SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012

HSC ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Answers should be in simplest exact form unless specified otherwise.
- Start each **NEW** section in a separate answer booklet.
- Each section is to be returned in a separate bundle.

Total Marks - 88

- Attempt Questions 1 - 6
- All questions are NOT of equal value.

Examiner: *A. Fuller*

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

<u>Section A</u>

Question 1 (15 marks)

(a) Find $\int \frac{5}{\cos^2 x} dx$. **1**

(b) Find the exact value of the following: **4**

(i) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(ii) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$

(iii) $\sin\left(2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$.

(c) Write the following in the form $a + ib$, where a and b are real: **6**

(i) $\overline{3 - 4i}$

(ii) $\frac{1}{3-4i}$

(iii) the two square roots of $3 - 4i$.

(d) (i) Express the following in the form $r(\cos\theta + i\ \sin\theta\ )$, **4**

where $r > 0$ and $-\pi < \theta \leq \pi$.

(α) $\sqrt{3} - i$

(β) $\left(\sqrt{3} - i\right)^7$

(ii) Hence, or otherwise, write $\left(\sqrt{3} - i\right)^7$ in the form $x + iy$,

where x and y are real.

Question 2 (15 marks)

(a) $P(x) = 3x^3 - 5x^2 + 4x + 2$. **4**

(i) Show that $(1 + i)$ is a root of $P(x) = 0$.

(ii) Explain why $(1 - i)$ is also a root of $P(x) = 0$.

(iii) Hence, or otherwise, factorise $P(x)$ over the Real field.

(b) Evaluate $\int_1^4 |2 - x|\, dx$. **2**

(c) (i) Show that $\frac{1}{4+5\sin^2 x} = \frac{2}{13-5\cos 2x}$. **4**

(ii) Hence, or otherwise, find $\int \frac{dx}{4+5\sin^2 x}$.

(d) Sketch the locus of the following on separate argand diagrams: **5**

(i) $|z + i| \leq 1$

(ii) $\mathfrak{Re}(z + iz) < 1$

(iii) $2|z| = z + \bar{z} + 4$

<u>Section B</u> **(Use a SEPARATE writing booklet)**

Question 3 (18 marks)

(a) If α, β, γ are the roots of the polynomial equation $2x^3 - 3x + 1 = 0$. **6**

(i) Find the value of the following:

(α) $\alpha\beta\gamma$

(β) $(1-\alpha)(1-\beta)(1-\gamma)$

(γ) $\alpha^2 + \beta^2 + \gamma^2$

(δ) $\alpha^4 + \beta^4 + \gamma^4$

(ii) Form a polynomial equation which has roots $\frac{1}{2\alpha+\beta+\gamma}, \frac{1}{\alpha+2\beta+\gamma}, \frac{1}{\alpha+\beta+2\gamma}$.

(b) (i) Write $\frac{2x^2+x+5}{(x-3)(x^2+4)}$ in the form $\frac{A}{x-3} + \frac{Bx+C}{x^2+4}$. **5**

(ii) Hence, or otherwise, find $\int \frac{2x^2+x+5}{(x-3)(x^2+4)}\,dx$.

(c) (i) State the domain and range of $y = \sin^{-1}\left(\frac{1}{x}\right)$, $x > 0$. **7**

(ii) Find $\frac{dy}{dx}$.

(iii) Show that $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ using the substitution $x = \sec\theta$.

(iv) Consider $y = \sec^{-1} x$ to be $x = \sec y$ for $0 \le y < \frac{\pi}{2}$.

Using the results from (ii) and (iii) write $\sec^{-1} x$ in terms of $\sin^{-1}\left(\frac{1}{x}\right)$.

Question 4 (15 marks)

(a) Given that $|z_1| = 15$ and $z_2 = -3 + 4i$. **4**

(i) Find the maximum value of $|z_1 + z_2|$.

(ii) Hence, find z_1 if $|z_1 + z_2|$ takes its maximum value.

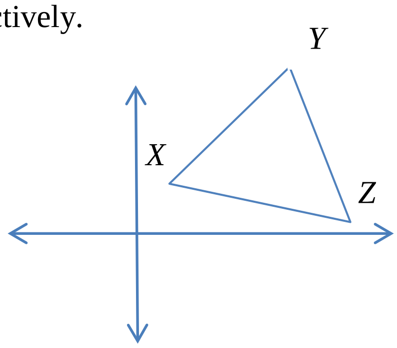
(b) (i) Show that $\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos^2 x}}\, dx = 2\ln\left(1 + \sqrt{2}\right)$ **5**

using the substitution $u = \cos x$.

