SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2006

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 60

- Attempt questions 1 – 3
- All questions are of equal value.

Examiner: *A.M.Gainford*

Question 1. (Start a new page.) (19 marks)

Marks

(a) For the complex number $z = \sqrt{3} - i$ find:

(i) $|z|$ **2**

(ii) $\arg z$.

(b) Given $w = 3 - 4i$ express each of the following in the form $a + ib$ (for real a and b). **4**

(i) $\overline{w}$

(ii) w^{-1}

(iii) the square roots of w

(c) For the real number θ consider $(\cos\theta + i\sin\theta)^4$.
Express $\cos 4\theta$ as a real polynomial in $\cos\theta$. **3**

(d) Sketch the region in the Argand diagram containing the points z for which: **4**

(i) $|\arg z| \leq \frac{\pi}{6}$

(ii) $z + \overline{z} < 4$

(iii) $2 < |z| < 4$

(c) Solve the equation $z^2 - 4iz - 3 = 0$. **2**

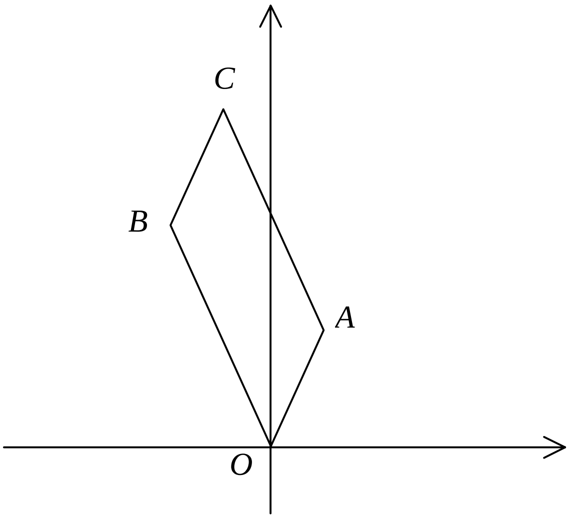
(d) Prove by mathematical induction that **4**

$\sin(n\pi + x) = (-1)^n \sin x$ for all positive integral n.

Question 2. (Start a new page.) (20 marks)

Marks

(a) In the Argand diagram below OACB is a parallelogram. The point A represents the complex number $1+i\sqrt{3}$, and $\angle \text{AOB} = \frac{\pi}{4}$. **4**



Given that $OB : OA = 2 : 1$, find in the form $a+ib$ (using exact values):

(i) the complex number represented by B

(ii) the complex number represented by C

(iii) the complex number represented by the vector AB

(b) Given that $z = 1-2i$ is a zero of the real polynomial $P(z) = z^3 - az^2 + bz - 20$: **4**

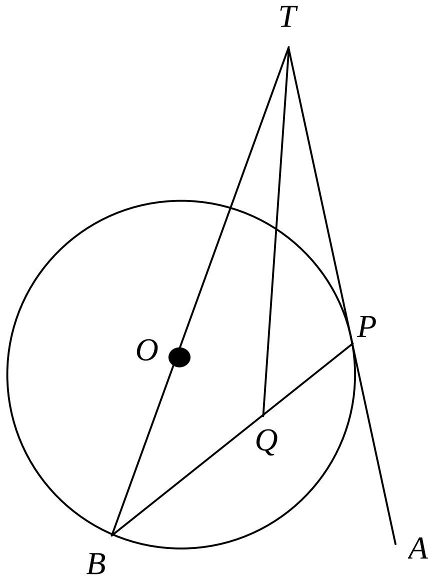
(i) State why $\overline{z}$ is also a zero of $P(z)$ (Quote the theorem.)

(ii) Find the values of a and b.

(c) If $2-i$ is a root of the equation $x^4 - 5x^3 + 3x^2 + 19x - 30 = 0$, find the other three roots. **3**

(d) If α, β, γ are the roots of the equation $x^3 + 4x^2 - 4 = 0$, find the monic cubic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, and hence or otherwise state the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. **3**

(e) TP is a tangent to the circle, centre O, and TQ bisects $\angle OTP$. **6**



Suppose that $\angle QTP = x$. Give reasons why:

(i) $\angle TOP = \frac{\pi}{2} - 2x$

(ii) $\angle TBP = \frac{\pi}{4} - x$, and

(iii) Find the size of $\angle TQP$.

