SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question.

Total Marks – 60

- Attempt questions 1 – 3
- All questions are of equal value.

Examiner: *A.M.Gainford*

Question 1. (Start a new page.) (20 marks)

Marks

(a) For the complex number $z = -2 + 2\sqrt{3}i$ find: **2**

(i) $|z|$

(ii) $\arg z$.

(b) Given $w = 5 + 12i$ express each of the following in the form $a + ib$ (for real a and b). **2**

(i) $\overline{w}$

(ii) w^{-1}

(c) Find the square roots of $32 - 24i$, giving your answer in the form $x + iy$. **2**

(d) Find in modulus-argument form all complex numbers z such that $z^3 + 1 = 0$, and plot them on the Argand diagram. **3**

(e) Sketch (on separate diagrams) the region in the Argand diagram containing the points z for which: **6**

(i) $0 \le \arg(z - i) \le \dfrac{\pi}{6}$

(ii) $|z - 2i| < 2$

(iii) $\arg\left(\dfrac{z - i}{z - 1}\right) = \dfrac{\pi}{2}$

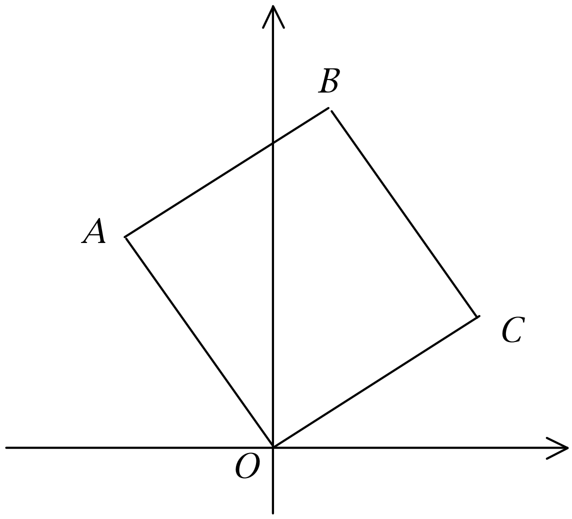
(f) Solve the equation $z^5 + z = 0$ over the field of complex numbers. **2**

(g) If α, β, γ are the roots of the equation $2x^3 + 5x^2 - 4x + 2 = 0$, find a cubic equation with roots $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$, and hence or otherwise state the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **3**

Question 2. (Start a new page.) (20 marks)

Marks

(a) In the Argand diagram at right $OABC$ is a square. The point A represents the complex number $-3+4i$. **3**



Find in the form $a+ib$:

(i) the complex number represented by C

(ii) the complex number represented by B

(iii) the complex number represented by the vector CA

(b) Find $\frac{d}{dx}\left(x\sin^{-1}x\right)$, and hence find $\int \sin^{-1}x\,dx$. **3**

(c) Given that $1+i$ is a root of the equation $x^4 - x^3 - 6x^2 + 14x - 12 = 0$: **4**

(i) explain why $1-i$ is also a root

(ii) find all roots of the equation.

(d) Prove that $\cos^{-1}x + \sin^{-1}x$ is a constant (independent of x). **2**

(e) Solve the quadratic equation $x^2 - (2+2i)x - 1 + 2i = 0$. **2**

(e) (i) Determine the domain of the function $y = \sin^{-1}(2x+1)$. **4**

(ii) Sketch the graph of $y = \sin^{-1}(2x+1)$.

(iii) Solve $\sin^{-1}(2x+1) = \cos^{-1} x$

(f) Which complex numbers are the squares of their conjugates? **2**

Question 3. (Start a new page.) (20 marks)

Marks

(a) (i) Find the least positive integer k such that $\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)$ is a solution of $z^k = 1$. **2**

(ii) Show that if the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are arbitrary integers. **1**

(b) When the polynomial $P(x) = x^3 + 3x^2 + ax + b$ is divided by $x-1$ the remainder is 5, and when $P(x)$ is divided by $x+2$ the remainder is 8. **2**

Find the values of a and b.

