SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for untidy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 90 Marks

- Attempt questions 1 - 3
- All questions are **NOT** of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 90
Attempt Questions 1 - 3
All questions are not of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (30 marks)	**Use a SEPARATE writing booklet**	**Marks**

(a) (i) For the complex number $z = \sqrt{3} - i$ find

(α) $|z|$; **1**

(β) $\arg z$. **1**

(ii) For $z = \sqrt{3} - i$, show on an Argand diagram (clearly labelled) $z, \overline{z}, z^2$ and $\frac{1}{z}$. **4**

(b) Sketch on an Argand diagram the region defined by

(i) $|\arg z| \le \frac{\pi}{4}$ and $z + \overline{z} < 6$ and $|z| > 3$ **3**

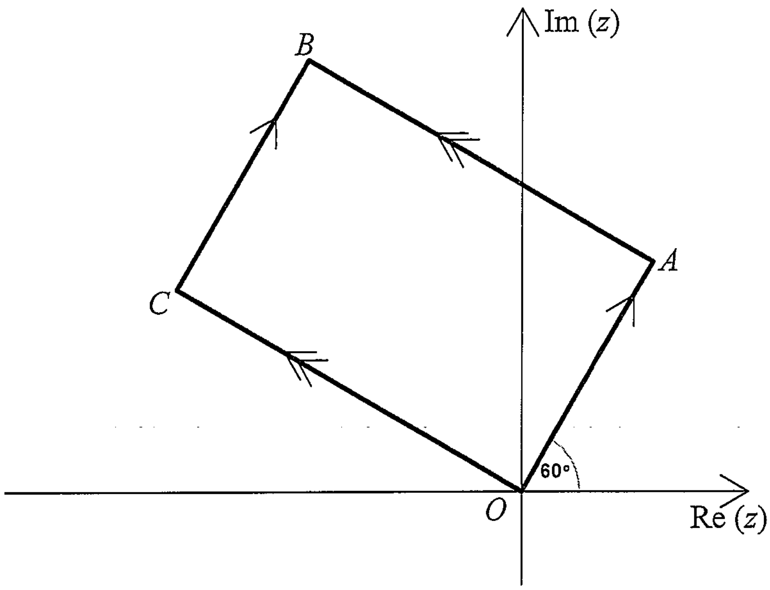
(ii) $\arg\left(\frac{z+2i}{z}\right) = -\frac{\pi}{3}$ **2**

(c) (i) Express $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ in modulus - argument form. **1**

(ii) Using (i) above, find the value(s) of n such that $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^n = 1$, where n is a positive integer. **3**

Question 1 is continued on page 3

(d)



The above figure shows a parallelogram $OABC$ in an Argand diagram.

$|OA| = 2$ and OA makes an angle of $60°$ with the positive real axis.

Let z_1, z_2 and z_3 be the complex numbers represented by vertices A, B and C respectively.

It is given that $z_3 = \left(\sqrt{3}i\right) z_1$.

(i) Find z_1 and z_3 in the form $x + iy$ **4**

(ii) Show that $\dfrac{z_2}{z_1} = 1 + \sqrt{3}i$. **4**

(iii) Let $\omega = \cos\theta + i\sin\theta$, where $0° \leq \theta < 360°$. Point E is a point on the Argand diagram representing the complex number ωz_3.

Find the value(s) of θ in each of the following cases:

(α) E represents the complex number z_3; **3**

(β) Points E, O and A lie on the same straight line. **4**

Question 2 starts on page 4

Question 2 (20 marks) **Use a SEPARATE writing booklet** **Marks**

(a) Find the constants p and q such that $x-2$ is a common factor of $x^3-x^2-2px+3q$ and qx^3-px^2+x+2. **3**

(b) $P(x)$ is a cubic polynomial with real coefficients. **5**

One zero of $P(x)$ is $1+2i$ and the constant term is -15.

Also, $P(2)=5$.

Write $P(x)$ in the form ax^3+bx^2+cx+d.

