Section A – (Start a new Booklet)

Question 1. (18 marks) Marks

(a) If $u = 4 + 2i$ and $v = 3 - 5i$ find: (4)

(i) $iu + \bar{v}$

(ii) $\text{Re}(\frac{1}{v})$

(b) (i) Express each of the complex numbers $z = 2i$ and $w = 1 + \sqrt{3}i$ in modulus – argument form. (2)

(ii) Indicate the points representing the complex numbers $z = 2i$ and $w = 1 + \sqrt{3}i$ on an Argand diagram. (2)

(iii) Find the exact values of $\arg\left(\frac{z}{w}\right)$ and $\arg(z + w)$ (4)

(c) The polynomial equation $x^3 - 3x^2 + x - 5 = 0$ has roots α, β and γ. Find the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (3)

(d) Find $\int x \tan^{-1} x \, dx$ (3)

End of Question 1.

Question 2. (16 marks) Marks

(a) If α, β and γ are roots of $x^3 - 4x - 1 = 0$ find the polynomial equation with roots $2\alpha, 2\beta$ and 2γ (answer in expanded form). (3)

(b) (i) Express $\sqrt{8-6i}$ in the form a + ib where a and b are real and a > 0. (3)

(ii) Hence solve the equation (3)

$$2z^2 + (1-3i)z - 2 = 0$$

expressing the answers in the form X + iY where X and Y are real.

(c) Let $A = 1 + 2i$ and $B = 2 - i$.

On an Argand diagram sketch the locus specified by:

(i) $\arg(z+B) = \frac{3\pi}{4}$ (2)

(ii) $|z-A| = |z-B|$ (2)

(d) Find $\int \frac{x+5}{\sqrt{x+6}} dx$ (3)

End of Question 2.

END OF SECTION A.

Section B – (Start a new Booklet)

Question 3. (14 marks) Marks

(a) Find:

(i) $\int \frac{dx}{\sqrt{3+2x-x^2}}$ (2)

(ii) $\int \frac{dx}{(x+1)(x^2+4)}$ (3)

(iii) $\int \frac{e^x + e^{2x}}{1+e^{2x}}\,dx$ (3)

(b) (i) Given that (4)

$$u_n = \int_0^1 (1-x^2)^{\frac{n}{2}}\,dx$$

prove that $u_n = \frac{n}{n+1}u_{n-2}$

(ii) Evaluate u_5 (2)

End of Question 3.

Question 4. (15 marks) Marks

(a) If $z = 1 + i$ is a root of the equation

$$z^3 + pz^2 + qz + 6 = 0$$

where p and q are real, find:

(i) the values of p and q (3)

(ii) all the roots of the equation. (1)

(b) Sketch the region in the Argand diagram consisting of those points representing the complex number z which simultaneously satisfy (4)

$|\arg z| < \frac{\pi}{3}$ and $z + \bar{z} < 4$ and $|z| > 2$

(c) The complex number z represents a variable point P in the Argand plane. If $|z| = |z - \alpha|$ where α is real evaluate $\arg(z^2 - z\alpha)$ (3)

(d) Given that w is the non-real root of $z^5 - 1 = 0$ with smallest positive argument find a real quadratic equation with roots w^2 and w^3. (4)

End of Question 4.

END OF SECTION B.

Section C – (Start a new Booklet)

Question 5. (12 marks) Marks

(a) Four medical tests A, B, C and D are carried out within 14 days on a patient. A must precede B and B must precede C and D. On the days when A and B are carried out the patient must not undergo any other test, but C and D can be carried out on the same day or on different days in any order.

In how many ways can the days for the test be chosen? (4)

(b) K is a positive real number and u, v are complex numbers. Show that the points on the Argand diagram which represent respectively the numbers: (4)

$$u,\ v,\ \frac{u-iKv}{1-iK}$$

form the vertices of a right-angled triangle.

(c) The quardatic equation

$$x^2-(2\cos\theta)x+1=0$$

has roots α and β.

