SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2007

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 1.5 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 76

- Attempt questions 1 – 3

Examiner: *P.Bigelow*

QUESTION 1 (23 marks) Marks

(a) Find

(i) $\int x \sec^2(1+x^2)dx$ [2]

(ii) $\int x^2 e^{2x} dx$ [2]

(b) Evaluate

(i) $\int_0^1 \frac{dx}{(1+x)\sqrt{1+x}}$ [2]

(ii) $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}dx$ [3]

(c)

(i) Show that $\frac{x^m + x^{m+2}}{1+x^2} = x^m$ [1]

(ii) An integral is defined by

$$\mathrm{I}_m = \int_0^1 \frac{x^m}{1+x^2}dx$$

where m is a non-negative integer.

(α) Evaluate I_0. [1]

(β) Show that $\mathrm{I}_m + \mathrm{I}_{m+2} = \frac{1}{m+1}$. [2]

(γ) Hence or otherwise evaluate I_2. [1]

(d) Five couples sit at a round table. How many seating arrangements are possible if

(i) there are no restrictions? [2]

(ii) each person sits next to their partner? [2]

(e) Let $z = \dfrac{1-i}{2+i}$. Find

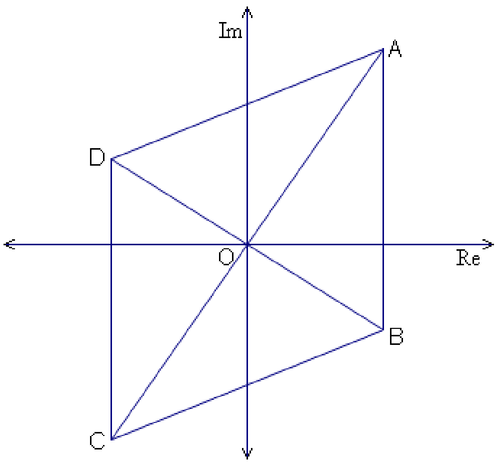
(i) $z + \dfrac{1}{z}$ [1]

(ii) $\bar{z}$ [1]

(f) ABCD is a rhombus, where diagonals meet at O. If A represents the complex number $3+4i$ and OD = 2 units find the complex number represented by

(i) C [1]

(ii) D [2]



[Not to scale]

End of Question 1

QUESTION 2 (26 marks)

Marks

(a) Sketch the following on separate Argand Diagrams.

(i) $|z+1|=3$ [1]

(ii) $\arg(z-1)=\dfrac{\pi}{4}$ [1]

(iii) $|z-1|=\operatorname{Re} z$ [2]

(iv) $\operatorname{Re}(z(\bar{z}+2))=3$ [2]

(b) Find the greatest and least values of $\arg z$ for which $|z-4i|=2$. [2]

(c)

(i) Express $1-\sqrt{3}\,i$ in mod-arg form. [1]

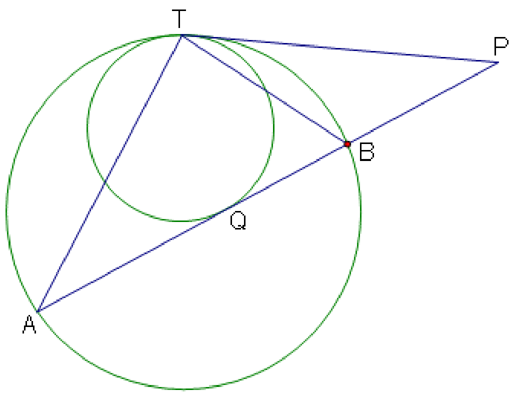
(ii) Hence find $\left(1-\sqrt{3}\,i\right)^6$ in the form $a+bi$. [1]

(d) PT is a common tangent to the circles which touch at T. PA is a tangent to the smaller circle at Q.

(i) Prove that ΔBTP is similar to ΔTAP. [2]

(ii) Hence show that $TP^2 = PA.PB$ [1]

(iii) If PT = t, QA = a and QB = b prove that $t=\dfrac{ab}{a-b}$. [2]



(e) ω is a complex cube root of unity. Evaluate $\left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)$. [2]

(f) If α, β and γ are roots of $2x^3-7x^2+5x-3=0$

(i) show that the equation with roots α^2, β^2 and γ^2 is $4x^3-29x^2-17x-9=0$. [2]

(ii) Hence or otherwise evaluate $\alpha^3+\beta^3+\gamma^3$. [2]

(g) The equation $x^3-3px+q=0$ (where $p>0$, $q\neq 0$ are both real) has three distinct non-zero real roots.