(c) Factorise $4x^4+1$ as a product of real quadratic polynomials. **3**

(d) If α and $-\alpha$ are both roots of $x^3+mx^2+nx+p=0$, show that $mn=p$. **4**

(e) (i) Explain briefly why any rational root of $x^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0=0$, must be an integer, where a_i $(i=0,1,2\ldots,n-1)$ are integers. **2**

(ii) Find the integral roots of $x^3-6x^2+6x+8=0$ and hence find all the roots **3**

Question 3 starts on page 5

Question 3 (40 marks) **Use a SEPARATE writing booklet** **Marks**

(a) (i) Evaluate $\displaystyle\int_1^5 \frac{dx}{x^2+5x+6}$ **3**

(ii) Find $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx$ **3**

(iii) Evaluate $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x^2\sqrt{1-x^2}}$ **3**

(iv) Find $\displaystyle\int x^2\cos x\,dx$ **3**

(b) (i) By means of the substitution $u = a - x$ prove that **2**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1-\sin 2x}\,dx$. **3**

(c) Let $y = x^n \sin x$, where n is a positive integer.

(i) Find $\dfrac{dy}{dx}$ **2**

(ii) Hence, show that $\displaystyle\int x^n\cos x\,dx = x^n\sin x - n\int x^{n-1}\sin x\,dx$, where $n \geq 1$. **2**

(iii) Use this result, in (ii) above, to show that $\displaystyle\int_0^{\pi} x^n\cos x\,dx = -n\int_0^{\pi} x^{n-1}\sin x\,dx$. **2**

(iv) Hence evaluate $\displaystyle\int_0^{\pi} x\cos x\,dx$. **2**

Question 3 continued on page 6

Question 3 continued **Marks**

(d) How many ways are there to split 4 red, 5 blue and 7 black balls among:

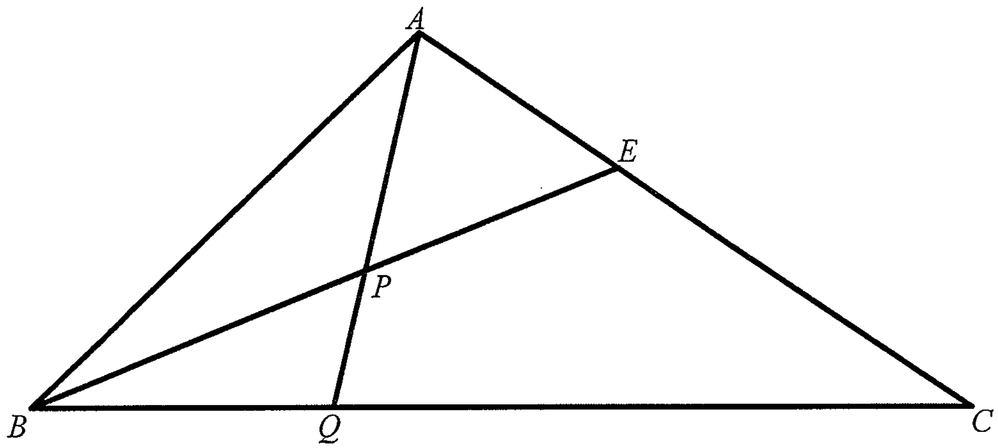
(i) Two boxes, without any restriction? **2**

(ii) Two boxes, with no box empty? **2**

(e) In the ΔABC below, BE bisects $\angle ABC$. **3**

APQ is a straight line such that $AP = AE$.

Prove that AB is a tangent to the circle that passes through the points A, Q and C.



(f) A straight line is drawn to the curve $y = x^4 - 4x^3 - 18x^2$ so that it is a common tangent at two distinct points on the curve.