(c) (i) Find the co-ordinates of intersection of the curves $y = \cos^{-1} x$ and $y = \sin^{-1} x$. **4**

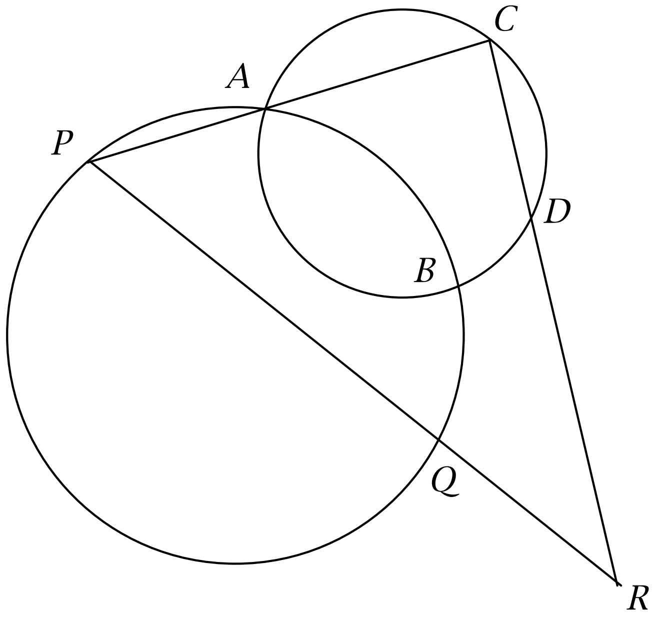
(ii) Find the acute angle between the curves $y = \cos^{-1} x$ and $y = \sin^{-1} x$ at the point of intersection. Give your answer in degrees, to the nearest minute.

(d) (i) Show that if $x = \alpha$ is a double root of the polynomial equation $A(x) = 0$, then $A(\alpha) = A'(\alpha) = 0$. **1**

(ii) Given that $ax^4 + 4bx + c = 0$ has a double root, prove that $ac^3 = 27b^4$. **2**

(e) If z is a non-zero complex number such that $z + \frac{1}{z}$ is real, prove that either $\text{Im}(z) = 0$ or $|z| = 1$. **3**

(f) In the diagram below *PAC, CDR*, *PQR* are straight lines. **5**



(i) Copy the diagram to your answer booklet.

(ii) Prove that $\angle RQD = \angle RBD$.

This is the end of the paper.

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x,\ x > 0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

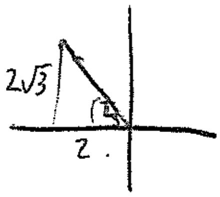
NOTE: $\ln x = \log_e x,\ \ x > 0$

QUESTION 1

(a) (i) $|-2+2\sqrt{3}i| = \sqrt{4+12}$

$= 4.$ 1

(ii)



$\arg(-2+2\sqrt{3}i) = \frac{2\pi}{3}.$ 1

(b)(i) $5-12i$ 1

(ii) $\frac{1}{5+12i} \times \frac{5-12i}{5-12i} = \frac{5-12i}{25+144}$

$= \frac{5}{169} - \frac{12}{169}i$ 1

(c) $32-24i = x^2-y^2+2xyi.$

$32 = x^2-y^2.$

$-24 = 2xy \Rightarrow y = -\frac{12}{x}.$ $\Rightarrow 144 = x^2y^2. \Rightarrow y^2 = \frac{144}{x^2}.$