(i) Show that the graph $y=x^3-3px+q$ has a relative maximum of $q+2p\sqrt{p}$ and a relative minimum of $q-2p\sqrt{p}$. [3]

(ii) Hence show that $q^2<4p^3$. [2]

End of Question 2

QUESTION 3 (27 marks) Marks

(a) The letters of KAYAK are arranged at random in a line.

(i) How many different letter sequences are possible? [2]

(ii) What is the probability that the sequence is the same from right to left as left to right? [2]

(b)

(i) Show that $z^5 + 1 = (z+1)(1 - z + z^2 - z^3 + z^4)$. [1]

(ii) Find in the form cisθ, the five complex solutions to $z^5 + 1 = 0$. [2]

(iii) If z is a solution to $z^5 + 1 = 0$ and $z \neq -1$, prove that $1 + z^2 + z^4 = z + z^3$. [1]

(iv) Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} - \frac{1}{2} = 0$. [3]

(c)

(i) find the values of the constants A and B such that $4x^4 + 1 \equiv (2x^2 + \mathrm{A}x + 1)(2x^2 + \mathrm{B}x + 1)$. [2]

(ii) Hence express the integer $2^{14} + 1$ as a product of prime factors. [2]

(d) Solve $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ if it has a root of multiplicity 3. [2]

(e)

(i) Prove $\int_0^c f(x)dx = \int_0^c f(c-x)dx$. [2]

(ii) Hence or otherwise find $\int_0^1 x(1-x)^{2007} dx$. [2]

(f) The diagram shows an isosceles triangle PAB. PM is the perpendicular bisector of $\angle \text{APB}$, which is β. PM bisects AB. A and B represent the complex numbers z_1 and z_2 respectively.

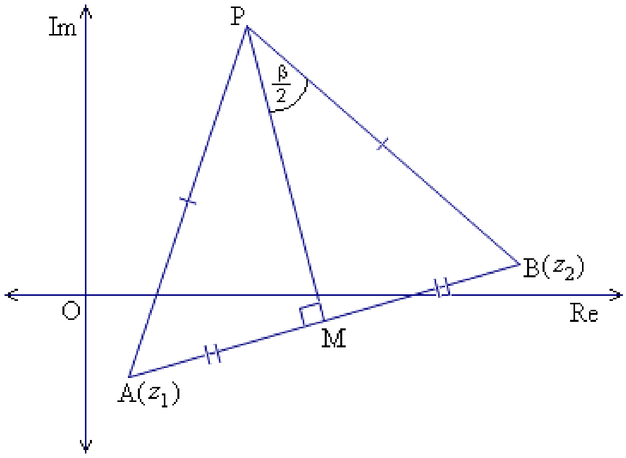
(i) Find the complex number represented by

(α) $\overrightarrow{\text{AM}}$ [1]

(β) $\overrightarrow{\text{MP}}$ [2]

(ii) Hence show that P represents the complex number

$$\frac{1}{2}\left(1 - i\cot\frac{\beta}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\beta}{2}\right)z_2$$ [3]



End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \;\; x > 0$

Solutions to Question (1)

(a)

(i) $\int x\sec^2(1+x^2)\,dx$ 2

$= \frac{1}{2}\tan(1+x^2)+c.$

(ii) $\int x^2\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)dx$

$= \frac{x^2e^{2x}}{2} - \int xe^{2x}dx$

Now

$\int xe^{2x}dx = \int x\frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)dx$

$= \frac{xe^{2x}}{2} - \frac{1}{4}\int 2e^{2x}dx$

$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4}$

$\therefore \int x^2e^{2x}dx$ 2

$= \frac{x^2e^{2x}}{2} - \frac{xe^{2x}}{2} + \frac{1}{4}e^{2x}+c$

$= \frac{e^{2x}}{4}(2x^2-2x+1)+c.$

$\frac{1}{2}xe^{2x}(x-1)+\frac{1}{4}e^{2x}+c$

(b) $\int_0^1 (1+x)^{-3/2}$

(i) $= \left[-2(1+x)^{-1/2}\right]_0^1$

$= -2\cdot\frac{1}{\sqrt{2}}+2$ 2

$= 2-\sqrt{2}.$ (0.586)