(i) If the equation of the tangent is $y = mx + b$ and its points of contact are $x = p$ and $x = q$, show that

(α) $p + q = 2$; **2**

(β) $p^2q^2 = -b$. **1**

(ii) Hence, or otherwise, find the equation of the common tangent. **5**

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ \ x \neq 0,\ \text{if}\ n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ \ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

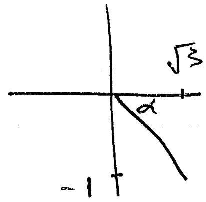
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Question 1

a) i) $z = \sqrt{3} - i$



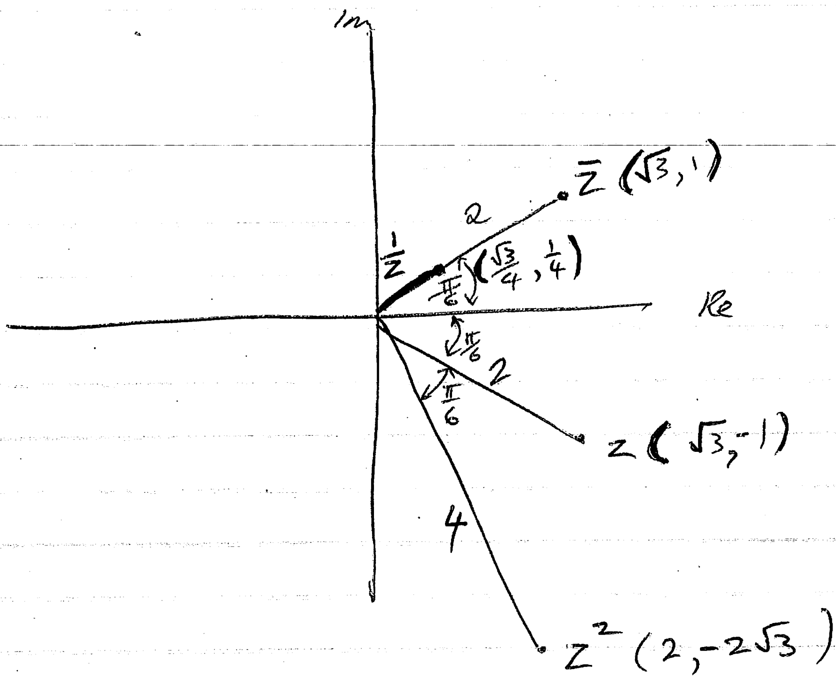
$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$

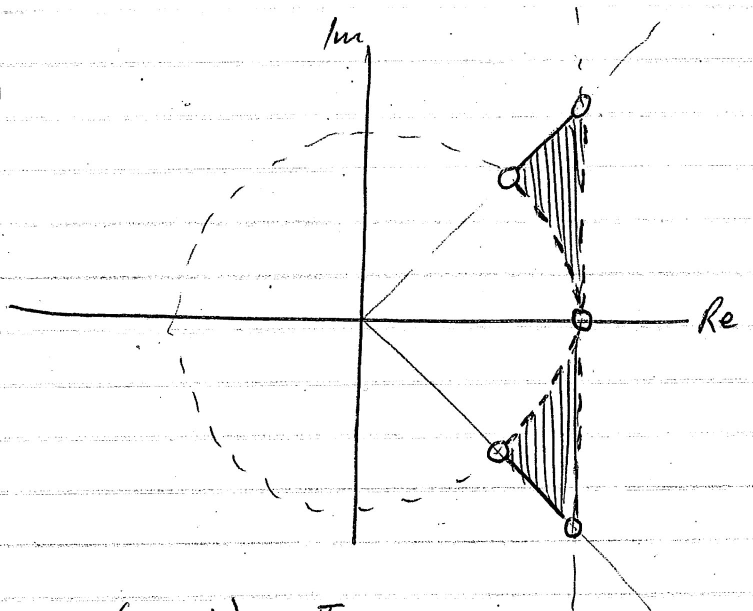
α) $|z| = 2$

β) $\arg z = -\frac{\pi}{6}$

ii)



b) i)