Question 3. (Start a new page.) (21 marks)

Marks

(a) (i) Using De Moivre's Theorem, find the six solutions to the equation $z^6 + 1 = 0$. Express your answers exactly in $a + i\,b$ form, and plot them on an Argand diagram. **4**

(ii) Hence or otherwise factorise $z^6 + 1$ completely: **3**

(α) over the complex field, and

(β) over the real number field (linear and/or irreducible quadratic factors.)

(b) Note that **4**

$$1 = 1$$
$$1 - 4 = -(1 + 2)$$
$$1 - 4 + 9 = 1 + 2 + 3$$
$$1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4)$$

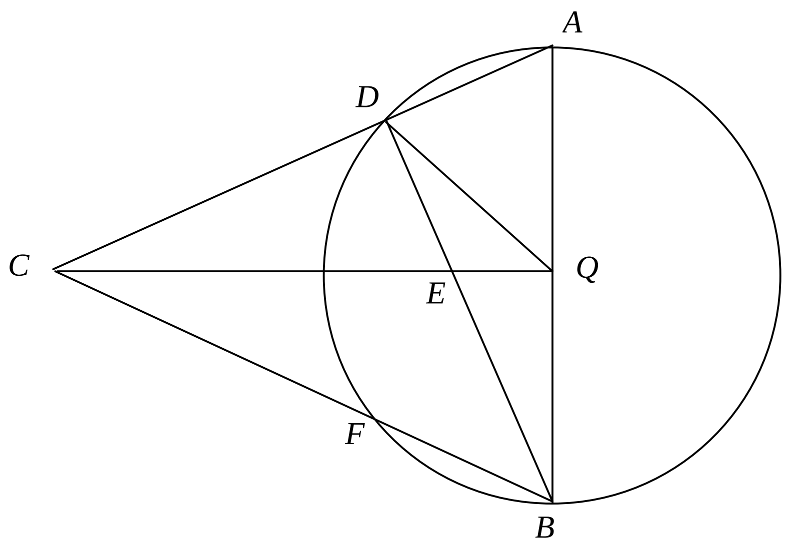
Propose a general hypothesis, and prove it by induction.

You may assume the result $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$.

(c) (i) Show that if $x = \alpha$ is a double root (of multiplicity 2 only) of the polynomial equation $A(x) = 0$, then $A(\alpha) = A'(\alpha) = 0$, and $A''(\alpha) \neq 0$. **2**

(ii) Find the value of k so that the equation $5x^5 - 3x^3 + k = 0$ has two equal positive roots. **3**

(d) AB is a diameter of the circle centre Q, $CQ \perp AB$ at Q, and AC and BC intersect the circle at D and F respectively.



(i) Prove that $\angle QDB = \angle QCA$. **3**

(ii) Hence or otherwise show that quadrilateral $CDQB$ is cyclic. **2**

This is the end of the paper.

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \; x > 0$

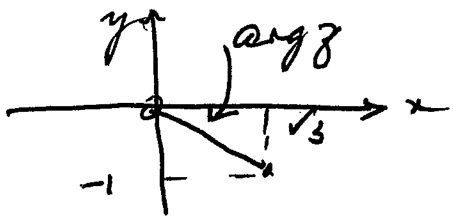
Question 1.

(a) (i) $|z| = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{3+1}$

$= \boxed{2.}$ ✓

(ii)



$\arg z = \boxed{-\frac{\pi}{6}.}$ ✓

(b) (i) $\bar{w} = \boxed{3 + 4i}$ ✓

(ii) $w^{-1} = \dfrac{1}{3-4i}$

$= \dfrac{3+4i}{25}$

$= \boxed{\dfrac{3}{25} + \dfrac{4}{25} i}$ ✓

(iii) let $\sqrt{3-4i} = a + ib$

$\therefore 3 - 4i = (a+ib)^2$

$= a^2 - b^2 + 2ab\,i$

$\therefore a^2 - b^2 = 3$ — ①

& $2ab = -4$ — ②

Also $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$= 9 + 16$

$= 25$

$\therefore a^2 + b^2 = 5$ — ③

① + ③

$2a^2 = 8$

$a^2 = 4$

$a = \pm 2$

$\therefore b = \mp 1$ ✓✓

$\therefore$ square roots are $\boxed{\pm(2 - i)}$

(c) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i \sin 4\theta$

or by expanding.