(i) Find expressions for α and β in terms of θ (2)

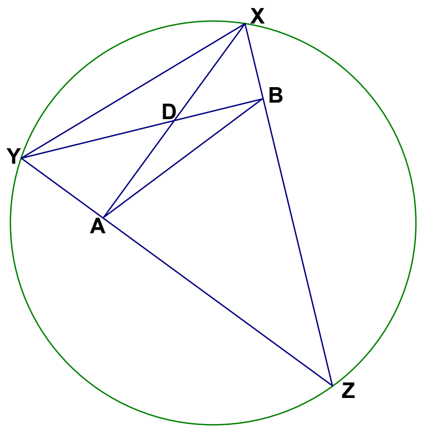
(ii) Show that $\alpha^{10}+\beta^{10}=2\cos(10\theta)$ (2)

End of Question 5.

Question 6. (15 marks) Marks

(a) When a cubic polynomial A(x) is divided by x^2+x+1 the remainder is 2x + 3. When A(x) is divided by x(x + 3) the remainder is 5(x + 1). Find the polynomial A(x) (3)

(b)



XY is a fixed chord of a circle. Z is a point on major arc XY. The perpendicular from X to the chord YZ meets YZ at A. The perpendicular from Y to the chord XZ meets XZ at B. Let $\angle XZY=\theta$.

(i) Copy the diagram into your booklet

(ii) Prove that ABXY is a cyclic quadrilateral. (1)

(iii) Show that AB = XY$\cos\theta$. (3)

(iv) Deduce that as Z moves on the major arc XY the length of AB is constant. (2)

(c) (i) Suppose that k is a double root of the polynomial equation P(x) = 0. Show that P'(k) = 0 (2)

(ii) If $x^3+3px^2+3qx+r=0$ has a double root prove that $(pq-r)^2=4(p^2-q)(q^2-pr)$ (4)

End of Question 6.

END OF SECTION C.

END OF THE EXAMINATION.

EXT 2 Task 1 2009 (1)

Question 1

(a)(i) $iu + \bar{v} = i(4+2i) + (3+5i)$

$= -2 + 4i + 3 + 5i$

$= 1 + 9i$ [2]

(ii) $Re\left(\frac{1}{v}\right) = Re\left[\frac{\bar{v}}{v\bar{v}}\right]$

$= Re\left[\frac{1}{|v|^2} \cdot \bar{v}\right]$

$= \frac{1}{|v|^2} \cdot Re(\bar{v})$

$= \frac{1}{34} \times 3$

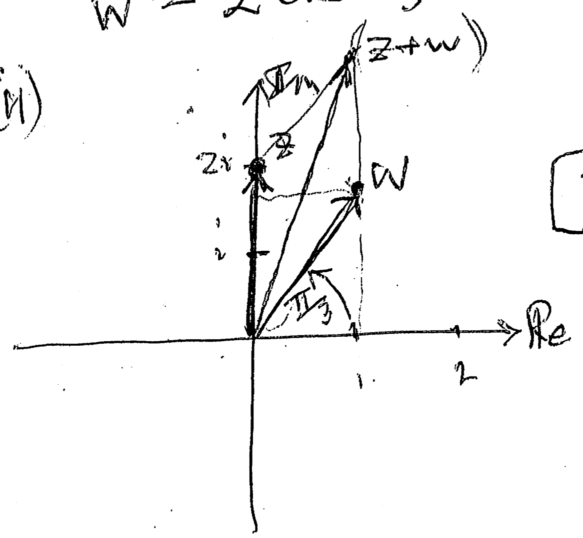
$= \frac{3}{34}$ [2]

(b) $z = 2i$, $w = 1 + \sqrt{3}i$

(i) $z = 2\operatorname{cis}\frac{\pi}{2}$ [2]

$w = 2\operatorname{cis}\frac{\pi}{3}$

(ii)



[2]

(iii) $\arg\left(\frac{z}{w}\right) = \arg z - \arg w$

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$ [2]

$\arg(z+w) = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

$= \frac{5\pi}{12}$ [2]

(c) $x^3 - 3x^2 + x - 5 = 0$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{1}{5}$ [3]