$|\arg z| \leq \frac{\pi}{4}$

$z + \bar{z} < 6$

$2x < 6$

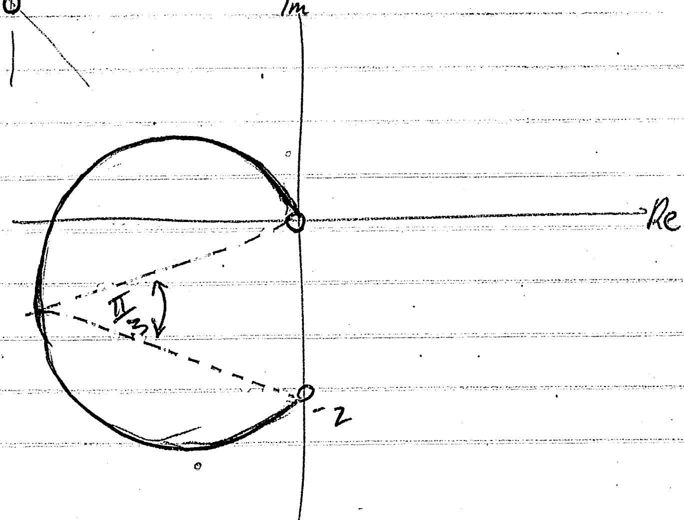
$x < 3$

$|z| > 3$

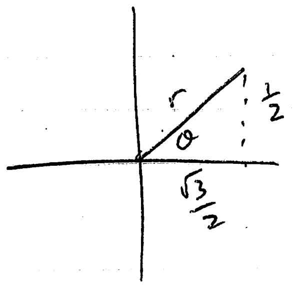
ii) $\arg\left(\frac{z+2i}{z}\right) = -\frac{\pi}{3}$

$\arg(z+2i) - \arg(z) = -\frac{\pi}{3}$

$\arg(z) - \arg(z+2i) = \frac{\pi}{3}$



c) i)



$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$r = 1$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

$\frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

ii) $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^n = 1$

$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

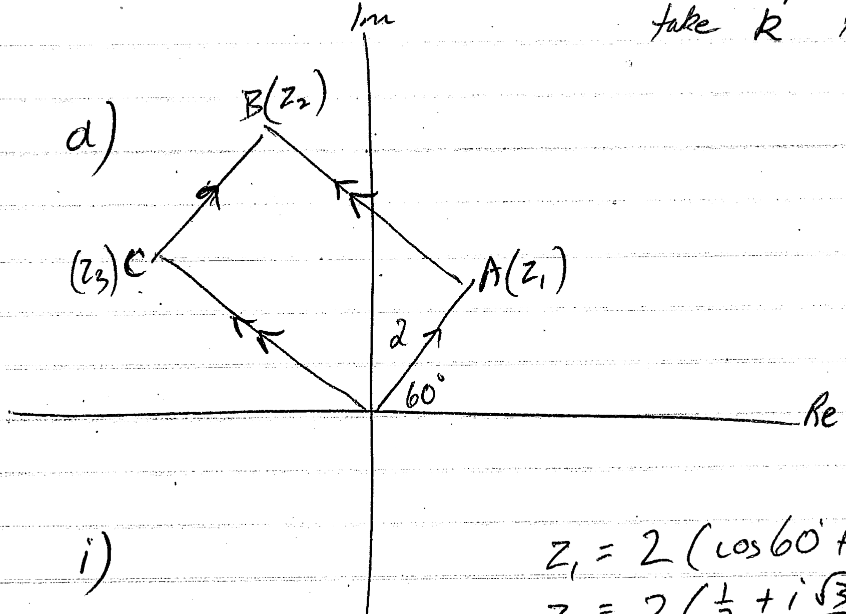
where k is an integer

$\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} = \text{cis}(0 + 2k\pi)$

$\frac{n\pi}{6} = 2k\pi$

$n = 12k$, since n needs to be a positive integer take k to be a positive integer.

d)



i)