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta . i^3\sin^3\theta + i^4\sin^4\theta$

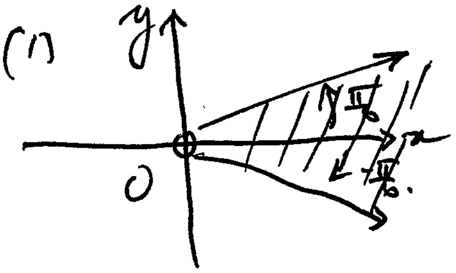
$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

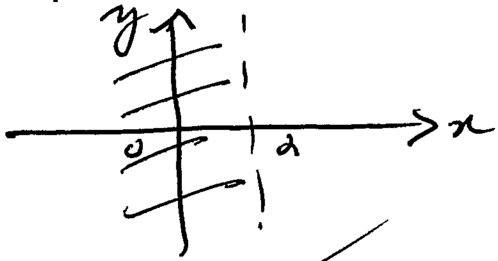
$= \boxed{8\cos^4\theta - 8\cos^2\theta + 1}$ ✓✓✓

(d) (i)

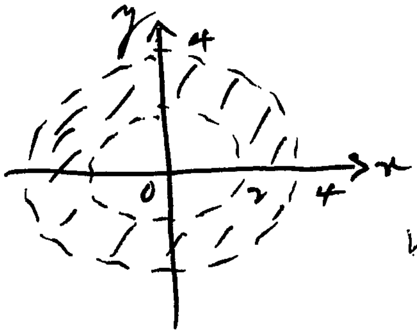


(ii) $z+\bar{z} = 2\,Re(z) < 4$

$\Rightarrow \quad x < 2$



(iii)



(e) $z^2 - 4iz - 3 = 0$

$$z = \frac{4i \pm \sqrt{-16+12}}{2} = \frac{4i \pm \sqrt{-4}}{2} = \frac{4i \pm 2i}{2}$$

$= 3i,\ i$

(f)

Step I. when $n=1$.

LHS $= \sin(\pi + x) = -\sin x$ RHS $= (-1)^1 \sin x = -\sin x$.

∴ Statement is true when $n=1$. (ie $S(1)$ is true).

Step II Assume $S(k)$ is true.

ie. $\sin(k\pi + x) = (-1)^k \sin x$.

Step III Assuming $S(k)$ is true, prove $S(k+1)$ is true.

ie. $\sin((k+1)\pi + x) = (-1)^{k+1} \sin x$.

LHS $= \sin[\pi + (k\pi + x)] = -\sin(k\pi + x)$

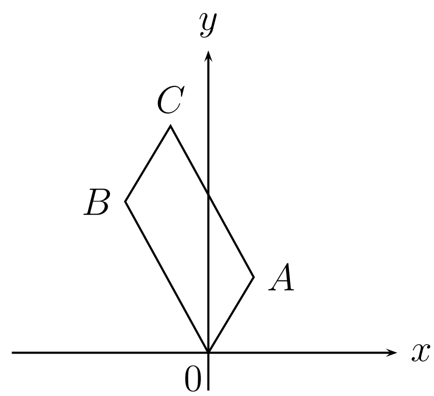
$= -(-1)^k \sin x$.

$= (-1)^{k+1} \sin x$

$=$ RHS.

Step IV Having assumed statement to be true for $n=k$ we proved it true for $n=k+1$. Also since it is true for $n=1$ it must be true for $n=2$ etc & hence true for all pos integers.

2. (a) In the Argand diagram $OACB$ is a parallelogram. The point A represents the complex number $1 + i\sqrt{3}$, and $\angle AOB = \dfrac{\pi}{4}$. Given that $OB : OA = 2 : 1$, find in the form $a + ib$ (using exact values):



i. the complex number represented by B. [2]

Solution: Method 1:
Rotation through $\frac{\pi}{4} \Longrightarrow$ multiplication by $\operatorname{cis}\frac{\pi}{4}$.

$$2(1+i\sqrt{3}) \times \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}(1+i+i\sqrt{3}-\sqrt{3},$$
$$= \sqrt{2}\left((1-\sqrt{3}) + i(1+\sqrt{3})\right).$$

$\therefore\ B$ is $(\sqrt{2}-\sqrt{6}) + i(\sqrt{2}+\sqrt{6})$.