$32 = x^2 - \frac{144}{x^2}.$

$x^4 - 32x^2 - 144 = 0.$

$x^2 = \frac{32 \pm \sqrt{1024 - 4 \times -144}}{2}.$

$= \frac{32 \pm 40}{2}.$

$= 36, -4.$

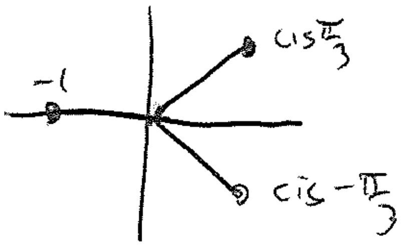
$x = \pm 6.$

when $x = 6$ $y = -2.$

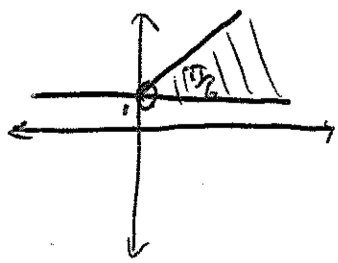
" $x = -6$ $y = 2.$

$\underline{-6+2i}, \underline{6-2i}$ 2

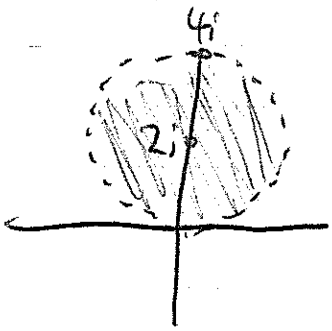
d) $\text{cis} \pm \frac{\pi}{3}$, $\text{cis}\,\pi = -1$.



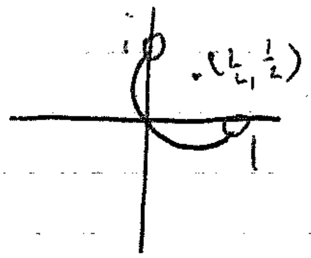
(e) (i)



(ii)



(iii)



(f) $z = 0$, $\text{cis} \pm \frac{\pi}{4}$, $\text{cis} \pm \frac{3\pi}{4}$.

(g) $x = \frac{1}{X}$

$2\left(\frac{1}{X}\right)^3 + 5\left(\frac{1}{X}\right)^2 - 4\left(\frac{1}{X}\right) + 2 = 0.$

$2X^3 - 4X^2 + 5X + 2 = 0.$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2.$

QUESTION 2

(a) (I) $(-3+4i)\div i = \quad 4+3i$

(ii) $-3+4i+4+3i = 1+7i$

(iii) $-3+4i-(4+3i) = -7+i$

(b) $$\frac{d}{dx}\left(x\sin^{-1}x\right) = x\times\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

$$\int\left(\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}\right)dx = x\sin^{-1}x$$

$$\int\left(\sin^{-1}x\right)dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{dx}$$

(c) (i) If a comlex number is a factor of a polynomial with real coefficients, its conjugate is also a factor.

(ii) $(x-1+i)(x-1-i)$ is a factor.

(x^2-2x+2) is a factor

$$\begin{array}{r|l} & x^2+x-6 \\ x^2-2x+2 & x^4-x^3-6x^2+14x-12 \\ & x^4-2x^3+2x^2 \\ & \text{-------} \\ & \quad x^3-8x^2+14x \\ & \quad x^3-2x^2+2x \\ & \quad \text{-----------} \\ & \qquad -6x^2+12x-12 \\ & \qquad -6x^2+12x-12 \\ & \qquad \text{-------------} \end{array}$$

Roots 1+i , 1-i, -3,2

(d)

$a=\cos^{-1}x, b=\sin^{-1}x$

$\cos a = x, \quad \sin b = x$

$\cos a = \sin b$

$a+b=\dfrac{\pi}{2}$

$\cos^{-1}x+\sin^{-1}x=\dfrac{\pi}{2}$ - constant

-

(e) $$x=\frac{2+2i\pm\sqrt{(-2-2i)^2-4\times1\times(-1-2i)}}{2}$$

$$=\frac{2+2i\pm\sqrt{4+8i-4+4-8i}}{2}$$

$$=\frac{2+2i\pm2}{2}$$

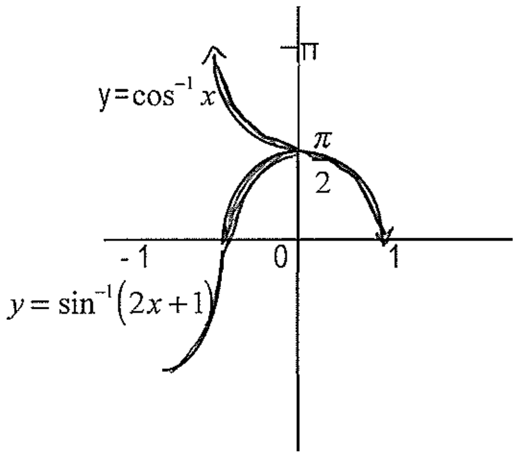
$x = 2+i \text{ or } i$

(f) (i)