$z_1 = 2(\cos 60° + i\sin 60°)$

$z_1 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$z_1 = 1 + \sqrt{3}i$

$$z_3 = (\sqrt{3}i)z_1$$
$$= (\sqrt{3}i)(1+\sqrt{3}i)$$
$$= -3+\sqrt{3}i$$

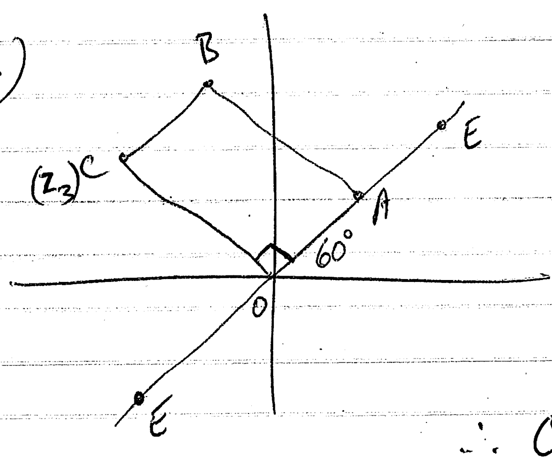
ii) $$z_2 = z_1 + z_3$$
$$= z_1 + (\sqrt{3}i)z_1$$
$$= z_1(1+\sqrt{3}i)$$

$$\frac{z_2}{z_1} = \frac{z_1(1+\sqrt{3}i)}{1+\sqrt{3}i}$$
$$= z_1$$
$$= 1+\sqrt{3}i$$

iii) α) $\omega = \cos\theta + i\sin\theta$ $\quad 0° \leq \theta < 360°$

$$\omega z_3 = z_3$$
$$\therefore \omega = 1$$
$$\cos\theta + i\sin\theta = 1$$
$$\theta = 0°$$

β)



since $\arg z_3 = 150°$

$\angle COA = 90°$

$$\therefore \theta = 90°, 270°$$

Question 2

(a) 3

$P(x) = x^3 - x^2 - 2px + 3q$

$P(2) = 2^3 - 2^2 - 4p + 3q = 0$

$\therefore -4p + 3q = -4 \quad \text{—} (1)$ 1

$Q(x) = qx^3 - px^2 + x + 2$

$Q(2) = q(2)^3 - 4p + 4 = 0$ 1

$\therefore -4p + 8q = -4 \quad \text{—} (2)$

$\Rightarrow q = 0$ and $p = 1$ 1

(b) 5 Since pol real $\Rightarrow 1-2i$ is also a zero

$\therefore x^2 - 2x + 5$ is quad. factor

$\Rightarrow P(x) = (kx-3)(x^2-2x+5)$

and $P(2) = (2k-3)(5) = 5$

$\therefore k = 2$

ie $P(x) = (2x-3)(x^2-2x+5)$

$\Rightarrow P(x) = 2x^3 - 7x^2 + 16x - 15$

(c) 3

$4x^4 + 1 = (4x^4 + 4x^2 + 1) - 4x^2$

$= (2x^2+1)^2 - (2x)^2$

$= (2x^2 - 2x + 1)(2x^2 + 2x + 1)$

(d) 4 let $\alpha, -\alpha, \beta$ be roots

sum $\alpha + (-\alpha) + \beta = -m \Rightarrow \boxed{\beta = -m}$ (1)

$\alpha(-\alpha) + \alpha\beta + (-\alpha)\beta = n \Rightarrow \boxed{\alpha^2 = -n}$ (2)

$\alpha(-\alpha)\beta = -p \Rightarrow \boxed{\alpha^2\beta = p}$ (3) 2

Subst. $\beta = -m$ in (3)

$\Rightarrow \alpha^2 = -\frac{p}{m}$ 1

From (2) $\alpha^2 = -n$

$-n = \frac{-p}{m}$ 1

$p = mn$

(e)

(i) If a pol. has a rational root $\frac{p}{q}$ then p divides the constant term and q divides the leading coeff. which in this case is $1 \Rightarrow q = 1$ 2

$\therefore$ root is p

(ii) $P(4) = 0 \Rightarrow (x-4)$ factor 1

$P(x) = (x-4)(x^2 - 2x - 2)$ 1

Roots are $x = 4$, $x = \frac{2 \pm 2\sqrt{3}}{2}$ 1 3

ie $x = 4$, $1 \pm \sqrt{3}$

Q3

(a) (i) Let $\frac{1}{x^2+5x+6} \equiv \frac{A}{x+3} + \frac{B}{x+2}$.

i.e. $1 \equiv A(x+2) + B(x+3)$

Let $x=-2$, $\boxed{1 = B}$

let $x=-3$ $1 = -A \Rightarrow \boxed{A=-1}$

Hence. $\int_1^5 \frac{dx}{x^2+5x+6} = \int_1^5 \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx$

$= \left[\ln(x+2) - \ln(x+3)\right]_1^5$

$= \ln 7 - \ln 8 - (\ln 3 - \ln 4)$

$= \ln \frac{7 \times 4}{3 \times 8}$

$= \boxed{\ln \frac{7}{6}}$.