(ii) $\frac{3x+7}{(x+1)(x+2)(x+3)}$ 3

$= \frac{a}{x+1}+\frac{b}{x+2}+\frac{c}{x+3}.$

$\therefore 3x+7 = a(x+2)(x+3) + b(x+1)(x+3) + c(x+1)(x+2)$

put $x=-1$, $a=2$

$x=-2$, $b=-1$

$x=-3$, $c=-1$

$\therefore \int_0^1 \frac{(3x+7)\,dx}{(x+1)(x+2)(x+3)}$

$= \left[2\ln(x+1)-\ln(x+2)-\ln(x+3)\right]_0^1$

$= \ln 2.$ (0.693)

(c) (i) $\frac{x^m(1+x^2)}{1+x^2} = x^m$ 1

(ii)

(α) $I_0 = \int_0^1 \frac{1}{1+x^2}dx$

$= \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$ 1

(β) $I_m + I_{m+2} = \int_0^1 \frac{x^m}{1+x^2}dx + \int_0^1 \frac{x^{m+2}}{1+x^2}dx$ 2

$= \int_0^1 \left(\frac{x^m+x^{m+2}}{1+x^2}\right)dx$

$= \int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1$

$= \frac{1}{m+1}.$

(γ)

$I_0+I_2 = 1$

$\frac{\pi}{4}+I_2 = 1$ 1

$\therefore I_2 = 1-\frac{\pi}{4}$

$= \frac{4-\pi}{4}$ (0.215)

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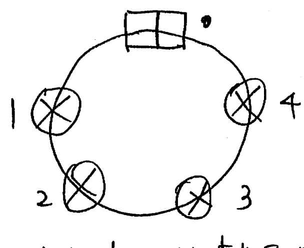
(d) (i) $9!$ 2

(ii) "Fix a couple for reference on a circle, the other couples can be arranged $4!$ ways



2

- each partner of each couple can swap without altering the arrangement

$\therefore$ Total arrangement

$= 4!\times 2^5.$ (768)

(e) (i) $\frac{1-i}{2+i}+\frac{2+i}{1-i}$

$= \frac{(1-i)^2+(2+i)^2}{(2+i)(1-i)}$

(e) $\frac{\cancel{1}-2i\cancel{-1}+4+4i-1}{(2+i)(1-i)}$

$= \frac{3+2i}{3-i}\times\left(\frac{3+i}{3+i}\right)$

$= \frac{9+9i-2}{10}$

$= \frac{7+9i}{10}$ 1

(ii) $\bar{z} = \frac{1+i}{2-i}\times\left(\frac{2+i}{2+i}\right)$

$= \frac{1+3i}{5}$ 1

(f)

(i) $\overrightarrow{OA} = z_1$

$= 3+4i$

$\therefore \overrightarrow{OC} = -\overrightarrow{OA}$

$= -3-4i$ 1

$\therefore C = -3-4i$

$|\overrightarrow{OD}| = 2.$

(ii) $\overrightarrow{OD} = \frac{2}{5}\,\overrightarrow{OA}\ \mathrm{cis}\frac{\pi}{2}$

$= \frac{2}{5}\left[(3+4i)i\right]$

$= \frac{2}{5}(-4+3i)$

$= -\frac{8}{5}+\frac{6i}{5}.$ 2

$\overrightarrow{OD} = 2\,\mathrm{cis}\left(\frac{\pi}{2}+\tan^{-1}\frac{4}{3}\right)$

[Total = 23]