(d) $\int x \tan^{-1} x \, dx$

$= \int \frac{d}{dx}\left(\frac{x^2}{2}\right) \cdot \tan^{-1} x \, dx$

$= \frac{x^2}{2} \cdot \tan^{-1} x - \int \frac{x^2}{2} \cdot \frac{d}{dx} \tan^{-1} x \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int x^2 \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

$= \frac{1}{2}(x^2+1) \tan^{-1} x - \frac{1}{2} x + C$ [3]

Question 2

(a) $x^3 - 4x - 1 = 0$

We seek eq'n w. roots 2α etc

Let $y = 2x$; so $x = \frac{y}{2}$

Substitute:

$\left(\frac{y}{2}\right)^3 - 4\left(\frac{y}{2}\right) - 1 = 0$

$\frac{y^3}{8} - 2y - 1 = 0$

$y^3 - 16y - 8 = 0$

Replace y with x.

$x^3 - 16x - 8 = 0$ [3]

Q2 (Cont'd) (2)

(b)(i) $\sqrt{8-6i} = 3-i$ (calculator)

OR

$\sqrt{8-6i} = a+ib$

$8-6i = (a+ib)^2$

$= a^2-b^2+2iab$

$\therefore a^2-b^2=8 \; ; \; 2ab=-6$

$ab=-3$

$b=-\frac{3}{a}$

$a^2-\left(\frac{-3}{a}\right)^2=8$

$a^4-9=8a^2$

$a^4-8a^2-9=0$

$(a^2-9)(a^2+1)=0$

$\therefore a=\pm 3$

$b=\mp 1$ [3]

$\therefore \sqrt{8-6i} = 3-i$ (Principal Root)

(ii) $2z^2+(1-3i)z-2=0$

$z=\dfrac{-(1-3i)\pm\sqrt{(1-3i)^2-4(2)(-2)}}{4}$

$=\dfrac{-1+3i\pm\sqrt{-8-6i+16}}{4}$

$=\dfrac{-1+3i\pm\sqrt{8-6i}}{4}$

$=\dfrac{-1+3i\pm(3-i)}{4}$

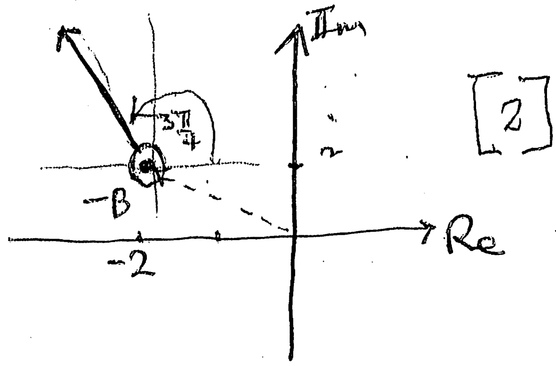
$=\frac{1}{4}(2+2i), \; \frac{1}{4}(-4+4i)$

$=\frac{1}{2}+\frac{1}{2}i, \; -1+i$ [3]

(C) $A=1+2i \; ; \; B=2-i$

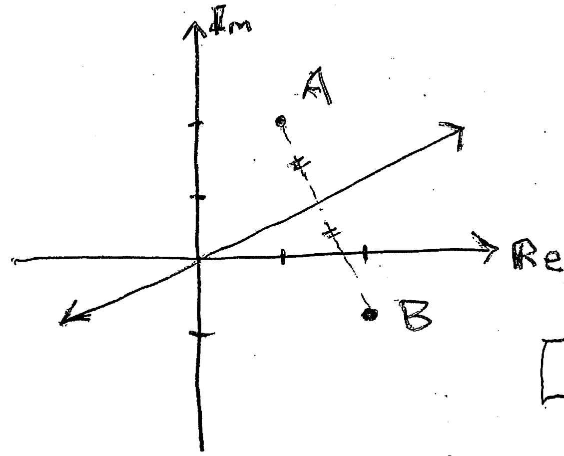
(i) $\arg(z+B)=\frac{3\pi}{4}$

ie $\arg(z-(-B))=\frac{3\pi}{4}$



[2]