Solution: Method 2:

$1 + i\sqrt{3} = 2\operatorname{cis}\dfrac{\pi}{3}$

$2 \times \operatorname{cis}\dfrac{\pi}{4} \times 2\operatorname{cis}\dfrac{\pi}{3} = 4\operatorname{cis}\dfrac{7\pi}{12}$

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3},\\ &= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2},\\ &= \tfrac{1}{4}(\sqrt{2}-\sqrt{6}).\end{aligned}$$

$$\begin{aligned}\sin\frac{7\pi}{12} &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3},\\ &= \frac{1}{\sqrt{2}} \times \frac{1}{2} + \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2},\\ &= \tfrac{1}{4}(\sqrt{2}+\sqrt{6}).\end{aligned}$$

$\therefore\ B$ is $(\sqrt{2}-\sqrt{6}) + i(\sqrt{2}+\sqrt{6})$.

ii. the complex number represented by C 1

Solution: C is $(1+\sqrt{2}-\sqrt{6})+i(\sqrt{2}+\sqrt{3}+\sqrt{6})$.

iii. the complex number represented by the vector AB 1

Solution: AB is $(\sqrt{2}-\sqrt{6}-1)+i(\sqrt{2}-\sqrt{3}+\sqrt{6})$.

(b) Given that $z = 1-2i$ is a zero of the real polynomial $P(z) = z^3 - az^2 + bz - 20$:

i. State why $\overline{z}$ is also a zero of $P(z)$ (quote the theorem). 1

Solution: Complex zeroes of polynomials with real coefficients occur in conjugate pairs.

ii. Find the values of a and b. 3

Solution: Method 1:
Two roots are $1-2i$ and $1+2i$. If the third root is γ, then the product of the roots $5 \times \gamma = 20$.
The roots are $1-2i$, $1+2i$, and 4.
Sum of roots: $1-2i+1+21+4 = a$,
so $a = 6$.
Roots two at a time $(\alpha\beta+\beta\gamma+\gamma\alpha)$: $5+4\times 2 = b$,
so $b = 13$.

Solution: Method 2:
Roots by inspection are $1-2i$, $1+2i$, 4.

$$\begin{aligned} P(4) &= 64-16a+4b-20 = 0,\\ i.e.\ b &= 4a-21.\\ \text{Now } 1-2i+1+2i+4 &= a,\\ a &= 6,\\ b &= 13. \end{aligned}$$

Solution: Method 3:

$$\begin{aligned} P(1-2i) &= (1-2i)^3 - a(1-2i)^2 + b(1-2i) - 20, \\ &= 1^3 + 3\times 1^2(-2i) + 3\times 1(-2i)^2 + (-2i)^3 \\ &\quad - a(1-4i-4) + b(1-2i) - 20, \\ &= 1 - 6i - 12 + 8i + 3a + 4ai + b - 2bi - 20, \\ &= -31 + 3a + b + i(2 + 4a - 2b) = 0. \end{aligned}$$

Now, equating coefficients,

$$\begin{aligned} 3a + b &= 31 \ldots \boxed{1} \\ 2a - b &= -1 \ldots \boxed{2} \\ \boxed{1} + \boxed{2} : \ 5a &= 30, \\ a &= 6, \\ b &= 31 - 18, \\ &= 13. \end{aligned}$$

(c) If $2 - i$ is a root of the equation $x^4 - 5x^3 + 3x^2 + 19x - 30 = 0$, find the other three roots. $\boxed{3}$

Solution: Method 1:

$$\begin{aligned} (x-2+i)(x-2-i) &= x^2 - 4x + 4 + 1, \\ &= x^2 - 4x + 5. \end{aligned}$$

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x^2 - 4x + 5 & x^4 - 5x^3 + 3x^2 + 19x - 30 \\ & -x^4 + 4x^3 - 5x^2 \\ \hline & -x^3 - 2x^2 + 19x \\ & x^3 - 4x^2 + 5x \\ \hline & -6x^2 + 24x - 30 \\ & 6x^2 - 24x + 30 \\ \hline & 0 \end{array}$$

$\therefore$ Other 3 roots are $2 + i$, 3, -2.