$\boxed{14}$ marks

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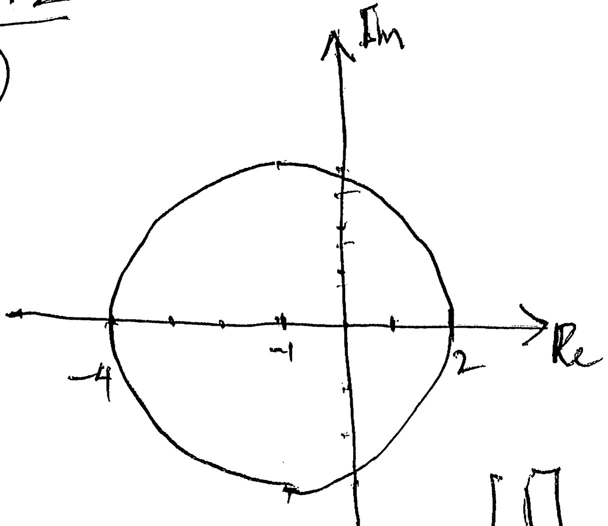
Question 2

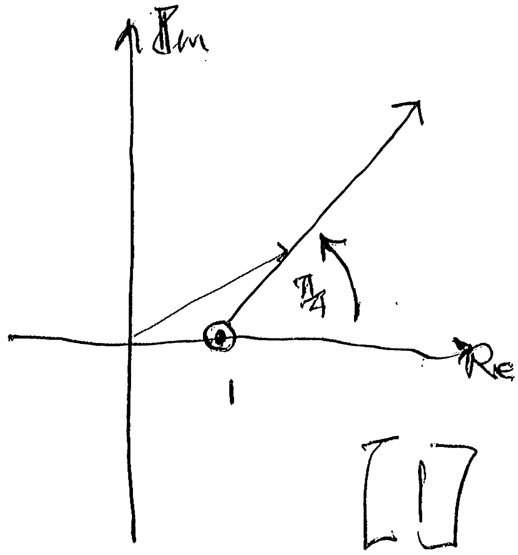
(a) (i)



[1]

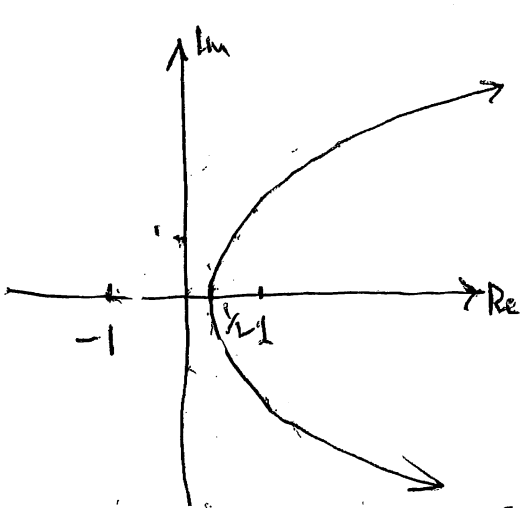
$|z+1|=3$

(ii)



[1]

(iii)



[2]

$|z-1| = \text{Re}\, z$

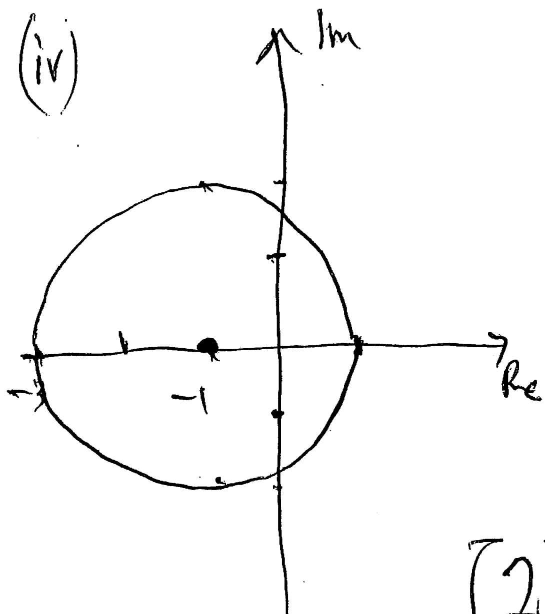
$|(x-1)+iy| = x$

$x^2-2x+1+y^2 = x^2$

$y^2 = 2x-1$

$y = \pm\sqrt{2x-1}$

(iv)



[2]