(ii) $|z-A|=|z-B|$



[2]

(d) $\int \frac{x+5}{\sqrt{x+6}}\,dx$ Let $u=x+6$, $du=dx$

$=\int \frac{u-1}{\sqrt{u}}\,du$

$=\int (u^{1/2}-u^{-1/2})\,du$

$=\frac{u^{3/2}}{3/2}-\frac{u^{1/2}}{1/2}+C$

$=\frac{2}{3}(x+6)^{3/2}-2(x+6)^{1/2}+C$

$=2\sqrt{x+6}\left(\frac{x}{3}+1\right)+C$ [3]

Solutions to Section (B) Ext (II) Assessment Task (1)

Question (3). (14 mks)

(i) $\int \frac{dx}{\sqrt{3+2x-x^2}}$

Now, $-(x^2-2x-3)$
$= -[(x-1)^2-4]$
by completing square

$\int \frac{dx}{\sqrt{4-(x-1)^2}}$ (2)

$= \sin^{-1}\left(\frac{x-1}{2}\right)+c$.

(ii) let
$\frac{1}{(x+1)(x^2+4)} = \frac{a}{x+1}+\frac{bx+c}{x^2+4}$.

$1 = a(x^2+4)+(bx+c)(x+1)$

put $x=-1$, $1=5a$

$\therefore a = \frac{1}{5}$

equate coefficients of x^2: $a+b=0$

$\Rightarrow b=-a=-1/5$

• put $x=0$
$1=4a+c \Rightarrow c = 1-4/5 = 1/5$

$\therefore \frac{1}{(x+1)(x^2+4)} = \frac{1}{5(x+1)} - \frac{x}{5(x^2+4)} + \frac{1}{5(x^2+4)}$.

$\therefore \int \frac{dx}{(x+1)(x^2+4)}$

$= \frac{1}{5}\int\left(\frac{1}{x+1}-\frac{x}{x^2+4}+\frac{1}{x^2+4}\right)dx$

$= \frac{1}{5}\ln|x+1| - \frac{1}{10}\ln(x^2+4) + \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right)+c$ (3)

(iii) $\int \frac{e^x+e^{2x}}{1+e^{2x}}dx$ "splitting"

$= \int \frac{e^x\,dx}{1+(e^x)^2} + \int \frac{e^{2x}\,dx}{1+e^{2x}}$

Let $u=e^x$, $du = e^x dx$

$\int \frac{du}{1+u^2} = \tan^{-1}(e^x)$.

Now, $\int \frac{e^{2x}dx}{1+e^{2x}}$

Let $u = 1+e^{2x}$
$du = 2e^{2x}dx$.
$= \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln u$
$= \frac{1}{2}\ln(1+e^{2x})$

$\therefore \int \frac{e^x+e^{2x}}{1+e^{2x}}dx$ (3)
$= \tan^{-1}(e^x)+\frac{1}{2}\ln(1+e^{2x})+c$

(b) (i) $u_n = \int_0^1 (1-x^2)^{n/2}dx$

From integration by parts:

$u_n = \int_0^1 (1-x^2)^{n/2}\frac{d}{dx}(x)\,dx$

$= \left[x(1-x^2)^{n/2}\right]_0^1 - \int_0^1 x\cdot\frac{n}{2}(1-x^2)^{n/2-1}(-2x)\,dx$

$= n\int_0^1 x^2(1-x^2)^{n/2-1}dx$

$= -n\int_0^1 -x^2(1-x^2)^{n/2-1}dx$.

3(b)(i) (cont.)