Domain $-1\le 2x+1\le 1$

$-2\le 2x\le 0$

$-1\le x\le o$



(f) (ii) From graph x =0

(g)

$x+iy=(x-iy)^2$

$= x^2-2xyi-y^2$

(1) $x=x^2-y^2$

$y=-2xy$

$1=-2x$

$x=-\dfrac{1}{2}$

From (1) $-\dfrac{1}{2}=\dfrac{1}{4}-y^2$

$y=\pm\dfrac{\sqrt{3}}{2}$

nos. are: 0, 1, $-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$

Question (3)

(a)

(i) $Z^K = 1$

Let $Z = \text{cis}\,\theta$

$Z^K = \cos K\theta + i\sin K\theta$

~~Equate~~ real part

$\cos K\theta = 1$

$\Longrightarrow K\theta = 2n\pi$ [2]

$\therefore \theta = \frac{2n\pi}{K}$

i.e. $\theta = 0, \frac{2\pi}{K}, \frac{4\pi}{K} \ldots$

$\Longrightarrow K = 7$

is the least positive integer.

(ii) W is a solution to $Z^n = 1$

$\therefore W^n = 1.$

Now $W^m = (W^n)^m$

$= 1^m$

$= 1$ [1]

$\Longrightarrow W^m$ is also a solution

(b) $P(x) = x^3 + 3x^2 + ax + b$

$P(1) = 5$

$P(-2) = 8$ [2]

$5 = 1 + 3 + a + b$

$8 = -8 + 12 - 2a + b$

$\therefore a + b = 1$ —— (1)

$-2a + b = 4$ —— (2)

(1) − (2) $3a = -3$

$\therefore a = -1$ —— (3)

Subst (3) into (1)

$\therefore b = 2$ —— (4)

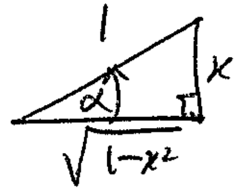
(c) (i) $\cos^{-1}x = \sin^{-1}x$

i.e. $\sin^{-1}x - \cos^{-1}x = 0$ —— (1)

Let $\alpha = \sin^{-1}x$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

$\beta = \cos^{-1}x$, $0 \le \beta \le \pi$

$\Longrightarrow \sin\alpha = x.$



$\cos\alpha = \sqrt{1-x^2}$

$\cos\beta = x$

$\sin\beta = \sqrt{1-x^2}.$

$\alpha - \beta = 0$

2

Take $\sin(\alpha - \beta) = 0$

$\sin\alpha\cos\beta - \cos\alpha\sin\beta = 0$

$x^2 - (\sqrt{1-x^2})^2 = 0$

$2x^2 = 1, \quad x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}, \quad \therefore x = \frac{1}{\sqrt{2}}$

$\therefore \left(\frac{1}{\sqrt{2}}, \pi/4\right) \quad y = \pi/4$

(c)

(ii) $y_1 = \cos^{-1} x$ [4]

$m_1 = \frac{dy_1}{dx} = -\frac{1}{\sqrt{1-x^2}}$

$m_2 = \frac{dy_2}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\left.\frac{dy_1}{dx}\right|_{x=\frac{1}{\sqrt{2}}} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$

$\left.\frac{dy_2}{dx}\right|_{x=\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{1/2}} = \sqrt{2}$