$\text{Re}(z(\bar{z}+2)) = 3$

$\text{Re}(z\bar{z}+2z) = 3$

$\text{Re}(x^2+y^2+2x+2iy) = 3$

$x^2+y^2+2x = 3$

$(x+1)^2+y^2 = 4$

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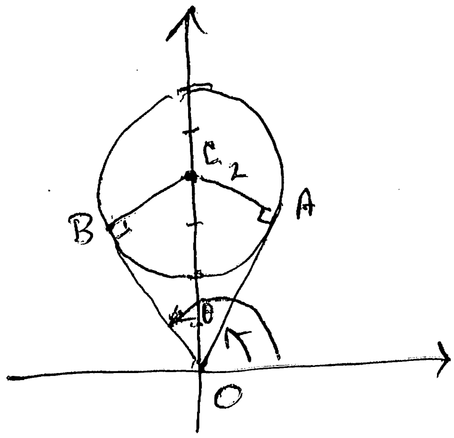
(b) $|z-4i|=2$



In $\triangle OAC$ $\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

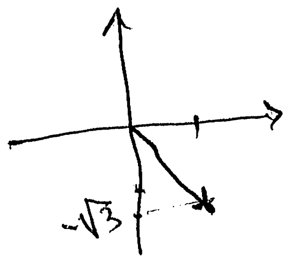
Hence

$\frac{\pi}{3} \leq \arg z \leq \frac{2\pi}{3}$

ie Greatest value $= \frac{2\pi}{3}$, Least value $= \frac{\pi}{3}$ [2]

(c)

(i) $z = 1-\sqrt{3}i$



$|z| = \sqrt{1+3}$
$= 2$

$\arg z = \tan^{-1}\frac{y}{x}$
$= \tan^{-1}\frac{-\sqrt{3}}{1}$
$= -\frac{\pi}{3}$

$\therefore z = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ [1]

(ii) $z^6 = \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^6$
$= 64\,\text{cis}\left(-6\times\frac{\pi}{3}\right)$
$= 64\,\text{cis}(-2\pi)$
$= 64\,\text{cis}\,0$

$\therefore (1-\sqrt{3}i)^6 = 64+0.i$ [1]

(d) (i) In the $\triangle$'s BTP, TAP

$\angle BTP = \angle TAP$ (alternate segment theorem)

$\angle P$ is common

$\therefore \triangle BTP \,|||\, \triangle TAP$ (equiangular) [2]

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(d) (ii) $\frac{PB}{PT} = \frac{PT}{PA}$ (corresponding sides of similar triangles) [1]

$\therefore PA \cdot PB = PT^2$

(iii) $PT = PQ$ (tangents from an external point are equal.)

$\therefore PB = t - b$, $PA = t + a$

Thus $(t-b)(t+a) = t^2$ (from above)

$t^2 + ta - tb - ab = t^2$

$t(a-b) = ab$ [2]

$t = \frac{ab}{a-b}$ QED

(e) Since $1, w, w^2$ are roots of $z^3 - 1 = 0$,

$1 + w + w^2 = 0$ (sum of roots)

$\therefore 1 + w = -w^2$

Thus Exp $= (1 - 3w + w^2)(1 + w - 8w^2)$

$= (1 - 3w - 1 - w)(-w^2 - 8w^2)$

$= (-4w)(-9w^2)$

$= 36w^3$

$= 36$ $(w^3 = 1)$ [2]

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(f)(i) $2x^3-7x^2+5x-3=0$

Transformation: Let $y=x^2$

$\therefore x=\sqrt{y}$

$2y\sqrt{y}-7y+5\sqrt{y}-3=0$

$\sqrt{y}(2y+5)=7y+3$

Square Both sides

$y(4y^2+20y+25)=49y^2+42y+9$

$\therefore 4y^3-29y^2-17y-9=0$

$\therefore$ Eq'n in x with roots $\alpha^2, \beta^2, \gamma^2$ [2]

$4x^3-29x^2-17x-9=0$

(ii) Since α, β, γ are zeroes

$2\alpha^3-7\alpha^2+5\alpha-3=0$

$2\beta^3-7\beta^2+5\beta-3=0$

$2\gamma^3-7\gamma^2+5\gamma-3=0$

By addition

$2(\alpha^3+\beta^3+\gamma^3)-7(\alpha^2+\beta^2+\gamma^2)+5(\alpha+\beta+\gamma)-9=0$