Solution: Method 2:
$2+i$ is also a root, so let the other two be $\alpha,\ \beta$.

$$\begin{aligned}
\text{Sum of roots: } 2+i+2-i+\alpha+\beta &= 5,\\
\alpha+\beta &= 1\ldots\boxed{1}.\\
\text{Product of roots: } (2+i)(2-i)\alpha\beta &= -30,\\
5\alpha\beta &= -30,\\
\therefore\ \beta &= -\tfrac{6}{\alpha}\ldots\boxed{2}.\\
\text{Sub. } \boxed{2} \text{ into } \boxed{1}:\ \alpha-\tfrac{6}{\alpha}-1 &= 0,\\
\alpha^2-\alpha-6 &= 0,\\
(\alpha-3)(\alpha+2) &= 0,\\
\therefore\ \alpha=3,\ \beta=-2,\ \text{or}\quad \alpha=-2,\ \beta=3.&
\end{aligned}$$

$\therefore$ Other 3 roots are $2+i$, 3, -2.

(d) If $\alpha,\ \beta,\ \gamma$ are the roots of the equation $x^3+4x^2-4=0$, find the monic cubic equation with roots $\dfrac{1}{\alpha},\ \dfrac{1}{\beta},\ \dfrac{1}{\gamma}$, and hence or otherwise state the value of $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$. $\boxed{3}$

Solution: Put $x=\dfrac{1}{y}$,

$$\begin{aligned}
\text{then}\quad \frac{1}{y^3}+\frac{4}{y^2}-4 &= 0,\\
4y^3-4y-1 &= 0.
\end{aligned}$$

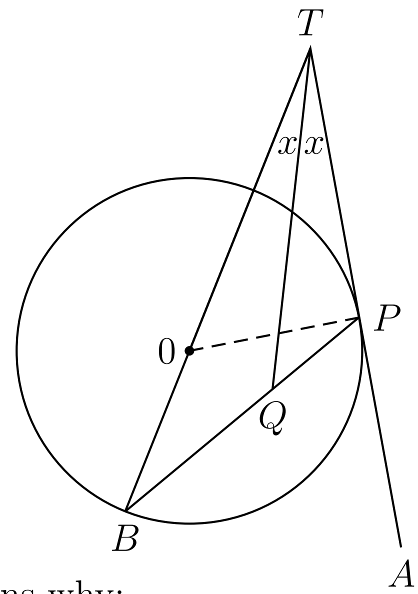
$\therefore$ Monic cubic is $x^3-x-\frac{1}{4}=0$.

And so $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=0$ (using sum of roots of $x^3-x-\frac{1}{4}=0$).

Alternative method:

$$\begin{aligned}
\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} &= \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} \text{ from } x^3+4x^2-4=0,\\
&= \frac{0}{4},\\
&= 0.
\end{aligned}$$

(e) TP is a tangent to the cirle, centre O, and TQ bisects $\angle OTP$.



Suppose that $\angle QTP = x$. Give reasons why:

i. $\angle TOP = \dfrac{\pi}{2} - 2x$, [3]

> **Solution:**
> $$\begin{aligned} \angle QTO &= x \quad (QT \text{ bisects } \angle QTP). \\ \angle OPT &= \frac{\pi}{2} \quad (\text{radius } \perp \text{ tangent at point of contact}). \\ \therefore\ \angle TOP &= \pi - \frac{\pi}{2} - 2x, \quad (\angle \text{ sum of } \triangle) \\ &= \frac{\pi}{2} - 2x. \end{aligned}$$

ii. $\angle TBP = \dfrac{\pi}{4} - x$, and [1]

> **Solution:** The angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

iii. find the size of $\angle TQP$. [2]

> **Solution:**
> $$\begin{aligned} \angle TQP &= \angle TBP + \angle BTQ \quad (\text{Ext. } \angle \text{ of a } \triangle), \\ &= \left(\frac{\pi}{4} - x\right) + x, \\ &= \frac{\pi}{4}. \end{aligned}$$