(ii) Hence, or otherwise, evaluate $\int_0^{\pi} \frac{x \sin x}{\sqrt{1+\cos^2 x}}\, dx$.

(c) In the diagram below, the points X, Y and Z correspond to the complex **6**

numbers x, y and z respectively.



Find the complex numbers represented by:

(i) the vector OX (where O is the origin)

(ii) the vector XZ

(iii) the point A such that $XYAZ$ is a parallelogram

(iv) the point C, the centroid of ΔXYZ.

Note: The centroid of a triangle is the point of intersection of the three medians.

You may assume that the centroid lies two-thirds along a median from the vertex.

Section C (Use a SEPARATE writing booklet)

Question 5 (13 marks)

(a) Let α be the complex root of the polynomial equation $z^7 = 1$ with **7**

the smallest positive argument.

Let $\theta = \alpha + \alpha^2 + \alpha^4$ and $\phi = \alpha^3 + \alpha^5 + \alpha^6$.

(i) Explain why $\alpha^7 = 1$ and $\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$.

(ii) Show that $\theta + \phi = -1$ and $\theta\phi = 2$.

(iii) Show that $\theta = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$ and $\phi = -\frac{1}{2} - i\frac{\sqrt{7}}{2}$.

(iv) Show that $-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} = -\frac{1}{2}$.

(b) 4 students have yet to be placed in a sport. **6**

There are 6 different sports to choose from. How many ways can this be done if:

(i) there are no restrictions

(ii) they must each be placed in different sports

(iii) no more than 2 can be placed in the same sport

(iv) 2 particular students can't play the same sport?

Question 6 (12 marks)

(a) $I_n = \int_0^1 x^n \sqrt{1-x}\ dx$ **8**

(i) Show that $I_n = \frac{2n}{2n+3}\ I_{n-1}$.

(ii) Use mathematical induction to prove that $I_n = \frac{n!(n+1)!}{(2n+3)!} 4^{n+1}$

for positive integers n.

(iii) Hence, find I_3.

(b) Prove that $ax^3 + 3bx^2 + 3cx + d$ has a triple zero if a, b, c, d are **4**

successive terms of a geometric series.

End of paper

EXT 2 - Section A 2012

Half-Yearly

1. (a) $\int \frac{5}{\cos^2 x}\,dx$

$= 5\int \sec^2 x\,dx$ (1)

$= 5\tan x + C$.

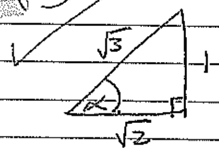
(b) (i) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ (1)

(ii) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \frac{-\pi}{6}$ (1) $\left(\frac{-\pi}{2} \le \tan^{-1}x \le \frac{\pi}{2}\right)$

(iii) $\sin\left(2\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

$= \sin(2\alpha)$ where $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\sin\alpha = \frac{1}{\sqrt{3}}$



$= 2\sin\alpha\cos\alpha$ (2)

$= 2\cdot\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}}$

$= \frac{2\sqrt{2}}{3}$

(c) (i) $\overline{3-4i} = 3+4i$ (1)

(ii) $\frac{1}{3-4i} = \frac{1}{3-4i}\times\frac{3+4i}{3+4i} = \frac{3+4i}{25}$ (1)

(iii) $\sqrt{3-4i} = a+bi$

$3-4i = (a+bi)^2$

$3-4i = a^2-b^2+2abi$

$\Rightarrow 3 = a^2-b^2$ (1)

$-4 = 2ab$ (2)

From (2) $b = \frac{-2}{a}$

Sub in (1)

$\Rightarrow 3 = a^2 - \frac{4}{a^2}$

$3a^2 = a^4 - 4$

$a^4 - 3a^2 - 4 = 0$

$(a^2-4)(a^2+1) = 0$

1 (c) (iii) cont $a = \pm 2$ or $a = \pm\sqrt{-1}$ → reject since $a, b \in R$

When $a = 2$, $b = -1$

$a = -2$, $b = 1$ (4)

$\therefore \pm\sqrt{3-4i} = 2-i$ or $-2+i$

1. (d) (i) (α) $\sqrt{3} - i$

Then $r = \sqrt{3+1} = 2$

and $\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = \frac{-\pi}{6}$

$\therefore \sqrt{3} - i = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$ (1)

(β) $(\sqrt{3}-i)^7 = (r\,\text{cis}\,\theta)^7$

$= \left[2\,\text{cis}\left(\frac{-\pi}{6}\right)\right]^7$

$= 2^7\,\text{cis}\left(7\times\frac{-\pi}{6}\right)$ by De M Th.

