of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 90
Attempt Questions 1 – 13
All questions are NOT of equal value

Answer each section in a SEPARATE writing booklet. Extra writing booklets are available.

SECTION A (Use a SEPARATE writing booklet)

Question 1 (8 marks) **Marks**

(a) Evaluate $(3+4i)\div(1+i)$ **2**

(b) If $(x+iy)(2+3i)=5+6i$ find x and y **2**

(c)
(i) Express $\dfrac{-1-i\sqrt{3}}{1-i}$ in modulus-argument form **2**

(ii) Hence evaluate $\left(\dfrac{-1-i\sqrt{3}}{1-i}\right)^6$ **2**

Question 2 (8 marks)

If P represents the complex number z, sketch the locus of P (on *separate* diagrams) if:

(i) $|z-1|=4$ **2**

(ii) $-1\le \text{Im}(z)\le 2$ **2**

(iii) $-\dfrac{\pi}{4}\le \arg(z)\le \dfrac{2\pi}{3}$ **2**

(iv) $\arg\left(\dfrac{z-i}{z-1}\right)=\dfrac{\pi}{6}$ **2**

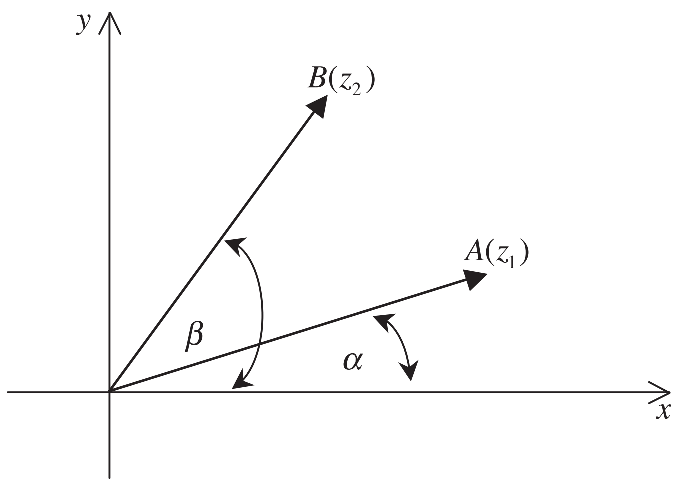
Question 3 (6 marks)

(a) If z is a non zero complex number such that $z+1/z$ is real, prove that $\text{Im}(z)=0$ or $|z|=1$ **3**

(b) Find the square roots of $-2-2i$. **3**

Leave your answer in modulus-argument form.

Question 4 (4 marks) **Marks**



In the diagram $\arg(z_1) = \alpha$ and $\arg(z_2) = \beta$. **4**
If $|z_1| = |z_2|$ prove that $\arg(z_1 z_2) = \arg\left((z_1 + z_2)^2\right)$

Question 5 (4 marks)

The point A in an Argand diagram represents the complex number $3 + 4i$. **4**

Find the complex number represented by B if ΔOAB is an equilateral triangle with B in the fourth quadrant.

O represents the complex number 0.

Leave your answer in the form $a + ib$.

SECTION B (Use a SEPARATE writing booklet)

Question 6 (7 marks) **Marks**

Given $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5$

(i) Find the zeros of $P(x)$ given that $1 + 2i$ is a zero. **3**

(ii) Express $P(x)$ in factored form:

(α) over the complex field; **2**

(β) over the real field. **2**

Question 7 (9 marks)

If α, β and γ are the roots of the equation $x^3 - 2x + 5 = 0$, find the equation which has roots:

(i) $2/\alpha,\ 2/\beta,\ 2/\gamma$; **3**

(ii) $\alpha^2, \beta^2, \gamma^2$; **3**

(iii) $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ **3**

Question 8 (6 marks) **Marks**

(a) Show that $f(x) = x^n - 1$ has no multiple roots, where n is an integer with $n > 1$. **3**