$\therefore \alpha^3+\beta^3+\gamma^3=\frac{1}{2}\left[7\left(\frac{29}{4}\right)-5\left(\frac{7}{2}\right)+9\right]$

$=\frac{169}{8}$ [2]

(9)

$x^3 - 3px + q = 0$ $\quad p > 0,\ q \neq 0$, real

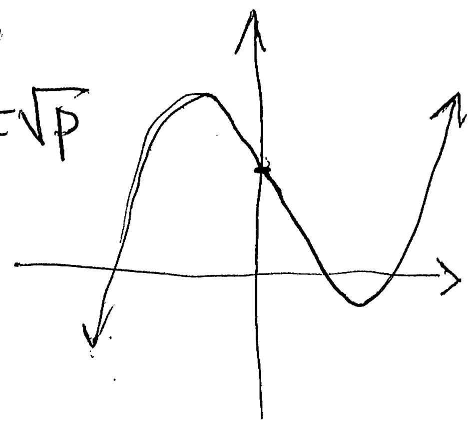
— distinct non-zero real roots

(i) $y = x^3 - 3px + q$

$y' = 3x^2 - 3p$

$y' = 0$ when $x^2 - p = 0$

$x = \pm\sqrt{p}$



$y'' = 6x$

At $x = \sqrt{p}$ $\quad y'' > 0$

$\therefore$ Relative minimum

$y(\sqrt{p}) = p\sqrt{p} - 3p\sqrt{p} + q$

$= q - 2p\sqrt{p}$

At $x = -\sqrt{p}$ $\quad y'' < 0$

$\therefore$ Relative maximum

$y(-\sqrt{p}) = -p\sqrt{p} + 3p\sqrt{p} + q$ [3]

$= q + 2p\sqrt{p}$

(ii) Since cubic has 3 non-zero real roots, the turning points lie on opposite sides of $y=0$.

$\therefore$ They are of different signs.

$\therefore$ Their product is < 0.

$\therefore (q + 2p\sqrt{p})(q - 2p\sqrt{p}) < 0$ [2]

$\therefore q^2 < 4p^3$

QED

3(a) (i) $\dfrac{5!}{2!\,2!} = 30$ (2)

(ii) $\dfrac{2}{30} = \dfrac{1}{15}$ (2)

(b) (i) $(z+1)(1-z+z^2-z^3+z^4)$

$= z - z^2 + z^3 - z^4 + z^5 + 1 - z + z^2 - z^3 + z^4$

$= z^5 + 1$ (1)

(ii) $(\text{cis}\,\theta)^5 + 1 = 0$

$\text{cis}\,5\theta = -1$

$\cos 5\theta = -1 \quad \sin 5\theta = 0$

$-\pi < \theta \leq \pi$

$-5\pi < 5\theta \leq 5\pi$

$\therefore 5\theta = -3\pi, -\pi, \pi, 3\pi, 5\pi$

$\therefore \theta = -\frac{3\pi}{5}, -\frac{\pi}{5}, \frac{\pi}{5}, \frac{3\pi}{5}, \pi$

$\therefore$ Solutions are (2)

$\text{cis}\left(-\frac{3\pi}{5}\right), \text{cis}\left(-\frac{\pi}{5}\right), \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\,\pi$

(iii) $z^5 + 1 = 0$

$(z+1)(1-z+z^2-z^3+z^4) = 0$

$\therefore 1 - z + z^2 - z^3 + z^4 = 0 \quad (\text{as } z \neq -1)$

$\therefore 1 + z^2 + z^4 = z + z^3$ (1)

(iv) Consider $z = \text{cis}\frac{\pi}{5}$

$1 + z^2 + z^4 = z + z^3$

$1 + \text{cis}\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} = \text{cis}\frac{\pi}{5} + \text{cis}\frac{3\pi}{5}$

Equating real parts

$1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = \cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$

$\therefore 1 + \cos\left(\pi - \frac{3\pi}{5}\right) + \cos\left(\pi - \frac{\pi}{5}\right) = \cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$

$\therefore 1 - \cos\frac{3\pi}{5} - \cos\frac{\pi}{5} = \cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$

$\therefore 2\cos\frac{\pi}{5} + 2\cos\frac{3\pi}{5} - 1 = 0$

$\therefore \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} - \frac{1}{2} = 0$ (3)