$= -n\int_0^1(1-x^2)(1-x^2)^{n/2-1}dx$
$+ n\int_0^1(1-x^2)^{n/2-1}dx$ (4)

$u_n = -n\int_0^1(1-x^2)^{n/2}dx + n\int_0^1(1-x^2)^{\frac{n-2}{2}}dx$

i.e. $u_n = -nu_n + nu_{n-2}$

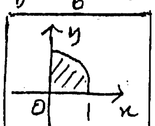
$\therefore (n+1)u_n = nu_{n-2}$

i.e. $u_n = \left(\frac{n}{n+1}\right)u_{n-2}$

(b) (ii) Now, $u_5 = \frac{5}{6}u_3$

$u_3 = \frac{3}{4}u_1$

$u_1 = \int_0^1\sqrt{1-x^2}\,dx$



$= \pi/4$ (from diagram)

(2) $\therefore u_3 = \frac{3\pi}{16} \Rightarrow u_5 = \frac{5\times3\pi}{6\times16}$

$\therefore u_5 = \frac{15\pi}{96} = \frac{5\pi}{32}$

Question (4) (15 Mks)

(a) If $1+i$ is a root, then
$(1+i)^3+p(1+i)^2+q(1+i)+6=0$

$(1+i)^2=2i$, $(1+i)^3=-2+2i$

$(q+4)+(2+2p+q)i=0$

i.e. $q+4=0$, $q=-4$

or $2+2p+q=0$ (3)

$\Rightarrow 2p=-2-q$
$=-2+4$
$=2$

$\therefore p=1,\ q=-4$.

$\therefore z^3+z^2-4z+6=0$ is the equation. (1)

$\Sigma\alpha_i=-1$

$(1+i)+(1-i)+\beta=-1$

$\therefore 2+\beta=-1$

$\Rightarrow \beta=-3$

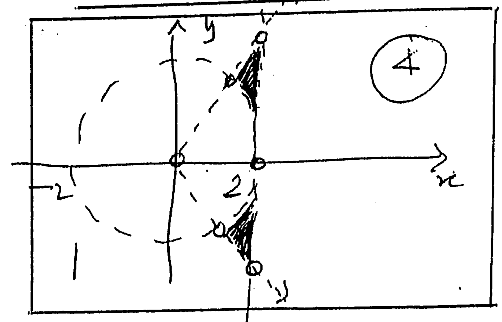
$\therefore$ roots are $1\pm i,\ -3$ (1)

(b) • $|\arg z|<\frac{\pi}{3}$

$-\frac{\pi}{3}<\arg z<\frac{\pi}{3}$

• $z+\bar z<4 \Rightarrow 2x<4$
$x<2$

• $|z|>2$



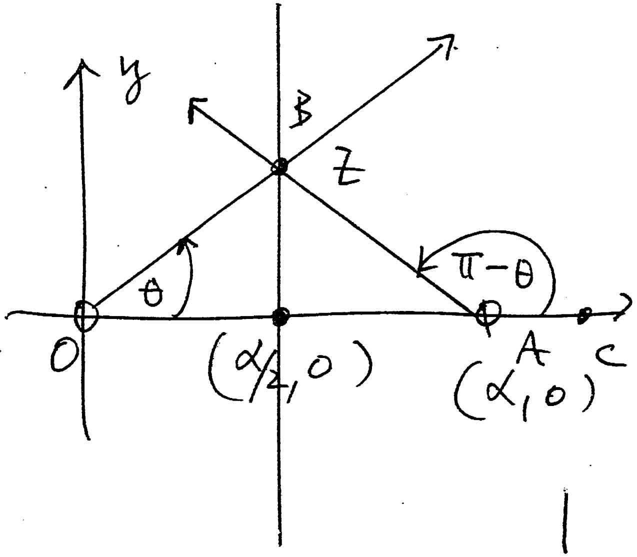
(4)

(c) $|z|=|z-\alpha|$. (3)

$\therefore$ locus is the perp. bisector joining $(0,0)$ and $(\alpha,0)$.

$\arg(z^2-\alpha z)=\arg[z(z-\alpha)]$

Question 4(c) (Cont.)



Let $\arg Z = \theta$

$\therefore \triangle OAB$ is Isosceles.

$\therefore \angle BAO = \theta$

i.e $\angle CAB = \pi - \theta$.