(b) If the roots of $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ show that **3**

$$(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\cdots(1-\alpha_{n-1}) = n$$

Question 9 (9 marks)

Consider $\omega = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

(i) Prove that $\omega^5 = 1$ and $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ **3**

(ii) Prove that $z = \omega + 1/\omega$ is a root of $z^2 + z - 1 = 0$ **3**

(iii) Hence prove that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ **3**

SECTION C (Use a SEPARATE writing booklet)

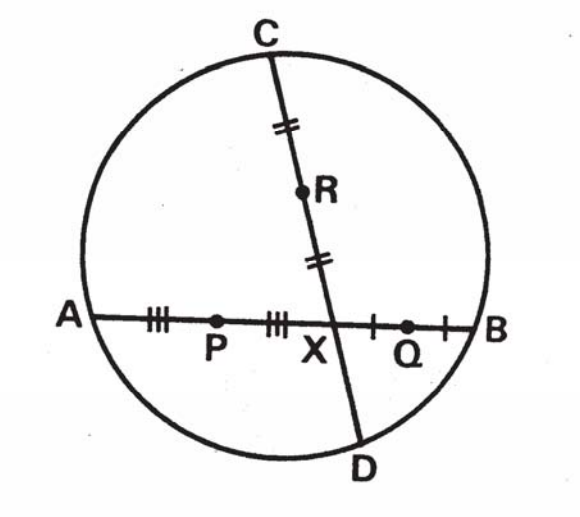
Question 10 (6 marks) **Marks**

Given that $y = x^3 - 3px + q$ where $p, q \in \mathbb{R}$

(i) Find the coordinates of the stationary points (in terms of p and q) of $y = f(x)$. **3**

(ii) Hence, find the relationship between p and q for $f(x) = x^3 - 3px + q$ to have 3 distinct real roots. **3**

Question 11 (6 marks)



6

AB and *CD* are two chords of a circle intersecting at a point *X*.
P, *Q* and *R* are the midpoints of *AX*, *XB* and *CX* respectively.

Prove that the circle *PQR* also bisects *DX*.

Question 12 (11 marks)

A and *B* are two points in an Argand diagram presenting the complex numbers $z_1 = -1$ and $z_2 = \cos\theta + i\sin\theta$ respectively, where $\frac{\pi}{2} < \theta < \pi$.

C is the point representing the complex number $z_3 = z_1 + z_2$.

(i) Sketch the quadrilateral *OACB* in an Argand diagram, where *O* is the point representing the complex number 0. **3**

Mark an angle in the diagram which is equal to θ .

(ii) Let $z_4 = z_2 - z_1$

(α) Show that $\frac{z_4}{z_3} = i\left(\frac{\sin\theta}{\cos\theta - 1}\right)$. Hence find $\arg\left(\frac{z_4}{z_3}\right)$. **4**

(β) Using (α) show that the diagonals of the quadrilateral *OACB* are perpendicular to each other. **4**

Question 13 (6 marks)

How many ways are there to place nine different rings on the four fingers of your right hand (excluding the thumb) if:

(i) the order of the rings on a finger does not matter? **3**

(ii) the order of the rings on a finger is considered? **3**

THIS IS THE END OF THE PAPER

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

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Sample Solutions

1(a) $\frac{3+4i}{1+i} \times \frac{1-i}{1-i} = \frac{3-3i+4i+4}{1+1}$

$= \frac{7}{2} + \frac{i}{2}$ ✓

(b) Method 1: $(x+iy)(2+3i) = 2x+3ix+2iy-3y$

Equating real & imaginary parts,

$2x - 3y = 5$ ——— ①

$3x + 2y = 6$ ——— ②

①×2: $4x - 6y = 10$ ——— ③

②×3: $9x + 6y = 18$ ——— ④

③+④: $13x = 28$ ✓

$x = 28/13$

$2y = 6 - 3(28/13)$

$y = -3/13$ ✓

Method 2: $x+iy = \frac{5+6i}{2+3i} \times \frac{2-3i}{2-3i}$

$= \frac{10-15i+12i+18}{4+9}$

$= \frac{28-3i}{13}$

$\therefore x = \frac{28}{13}, \; y = \frac{-3}{13}$

1(c) (i) $-1-i\sqrt{3} = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

$1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore \dfrac{-1-i\sqrt{3}}{1-i} = \sqrt{2}\,\text{cis}\left(\dfrac{-5\pi}{12}\right)$ ✓

$\frac{-2}{3}+\frac{1}{4}=\frac{-8+3}{12}$

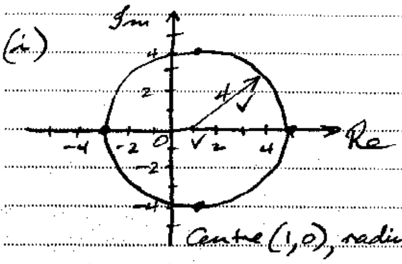
$\frac{1}{2}$ for $\frac{-1+\sqrt{3}}{2} - i\left(\frac{1+\sqrt{3}}{2}\right)$

(ii) $\left(\dfrac{-1-i\sqrt{3}}{1-i}\right)^6 = 8\,\text{cis}\left(\dfrac{-30\pi}{12}\right)$ ✓

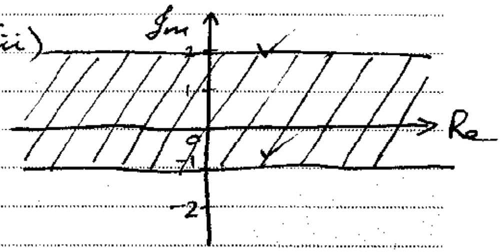
$= 8\,\text{cis}\left(-\frac{5\pi}{2}\right)$

$= -8i$ ✓

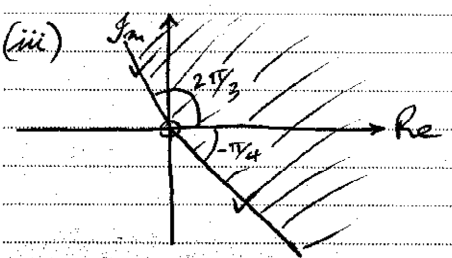
2 (i)



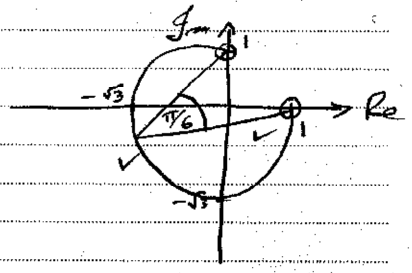
(ii)



(iii)



(iv)



3(a) Let $z = a+ib$, then

$z+\frac{1}{z} = a+ib + \dfrac{1}{a+ib}\times\dfrac{a-ib}{a-ib}$

$= a+ib + \dfrac{a-ib}{a^2+b^2}$

As $z+\frac{1}{z}$ is real, $b - \dfrac{b}{a^2+b^2} = 0$ ✓

ie $b\left(1-\dfrac{1}{a^2+b^2}\right) = 0$

so $b=0$ or $a^2+b^2=1$ ✓

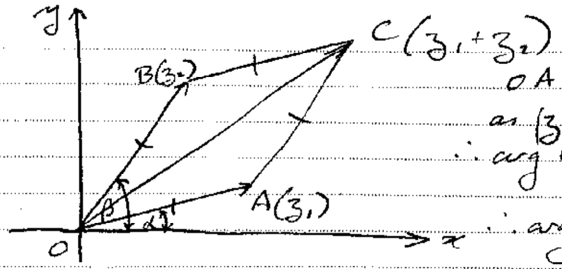
$\therefore \text{Im}(z) = 0$ or $|z| = 1$. ✓