(ii) $\int \frac{1+\sin x}{1+\cos x}\, dx = \int \frac{1+\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$ where $t = \tan\frac{x}{2}$

$= \int \frac{1+t^2+2t}{1+t^2-(1-t^2)} \times \frac{2dt}{1+t^2}$

$= \int \frac{1+t^2+2t}{2(1+t^2)} \cdot 2dt$

$= \int \left(\frac{1+t^2}{1+t^2} + \frac{2t}{1+t^2}\right) dt$.

$= \int \left(1 + \frac{2t}{1+t^2}\right) dt$.

$= t + \ln(1+t^2) + c$

$= \boxed{\tan\frac{x}{2} + \ln\left(1+\tan^2\frac{x}{2}\right) + c}$

(iii) $$\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{dx}{x^2\sqrt{1-x^2}}$$ let $x = \cos\theta$

$dx = -\sin\theta\, d\theta.$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{-\sin\theta\, d\theta.}{\cos^2\theta\sqrt{1-\cos^2\theta}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin\theta\, d\theta.}{\cos^2\theta \cdot \sin\theta}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{d\theta}{\cos^2\theta.}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2\theta\, d\theta.$$

$$= \Big[\tan\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 1 - \frac{1}{\sqrt{3}}.$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}} \quad \underline{\text{OR}}\left(\frac{3-\sqrt{3}}{3}\right)$$

(iv) $$I = \int x^2 \cos x\, dx.$$

$$= \int x^2 \frac{d}{dx}(\sin x)\, dx.$$

$$= x^2\sin x - \int 2x \sin x\, dx$$

$$= x^2\sin x - 2\int x \frac{d}{dx}(-\cos x)\, dx.$$

$$= x^2\sin x - 2\left[-x\cos x - \int -\cos x\, dx\right]$$

$$= x^2\sin x - 2\left[-x\cos x + \int \cos x\, dx\right]$$

$$= x^2\sin x - 2\left[-x\cos x + \sin x\right]$$

$$= x^2\sin x + 2x\cos x - 2\sin x + c.$$

(b) (i) R.T.P $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

$LHS = \int_0^a f(x)\,dx$ let $u = a-x$.

$\therefore x = a-u$.

$dx = -du$

$= -\int_a^0 f(a-u)\,du$.

$= \int_0^a f(a-x)\,dx$.

$= RHS$.

(ii) $\int_0^{\frac{\pi}{4}} \sqrt{1-\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{1-\sin 2(\frac{\pi}{4}-x)}\,dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{1-\sin(\frac{\pi}{2}-2x)}\,dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{1-\cos 2x}\,dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{1-(1-2\sin^2 x)}\,dx$

$= \int_0^{\frac{\pi}{4}} \sqrt{2\sin^2 x}\,dx$.

$= \sqrt{2}\int_0^{\frac{\pi}{4}} \sin x\,dx$.

$= \sqrt{2}\left[-\cos x\right]_0^{\frac{\pi}{4}}$.

$= \sqrt{2}\left[-\frac{1}{\sqrt{2}} - -1\right]$

$= -1 + \sqrt{2}$

$= \boxed{\sqrt{2}-1}$

(c) (i) $\frac{d}{dx}(x^n \sin x) = x^n \cos x + n x^{n-1} \sin x$

(ii) now $\int (x^n \cos x + n x^{n-1} \sin x)\,dx = x^n \sin x$

$\therefore \int x^n \cos x\,dx + n\int x^{n-1} \sin x\,dx = x^n \sin x$

$\therefore \int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1} \sin x\,dx$

(iii) $\therefore \int_0^\pi x^n \cos x\,dx = \left[x^n \sin x\right]_0^\pi - n\int_0^\pi x^{n-1} \sin x\,dx$