$\therefore \arg Z + \arg(Z - \alpha)$

$= \theta + \pi - \theta$

$= \pi$.

i.e $\boxed{\arg(Z^2 - \alpha Z) = \pi}$

(4)

(d) $Z^5 - 1 = 0$

$\therefore Z^5 = 1 = \cos(2k\pi) + i\sin(2k\pi)$

$$\therefore \boxed{Z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right) \quad k = 0, 1, 2, 3, 4.}$$

$Z_1 = \text{cis}(0) = 1$

$\omega = Z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$\omega^2 = Z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$\omega^3 = Z_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$= \text{cis}\left(2\pi - \frac{4\pi}{5}\right)$

$= \text{cis}\left(-\frac{4\pi}{5}\right)$

$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$\omega^4 = Z_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

$= \text{cis}\left(2\pi - \frac{2\pi}{5}\right)$

$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$

$$\boxed{\omega^2 + \omega^3 = 2\cos\frac{4\pi}{5}}$$

$$\boxed{\omega^2 \cdot \omega^3 = \omega^{2+3} = \omega^5 = 1}$$

$Z^2 - (\omega^2 + \omega^3) Z + (\omega^2 \cdot \omega^3) = 0$

$\therefore$ real quad. equation whose roots are ω^2, ω^3 is

$$\boxed{Z^2 - \left(2\cos\frac{4\pi}{5}\right) Z + 1 = 0}$$

2009 Mathematics Extension 2 Assessment 1: **Section C** solutions

5. (a) Four medical tests A, B, C, and D are carried out within fourteen days on a patient. A must precede B and B must precede C and D. On the days when A and B are carried out the patient must not undergo any other tests, but C and D can be carried out on the same day or on different days in any order. 4

In how many ways can the days for the tests be chosen?

Solution: Method 1:

Case 1— consider when C and D are on different days. Ways of choosing the days is $\binom{14}{4}$. Then we know that the first day is A and the second is B so we have to order C and D, *i.e.* 2! ways. So the number of ways if 4 different days is $\binom{14}{4} \times 2!$.

Case 2— consider when C and D are on the same day. Ways of choosing the days is $\binom{14}{3}$.

$$\begin{aligned} \therefore \text{ Total ways} &= \binom{14}{4} \times 2! + \binom{14}{3}, \\ &= 2366. \end{aligned}$$

Method 2:

A	B												

If A and B are on the first 2 days, C can be in any of 12 places AND D can be in any of 12 places $\Longrightarrow$ Ways $= 12^2$.

	A	B											

If A and B are on the second 2 days, C can be in any of 11 places AND D can be in any of 11 places $\Longrightarrow$ Ways $= 11^2 \ldots$

Hence the total number of ways with A and B together is $12^2 + 11^2 + 10^2 + \cdots + 2^2 + 1 = 650$.

A		B											

If A and B are over the first 3 days, C can be in any of 11 places AND D can be in any of 11 places $\Longrightarrow$ Ways $= 11^2$.

	A		B										

If A and B are over the second 3 days, C can be in any of 10 places AND D can be in any of 10 places $\Longrightarrow$ Ways $= 10^2 \ldots$

Hence the total number of ways with A and B over three days is $11^2 + 10^2 + 9^2 \cdots + 2^2 + 1 = 506$.

$\vdots$

Therefore, the total number of ways with A and B in all possible positions is $650 + 506 + 385 + \cdots + 5 + 1 = 2366$.

(b) K is a positive real number and u, v are complex numbers. Show that the points on the Argand diagram which represent respectively the numbers: [4]

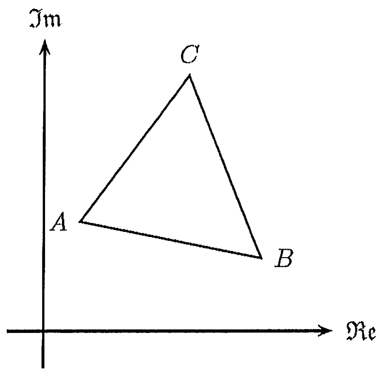
$$u,\ v,\ \frac{u - iKv}{1 - iK}$$

form the vertices of a right-angled triangle.