Question 3

(a) $z^6 + 1 = 0$

(i) $\Rightarrow z^6 = -1$

let $z = cis\theta$

$\Rightarrow cis 6\theta = cis[\pi + 2n\pi]$

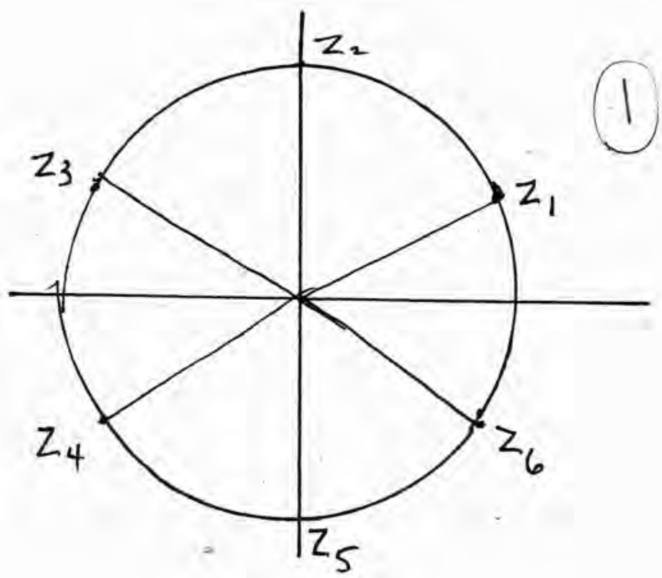
$\therefore$ for $n = 0, 1, 2, 3, 4, 5$ roots

are $cis\frac{\pi}{6}$, $cis\frac{3\pi}{6}$, $cis\frac{5\pi}{6}$

$cis\frac{7\pi}{6}$, $cis\frac{9\pi}{6}$, $cis\frac{11\pi}{6}$ (3)

$z_1 \pm z_6$ $\quad z_3 \pm z_4$ $\quad \pm z_2$

ie $\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$, $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$, $\pm i$



(1)

(ii)

α)

$z^6+1 = \left[z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right]\left[z - i\right]\left[z + \frac{1}{2} - \frac{i\sqrt{3}}{2}\right]\ldots$ ($1\frac{1}{2}$)

$\times \left[z + \frac{1}{2} + \frac{i\sqrt{3}}{2}\right]\left[z + i\right]\left[z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right]$

β) ($1\frac{1}{2}$)

~~$z^6+1 = [z^2 - z + 1][z^2 + 1][z^2 + z + 1]$~~

$z^6 + 1 = [z^2 - \sqrt{3}z + 1][z^2 + \sqrt{3}z + 1][z^2 + 1]$

(b) Proposition is for $n \geq 1$ (4)

$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1} \cdot n^2 = (-1)^{n+1} \cdot \frac{n(n+1)}{2}$

When $n = 1$

LHS $= (-1)^{1+1} \cdot 1^2$

$= 1$

RHS $= (-1)^{1+1} \cdot 1\frac{(2)}{2}$

$= 1$

true for $n = 1$

Assume

$1^2 - 2^2 + 3^2 - 4^2 + (-1)^{k+1} \cdot k^2 = (-1)^{k+1} \cdot \frac{k(k+1)}{2}$

RTP

$1^2 - 2^2 + \cdots + (-1)^{k+1} \cdot k^2 + (-1)^{k+2} \cdot (k+1)^2 =$

$(-1)^{k+2} \cdot \frac{(k+1)(k+2)}{2}$

LHS $= (-1)^{k+1} \cdot \frac{k(k+1)}{2} + (-1)^{k+2} \cdot (k+1)^2$

$= \frac{1}{2}(-1)^{k+1} \cdot (k+1)\left[k + (-1)^1 \cdot (k+1) \cdot 2\right]$

$= \frac{1}{2}(-1)^{k+1}(k+1)\left[-(k+2)\right]$

$= \frac{(k+1)(k+2)}{2} \cdot (-1)^{k+1} \cdot (-1)^1$

$= \frac{(k+1)(k+2)}{2} \cdot (-1)^{k+2}$

Question 3

(c) (i)

let $A(x) = (x-\alpha)^2 . Q(x)$ (½)