(b) $-2-2i = \sqrt{8}\,\text{cis}\left(-\frac{3\pi}{4}+2n\pi\right)$ ✓

$\therefore \sqrt{-2-2i} = 8^{1/4}\,\text{cis}\left(-\frac{3\pi}{8}+n\pi\right)$

$= 8^{1/4}\,\text{cis}\left(-\frac{3\pi}{8}\right),\ 8^{1/4}\,\text{cis}\left(\frac{5\pi}{8}\right)$ ✓ ✓

4



OACB is a rhombus as $|z_1| = |z_2|$ ✓

$\therefore \arg C = \frac{\alpha+\beta}{2} = \arg(z_1+z_2)$

$\therefore \arg(z_1+z_2)^2 = 2\left(\frac{\alpha+\beta}{2}\right)$ ✓

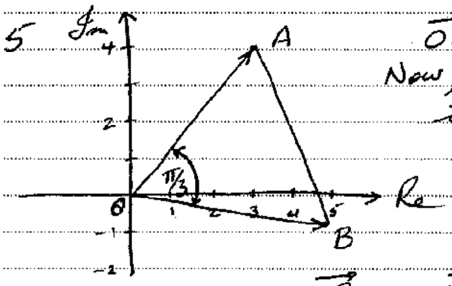
$= \alpha+\beta$

(using deMoivre's thm.)

Now, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ ✓
$= \alpha + \beta$

$\therefore \arg(z_1 z_2) = \arg(z_1+z_2)^2$. ✓

5



$\overrightarrow{OA} = 3+4i$

Now, unit vector with arg $-\pi/3$ is $cis(-\pi/3) = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ ✓

$\overrightarrow{OB} = \overrightarrow{OA} \times cis(-\pi/3)$ ✓

$= (3+4i)\times\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$ ✓

$= \frac{1}{2}\left(3 - 3i\sqrt{3} + 4i + 4\sqrt{3}\right)$

$= \frac{3+4\sqrt{3}}{2} + \frac{i}{2}\left(4 - 3\sqrt{3}\right)$ ✓

$\approx 4{\cdot}9641 - 0{\cdot}5981\,i$

Question 6

$P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5$

(i) Since $x = 1 + 2i$ is a zero **AND** the coefficients are real then $x = 1 - 2i$ is also a zero (*conjugate root theorem*).

So $\left(x - (1 + 2i)\right)\left(x - (1 - 2i)\right) = x^2 - 2x + 5$ is a factor of $P(x)$.

So $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5 = (x^2 - 2x + 5)(x^2 + bx + 1)$

Collecting terms of degree 1 and comparing coefficients we get

$-2 + 5b = -2 \quad b = 0$

[Or by applying long division methods ie
$(x^4 - 2x^3 + 6x^2 - 2x + 5) \div (x^2 - 2x + 5)$ etc….]

Thus $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5 = (x^2 - 2x + 5)(x^2 + 1)$, so the zeros of $P(x)$ are $x = 1 + 2i, 1 - 2i, \pm i$

(ii) (α)

$$P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5 = \left(x - (1 + 2i)\right)\left(x - (1 - 2i)\right)\left(x - i\right)\left(x + i\right)$$

(β) $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5 = (x^2 - 2x + 5)(x^2 + 1)$

Question 7

$x^3 - 2x + 5 = 0$

(i) Apply the transformation $y = \dfrac{2}{x} \quad x = \dfrac{2}{y}$

So $\left(\dfrac{2}{y}\right)^3 - 2\left(\dfrac{2}{y}\right) + 5 = 0 \quad \dfrac{8}{y^3} - \dfrac{4}{y} + 5 = 0 \quad 5y^3 - 4y^2 + 8 = 0$

So the equation with roots $2/\alpha, 2/\beta, 2/\gamma$ is $\boxed{5x^3 - 4x^2 + 8 = 0}$