Solution: Let A, B, C be the points u, v, $\dfrac{u-iKv}{1-iK}$ respectively on an Argand diagram.

$$\begin{aligned}
\overrightarrow{AB} &= v - u.\\
\overrightarrow{BC} &= \frac{u - iKv}{1 - iK} - v,\\
&= \frac{u - iKv - v + iKv}{1 - iK},\\
&= \frac{u - v}{1 - iK}.\\
\overrightarrow{AC} &= \frac{u - iKv}{1 - iK} - u,\\
&= \frac{u - iKv - u + iKu}{1 - iK},\\
&= \frac{iK(u - v)}{1 - iK}.\\
\therefore\ \overrightarrow{AC} &= iK\ \overrightarrow{BC}\ .
\end{aligned}$$

i.e., $\overrightarrow{AC}$ is rotated 90° from $\overrightarrow{BC}$.



(c) The quadratic equation

$$x^2 - (2\cos\theta)x + 1 = 0$$

has roots α and β.

(i) Find expressions for α and β in terms of θ. [2]

Solution: Using the quadratic equation to find x,

$$\begin{aligned}
x &= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2},\\
&= \cos\theta \pm \sqrt{-(1 - \cos^2\theta)},\\
&= \cos\theta \pm i\sqrt{\sin^2\theta},\\
&= \cos\theta \pm i\sin\theta.\\
\text{So } \alpha &= \cos\theta + i\sin\theta,\\
\text{and } \beta &= \cos\theta - i\sin\theta,\\
&= \cos(-\theta) + i\sin(-\theta).
\end{aligned}$$

(ii) Show that $\alpha^{10} + \beta^{10} = 2\cos(10\theta)$. [2]

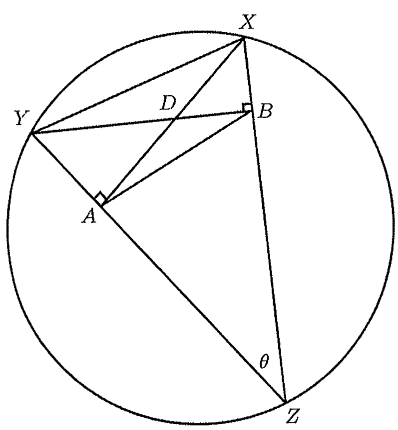
Solution:

$$\begin{aligned}
\alpha^{10} &= (\cos\theta + i\sin\theta)^{10},\\
&= \cos 10\theta + i\sin 10\theta \text{ by deMoivre's theorem},\\
\beta^{10} &= \big(\cos(-\theta) + i\sin(-\theta)\big)^{10},\\
&= \cos(-10\theta) + i\sin(-10\theta) \text{ by deMoivre's theorem},\\
\therefore\ \alpha^{10} + \beta^{10} &= \cos 10\theta + i\sin 10\theta + \cos(-10\theta) + i\sin(-10\theta),\\
&= \cos 10\theta + i\sin 10\theta + \cos 10\theta - i\sin 10\theta,\\
&= 2\cos(10\theta).
\end{aligned}$$

6. (a) When a cubic polynomial $A(x)$ is divided by x^2+x+1 the remainder is $2x+3$. When $A(x)$ is divided by $x(x+3)$ the remainder is $5(x+1)$. Find the polynomial $A(x)$. [3]

Solution:
$$\begin{aligned} A(x) &= (x^2+x+1)(ax+b)+2x+3,\\ &= x(x+3)(cx+d)+5(x+1),\\ \text{Put } x &= 0,\\ b+3 &= 5,\\ b &= 2. \end{aligned}$$
Equating coefficients of x^3, $a=c$.
Equating coefficients of x^2, $a+2=3c+d$ or $2a=2-d$.
Equating coefficients of x, $2+a+2 = 3d+5$,
$$a = 3d+1.$$
$$\begin{aligned} \text{Now } 2a &= 2-\frac{a-1}{3},\\ 6a &= 6-a+1,\\ 7a &= 7,\\ a=c &= 1.\\ \therefore\ A(x) &= (x^2+x+1)(x+2)+2x+3,\\ &= x^3+x^2+x+2x^2+2x+2+2x+3,\\ &= x^3+3x^2+5x+5. \end{aligned}$$

(b)



XY is a fixed chord of a circle. Z is a point on the major arc XY.
The perpendicular from X to the chord YZ meets YZ at A.
The perpendicular from Y to the chord XZ meets XZ at B.
Let $\angle XZY = \theta$.