$A'(x) = (x-\alpha)^2 . Q'(x) + 2(x-\alpha)Q(x)$

$\Rightarrow A(\alpha) = (\alpha-\alpha)^2 Q(\alpha) = 0$

$A'(\alpha) = (\alpha-\alpha)^2 Q'(x) + 2(\alpha-\alpha)Q(\alpha)$

$= 0$ (½)

$A''(x) = (x-\alpha)^2 Q''(x) + 2(x-\alpha).Q'(x)$

$+ 2(x-\alpha)Q'(x) + 2Q(x)$

$A''(\alpha) = 0 + 0 + 0 + 2Q(\alpha)$ (½)

and $Q(\alpha) \neq 0$ [since double root]

$\Rightarrow A''(\alpha) \neq 0$ (½)

(ii) $A(x) = 5x^5 - 3x^3 + k$

$A'(x) = 25x^4 - 9x^2 = 0$ (1)

when $x = 0$ or $x = \pm\frac{3}{5}$

$A'(0) = 0$ but $A(0) = k$

$\therefore x = 0$ not double root (unless $k = 0$)

$A'\left(\frac{3}{5}\right) = 0$

$A''(x) = 100x^3 - 18x$

$A''\left(\frac{3}{5}\right) = 10.8$

} $x = \frac{3}{5}$ double root (1)

$\therefore A\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^5 - 3\left(\frac{3}{5}\right)^3 + k = 0$

$\boxed{k = 0.2592}$

~~Also A(~~

(1) $k = \frac{162}{625}$

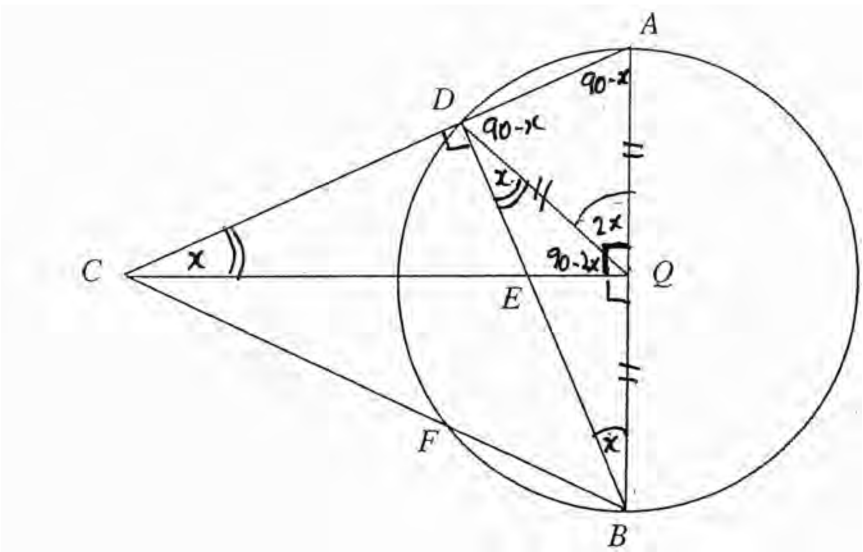
0.3888
0.6480
~~408~~

0.3888

$\frac{3}{100}$ 45000

1300

(d) AB is a diameter of the circle centre Q, $CQ \perp AB$ at Q, and AC and BC intersect the circle at D and F respectively.



(i) Prove that $\angle QDB = \angle QCA$. **3**

(ii) Hence or otherwise show that quadrilateral $CDQB$ is cyclic. **2**

This is the end of the paper.

(i) let $Q\hat{D}B = x$ ✓

$Q\hat{B}E = x$ (isosceles ΔDQB)

$A\hat{Q}D = 2 \times A\hat{B}D$ (angle at centre th.)

ie $A\hat{Q}D = 2x$

$Q\hat{D}A = 90 - x$ (base angle isosc. ΔQDA)

$\therefore Q\hat{C}A = x$ (angle sum ΔQCA)

(ii) $C\hat{D}E = 90°$

($A\hat{D}E + C\hat{D}E = 180°$ and $A\hat{D}E = 90°$)

$C\hat{Q}B = 90°$ (adj to $C\hat{Q}A$ on str. line)

$\therefore C\hat{D}E = C\hat{Q}B = 90°$

Both angles are subtended at cir. of circle through C, D, Q, B by same chord BC.

(BC is the diameter of circle)