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$ 2

$= \left|\frac{-\sqrt{2}-\sqrt{2}}{1+(-2)}\right|$

$= 2\sqrt{2}$

$\therefore \theta = \tan^{-1} 2\sqrt{2} = 70^\circ 32'$

(d) If $x = \alpha$

(i) is a double root of $A(x)$

then

$A(x) = (x-\alpha)^2 Q(x)$

$\therefore A(\alpha) = (\alpha-\alpha)^2 Q(\alpha)$

$= 0$

$A'(x) = 2Q(x)(x-\alpha) + (x-\alpha)^2 Q'(x)$

$\therefore A'(\alpha) = 2Q(\alpha)(\alpha-\alpha) + (\alpha-\alpha)^2 Q'(\alpha) = 0$

$\therefore A(\alpha) = A'(\alpha) = 0$ [1]

(ii) $P(x) = ax^4 + 4bx + c$

$P(\alpha) = 0$ and $P'(\alpha) = 0$

$a\alpha^4 + 4b\alpha + c = 0$ —— ①

and $4a\alpha^3 + 4b = 0$ —— ②

From ② $\alpha^3 = -\frac{b}{a}$

—— ③

From ①

$a(\alpha^3)(\alpha) + 4b(\alpha) + c = 0$

$-b\alpha + 4\alpha = -c$

$\therefore \alpha = -\frac{c}{3b}$

—— ④

Subst ④ into ①

$a\left(\frac{c^4}{81b^4}\right) - 4b\left(\frac{c}{3b}\right) + c = 0$

$\Longrightarrow ac^3 = 27b^4$ [2]

(e) let $z = x + iy$

$\frac{1}{z} = \frac{x-iy}{x^2+y^2}$ [3]

$\therefore z + \frac{1}{z} = x + iy + \frac{x-iy}{x^2+y^2}$

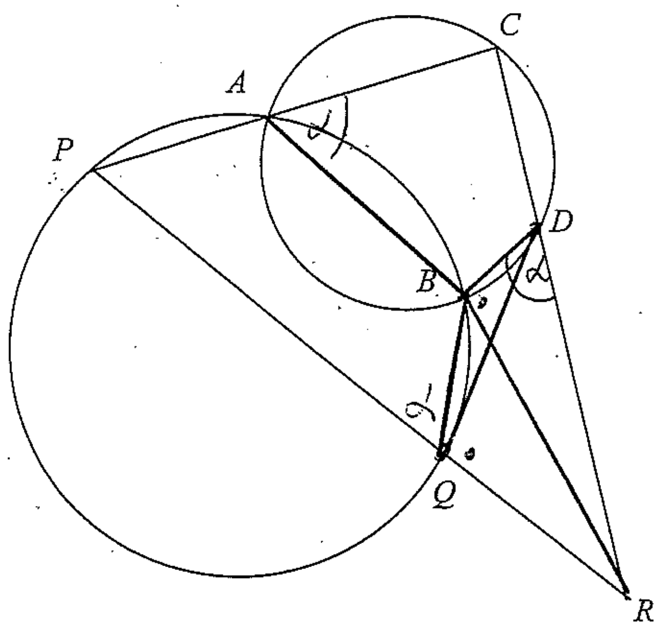
If $z + \frac{1}{z}$ is real

$\Longrightarrow \operatorname{Im}\left(z+\frac{1}{z}\right) = 0$

$iy\left(1 - \frac{1}{x^2+y^2}\right) = 0$

$\Longrightarrow y = 0$ or $x^2+y^2 = 1$

i.e. $|z|^2 = 1$



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(i) Copy the diagram to your answer booklet.

(ii) Prove that $\angle RQD = \angle RBD$.

Let $\angle PQB = \alpha$ ∴ $\angle CAB = \alpha$.
(exterior angle of cyclic quad ABQP)

Similarly, $\angle BDR = \alpha$.

3 In quad. BDRQ

$\angle BDR = \angle PQB = \alpha$.

∴ BDRQ is a cyclic quad

∴ $\angle RQD = \angle RBD$

2 (Converse of angles in the same segment).