(c) (i) $(2x^2 + Ax + 1)(2x^2 + Bx + 1)$

$= 4x^4 + 2Bx^3 + 2x^2 + 2Ax^3 + ABx^2 + Ax + 2x^2 + Bx + 1$

$= 4x^4 + (2B + 2A)x^3 + (AB + 4)x^2 + (A+B)x + 1$

$\therefore A + B = 0 \Rightarrow B = -A$

$AB + 4 = 0$

$\therefore -A^2 + 4 = 0$

$\therefore A^2 = 4$

$\therefore A = \pm 2 \qquad B = \mp 2$ (2)

(ii) $2^{14} + 1$

$= 4.2^{12} + 1 \Rightarrow x = 2^3 = 8$

$= (128 + 16 + 1)(128 - 16 + 1)$

$= 145 \times 113$

$= 5 \times 29 \times 113$ (2)

(d) $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

Consider $24x^2 + 54x + 12 = 0$

$\therefore 4x^2 + 9x + 2 = 0$

$\therefore (4x+1)(x+2) = 0$

$\therefore x = -2$ or $x = -\frac{1}{4}$

$P(-2) = 32 - 72 + 24 + 40 - 24$

$= 0$

$\therefore -2$ is the triple root.

Sum of roots: $-6 + \beta = -\frac{9}{2}$

$\therefore \beta = 1\frac{1}{2}$

$\therefore$ Solutions are -2 and $1\frac{1}{2}$

triple root.

(2)

(e)(i) $\int_0^c f(c-x)\,dx$ Let $u = c-x$

$\therefore du = -dx$

$= \int_c^0 f(u)(-du)$ If $x=0$ $u=c$

$= \int_0^c f(u)\,du$ If $x=c$ $u=0$

$= \int_0^c f(x)\,dx$

2

(ii) $\int_0^1 x(1-x)^{2007}\,dx$

$= \int_0^1 (1-x)x^{2007}\,dx$

$= \int_0^1 \left(x^{2007} - x^{2008}\right) dx$

$= \left[\frac{1}{2008}x^{2008} - \frac{1}{2009}x^{2009}\right]_0^1$

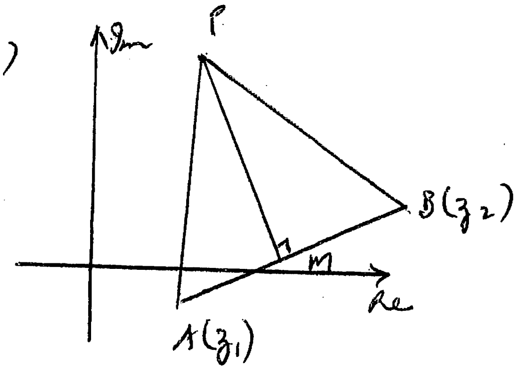
$= \left[\frac{1}{2008} - \frac{1}{2009}\right] - [0-0]$

$= \frac{1}{2008.2009}$

$= \frac{1}{4034072}$

2

(f) (i)



AB represents $z_2 - z_1$

$\therefore \overrightarrow{AM}$ represents $\frac{1}{2}(z_2 - z_1)$.

1

(ii) $\frac{AM}{PM} = \tan\frac{\beta}{2}$

$\therefore PM = AM\cot\frac{\beta}{2}$

$\therefore \overrightarrow{MP}$ represents

$\frac{1}{2}(z_2 - z_1)\cdot\cot\frac{\beta}{2}\cdot i$

$= \frac{i}{2}(z_2 - z_1)\cot\frac{\beta}{2}$

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(iii) $\therefore$ P represents

$z_1 + \frac{1}{2}(z_2 - z_1) + \frac{i}{2}(z_2 - z_1)\cot\frac{\beta}{2}$

$= z_1 + \frac{1}{2}z_2 - \frac{1}{2}z_1 + \frac{i}{2}\cot\frac{\beta}{2}z_2$

$- \frac{i}{2}\cot\frac{\beta}{2}z_1$

$= \left(\frac{1}{2} - i\cot\frac{\beta}{2}\right)z_1 + \left(\frac{1}{2} + i\cot\frac{\beta}{2}\right)z_2$

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