(i) Copy the diagram into your booklet.

Solution: See above.

(ii) Prove $ABXY$ is a cyclic quadrilateral. [1]

Solution:
$$\begin{aligned} \angle XAY &= 90^\circ\ (XA \perp YZ),\\ \angle YBX &= 90^\circ\ (YB \perp XZ),\\ \therefore\ \angle XAY &= \angle YBX. \end{aligned}$$
$\therefore$ $ABXY$ is a cyclic quadrilateral (angles on arc XY are equal).

(iii) Show that $AB = XY\cos\theta$. [3]

Solution:
$$\begin{aligned}\angle ABZ &= \angle XYZ \text{ (exterior } \angle \text{ theorem for cyclic quad.)}\\ \angle BZA &= \angle YZX \text{ (common)},\\ \therefore\ \triangle ABZ &\,|||\, \triangle XYZ \text{ (equiangular)},\\ \therefore\ \frac{AB}{XY} &= \frac{ZA}{ZX} \text{ (corresp. sides of similar } \triangle\text{s)},\\ \text{but } \frac{ZA}{ZX} &= \cos\theta,\\ \therefore\ \frac{AB}{XY} &= \cos\theta,\\ \text{so } AB &= XY\cos\theta.\end{aligned}$$

(iv) Deduce that as Z moves on the major arc XY the length of AB is constant. [2]

Solution: XY is constant (given),
θ is constant ($\angle$s at circumf. on same arc).
$\therefore\ \cos\theta$ is constant,
$$\begin{aligned}\therefore\ AB &= \text{constant} \times \text{constant},\\ &= \text{constant}.\end{aligned}$$

(c) (i) Suppose that k is a double root of the polynomial equation $P(x) = 0$. Show that $P'(k) = 0$. [2]

Solution: Put $P(x) = (x-k)^2Q(x) = 0$, where $Q(x)$ is a polynomial.
$$\begin{aligned}P'(x) &= 2(x-k)Qx + (x-k)^2Q'(x),\\ &= (x-k)\{2Q(x) + (x-k)Q'(x)\}.\\ \therefore\ P'(k) &= (k-k)\{2Q(k) + (k-k)Q'(k)\},\\ &= 0.\end{aligned}$$

(ii) If $x^3 + 3px^2 + 3qx + r = 0$ has a double root, prove that $(pq - r)^2 = 4(p^2 - q)(q^2 - pr)$. [4]

Solution:
$$\begin{aligned}\text{Let } P(x) &= x^3 + 3px^2 + 3qx + r = 0 \quad \boxed{1}\\ P'(x) &= 3x^2 + 6px + 3q = 0,\\ \text{i.e., } x^2 + 2px + q &= 0 \quad \boxed{2}\\ x \times \boxed{2}:\quad x^3 + 2px^2 + qx &= 0 \quad \boxed{3}\\ \boxed{1} - \boxed{3}:\quad px^2 + 2qx + r &= 0, \quad \boxed{4}\\ p \times \boxed{2}:\quad px^2 + 2p^2x + qp &= 0, \quad \boxed{5}\\ \boxed{5} - \boxed{4}:\quad 2(p^2 - q)x + qp - r &= 0,\\ x &= \frac{r - pq}{2(p^2 - q)}.\\ \text{Sub. in } \boxed{2}:\quad \frac{(r-pq)^2}{4(p^2-q)^2} + \frac{2p(r-pq)}{2(p^2-q)} + q &= 0,\\ \frac{(pq-r)^2}{4(p^2-q)^2} &= \frac{p(pq-r)}{p^2-q} - q,\\ (pq-r)^2 &= 4(p^2-q)(pq-r)p - 4q(p^2-q)^2,\\ &= 4(p^2-q)(p^2q - pr - p^2q + q^2),\\ &= 4(p^2-q)(q^2-pr).\end{aligned}$$