(ii) Apply the transformation $y = x^2$

NB $x^3 - 2x + 5 = 0 \quad x(x^2 - 2) = -5$

So square both sides and we get $x^2(x^2 - 2)^2 = 25 \quad y(y - 2)^2 = 25$

So the equation with roots $\alpha^2, \beta^2, \gamma^2$ is given by

$\boxed{x(x - 2)^2 = 25 \text{ OR } x^3 - 4x^2 + 4x - 25 = 0}$

(iii) We have to use the fact that $\alpha+\beta+\gamma=0 \quad \alpha+\beta=-\gamma$ and so on. So use the transformation $y=-x \quad x=-y$

So we get $(-y)^3-2(-y)+5=0 \quad -y^3+2y+5=0$.

So the equation with roots $\alpha+\beta, \beta+\gamma, \gamma+\alpha$ is $\boxed{x^3-2x-5=0}$

Question 8

(a) $f(x)=x^n-1 \quad f'(x)=nx^{n-1}$

Suppose α is a multiple root then $f(\alpha)=f'(\alpha)=0 \quad n\alpha^{n-1}=0 \quad \alpha=0$

BUT $f(0)\neq 0$, so $f(x)$ cannot have any multiple roots.

(b) $x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})$

By long division; series; or other means $x^n-1=(x-1)(1+x+x^2+\ldots+x^{n-1})$

So $x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})=(x-1)(1+x+x^2+\ldots+x^{n-1})$

So $(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})=1+x+x^2+\ldots+x^{n-1}$, sub $x=1$ into both sides

$$(1-\alpha_1)(1-\alpha_2)...(1-\alpha_{n-1})=(1+1+1^2+\ldots+1^{n-1})=n \quad \textbf{QED}$$

Question 9

$$\omega=\text{cis}\frac{2\pi}{5}\neq 1$$

(i) $\omega^5=\text{cis}\ \ 5\times\frac{2\pi}{5}\ \ =\text{cis}2\pi=1$. [de Moivre's Theorem – DMT]

$\omega^5-1=(\omega-1)\left(1+\omega+\omega^2+\omega^3+\omega^4\right)=0$

$\because \omega\neq 1 \quad 1+\omega+\omega^2+\omega^3+\omega^4=0$

OR $1+\omega+\omega^2+\omega^3+\omega^4=\frac{\omega^5-1}{\omega-1}=\frac{1-1}{\omega-1}=0$ [using geometric series]

(ii) $1+\omega+\omega^2+\omega^3+\omega^4=0 \quad \frac{1}{\omega^2}+\frac{1}{\omega}+1+\omega+\omega^2=0$ [$\div\omega^2$]

Let $z=\omega+\frac{1}{\omega} \quad (z^2-2)+1+z=0 \quad z^2+z-1=0$

[$\because \frac{1}{\omega^2}+\omega^2=\ \left(\omega+\frac{1}{\omega}\right)^2-2$]

(ii) **Alternative solution**

Examine z^2+z-1 when $z=\omega+\frac{1}{\omega}$

z^2+z-1

$=\left(\omega+\frac{1}{\omega}\right)^2+\omega+\frac{1}{\omega}-1$

$=\omega^2+2+\frac{1}{\omega^2}+\omega+\frac{1}{\omega}-1$

$=\frac{\omega^4+\omega^3+\omega^2+\omega+1}{\omega^2}$

$=\frac{0}{\omega^2}$

$=0$

So $z=\omega+\frac{1}{\omega}$ is a solution of $z^2+z-1=0$

(iii) $|\omega|=1 \quad \frac{1}{\omega}=\bar{\omega} \quad \omega+\frac{1}{\omega}=2\operatorname{Re}\omega=2\cos\frac{2\pi}{5}$