$= 0 - 0 - n\int_0^\pi x^{n-1} \sin x\,dx$

$\therefore \int_0^\pi x^n \cos x\,dx = -n\int_0^\pi x^{n-1} \sin x\,dx$

(iv). let $n = 1$

$\int_0^\pi x \cos x\,dx = -\int_0^\pi x^0 \sin x\,dx$

$= -\left[-\cos x\right]_0^\pi.$

$= \left[\cos x\right]_0^\pi$

$= \cos\pi - \cos 0$

$= -1 - 1$

$= \boxed{-2}$

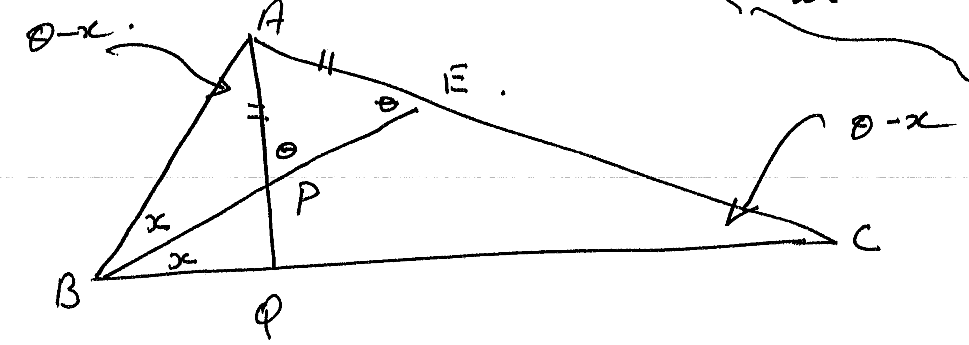
5.

(d) (i) $\binom{5}{1} \times \binom{6}{1} \times \binom{8}{1} = 5 \times 6 \times 8$
$= \boxed{240}$.

(ii) $240 - 2 = \boxed{238}$.

NB if the number of balls were a, b & c. then answer is $\binom{a+1}{1} \times \binom{b+1}{1} \times \binom{c+1}{1}$ which is not the same as ${}^{a+b+c}P_2$. so ${}^{16}P_2 = 240$ was unacceptable

(e)



Since $\triangle APE$ is isosceles

$\angle APE = \angle AEP = \theta$ (base angles of an isosceles triangle are equal.)

$\angle ABE = \angle CBE$ (data).

now $\angle BAQ = \theta - x$ (exterior angle equal to the sum of the interior opposite angles)

Similarly $\angle BCA = \theta - x$ (" ")

$\therefore \angle BAQ = \angle BCA$.

ie angle between tangent and chord is equal to the angle in the alternate segment
[NB actually the converse is required – didn't expect a proof!].

(f) (i) Let $y = x^4 - 4x^3 - 18x^2 = mx + b$.

$\Rightarrow \quad x^4 - 4x^3 - 18x^2 - mx - b = 0$

has a common tangent at two distinct points. (then these roots will be double roots)

ie. p, p, q, q.

now $S_1 = p + p + q + q = 4$ (ie. $-\frac{b}{a}$)

$2(p+q) = 4$

(α) $\boxed{p + q = 2}$ (A)

$S_4 = p \times p \times q \times q = -b$

(β) ie $\boxed{p^2 q^2 = -b}$ (B). (ie $\frac{e}{a}$)

(ii) now $S_2 = \boxed{p^2 + q^2 + 4pq = -18}$ (C). (ie $\frac{c}{a}$)

$S_3 = 2p^2 q + 2q^2 p = m$ (ie $-\frac{d}{a}$)

$2pq(p+q) = m$

$4pq = m$

$\boxed{pq = \frac{m}{4}}$

from (C) $(p+q)^2 + 2pq = -18$.

$4 + \frac{m}{2} = -18$

$\frac{m}{2} = -22$

$\boxed{m = -44}$

$\therefore pq = \frac{-44}{4} = -11 \quad \therefore (-11)^2 = -b$

$b = -121$

$\therefore \boxed{y = -44x - 121.}$