$z^2+z-1=0 \quad z=\frac{-1\pm\sqrt{5}}{2}$

$\because \cos\frac{2\pi}{5}>0 \quad 2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2}$

$\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$

[NB the other solution of the quadratic is $2\cos\frac{4\pi}{5}$]

SECTION C

Q(10) $y = x^3 - 3px + q$

(i) $y' = 3x^2 - 3p = 0 \therefore$ st. pts when $x = \pm\sqrt{p}$

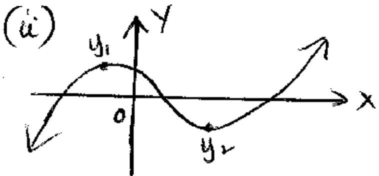
When $x = \sqrt{p}$, $y = -2p^{3/2} + q$

When $x = -\sqrt{p}$, $y = 2p^{3/2} + q$

$\therefore$ St. pts $(\sqrt{p}, q - 2p^{3/2})$

$(-\sqrt{p}, q + 2p^{3/2})$

(ii)



For 3 distinct real roots

$y_1 y_2 < 0 \Rightarrow (q - 2p^{3/2})(q + 2p^{3/2}) < 0$

$\therefore \quad q^2 < 4p^3$

or $|q| < 2p\sqrt{p}$

Q(11) Let E be the point where circle PQR meets DX.

Now $CX.DX = AX.BX$

(product of chords of circle)

$\therefore (2RX).DX = (2PX)(2QX)$

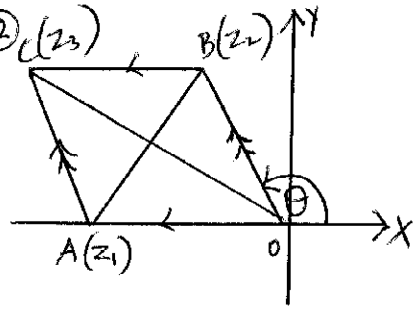
$\Rightarrow (RX)\frac{1}{2}(DX) = (PX)(QX)$

However $(RX)(EX) = (PX)(QX)$

$(EX) = \frac{1}{2}(DX)$

Q(12)

(i)



(ii) $z_4 = z_2 - z_1$

$= (\cos\theta + 1) + i\sin\theta$

(α) $z_3 = z_1 + z_2 = (-1 + \cos\theta) + i\sin\theta$

$\therefore \frac{z_4}{z_3} = \frac{(\cos\theta + 1) + i\sin\theta}{(\cos\theta - 1) + i\sin\theta} \cdot \frac{(\cos\theta - 1) - i\sin\theta}{(\cos\theta - 1) - i\sin\theta}$

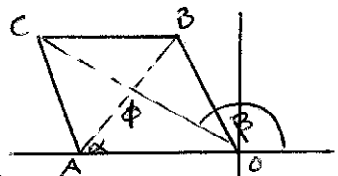
$= \frac{-2i\sin\theta}{2(1 - \cos\theta)}$ on simplifying

$= -i\left[\frac{\sin\theta}{1 - \cos\theta}\right]$

and $\arg\left(\frac{z_4}{z_3}\right) = -\frac{\pi}{2}$ since

$\frac{\sin\theta}{1 - \cos\theta} < 0$ for $\frac{\pi}{2} < \theta < \pi$

(β)



$\arg\left(\frac{z_4}{z_3}\right) = \arg z_4 - \arg z_3 = -\frac{\pi}{2}$

$\Rightarrow \arg z_3 - \arg z_4 = \frac{\pi}{2}$

$\arg z_3 = \beta$, $\arg z_4 = \alpha$

$\arg z_3 - \arg z_4 = \beta - \alpha = \phi$

$\therefore \phi = \frac{\pi}{2}$

Diagonals are $\perp$

or careful expl. using × by i

Q(13) (i) 4^9 (ii) $\frac{12!}{3! \times 9!} \times 9! = \frac{12!}{3!}$