$= 128\,\text{cis}\left(\frac{-7\pi}{6}\right)$

$= 128\,\text{cis}\,\frac{5\pi}{6}$ (2)

$= 128\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

1 (d) (ii) $\left(\sqrt{3}-i\right)^7 = 128\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$$= 128\left(\frac{-\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$$

$$= 64\left(-\sqrt{3} + i\right)$$

$$= -64\sqrt{3} + 64i$$

1

Q2. $P(x) = 3x^3 - 5x^2 + 4x + 2$

(i) Let $x = 1+i$.

$$x^2 = (1+i)^2 = 1 - 1 + 2i = 2i$$
$$x^3 = 2i(1+i) = 2i - 2$$

$$P(1+i) = 3(2i-2) - 5(2i) + 4(1+i) + 2$$

1

$$= 6i - 6 - 10i + 4 + 4i + 2$$
$$= 0$$

(ii) By the conjugate root theorem if $(a+ib)$ is a complex root of $P(x)$ where $P(x)$ has real coefficients then $(a-ib)$ is also a root.

1

(iii) Then $(x-(1-i))(x-(1+i))$ is a factor of $P(x)$

$\Rightarrow x^2 - x(1+i) - x(1-i) + 2$ is a factor

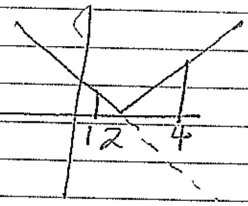
$\Rightarrow x^2 - 2x + 2$ is a factor.

2

$$3x^3 - 5x^2 + 4x + 2 = (x^2-2x+2)(3x+1)$$ by observation

$\therefore P(x) = (x^2-2x+2)(3x+1)$ over R

2 (b) $\int_1^4 |2-x|\,dx$



$$= \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$$

$$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$= \left[(4-2) - \left(2-\frac{1}{2}\right)\right] + \left[(8-8) - (2-4)\right]$$

$$= \frac{1}{2} + 2$$

$$= 2\tfrac{1}{2} \text{ or } \tfrac{5}{2}$$

2

(c) (i) Show $\dfrac{1}{4+5\sin^2 x} = \dfrac{2}{13-5\cos 2x}$

$$\text{RHS} = \frac{2}{13-5\cos 2x}$$

$$= \frac{2}{13-5(2\cos^2 x - 1)}$$

$$= \frac{2}{13-10\cos^2 x + 5}$$

$$= \frac{2}{18-10\cos^2 x}$$

$$= \frac{1}{9-5(1-\sin^2 x)}$$

2

$$= \frac{1}{9-5+5\sin^2 x}$$

$$= \frac{1}{4+5\sin^2 x} = \text{LHS} \quad \#$$

2(c)(ii) $\int \frac{dx}{4+5\sin^2 x}$

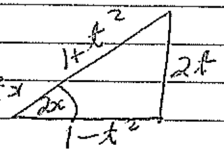
$= 2\int \frac{1}{13-5\cos 2x}\,dx$

$= 2\int \frac{1}{-5\cos 2x+13}\,dx$

Let $t = \tan x$

$\frac{dt}{dx} = \sec^2 x = 1+\tan^2 x = 1+t^2$

$dx = \frac{dt}{1+t^2}$



$= 2\int \frac{1}{-5\times\frac{1-t^2}{1+t^2}+13} \cdot \frac{dt}{1+t^2}$

$= 2\int \frac{dt}{-5(1-t^2)+13(1+t^2)}$

$= 2\int \frac{dt}{-5+5t^2+13+13t^2} = 2\int \frac{dt}{18t^2+8}$

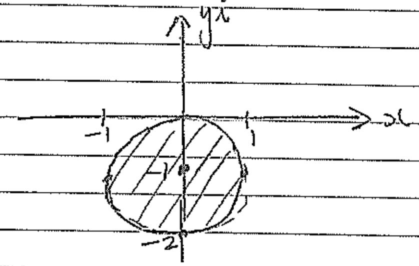
$= \int \frac{dt}{4+9t^2} = \int \frac{dt}{4+(3t)^2}$

$= \frac{1}{3} \cdot \frac{1}{2} \tan^{-1}\left(\frac{3t}{2}\right) + c$

$= \frac{1}{6} \tan^{-1}\left(\frac{3}{2}\tan x\right) + C$

2

2(d)(i) $|z+i| \le 1$



1

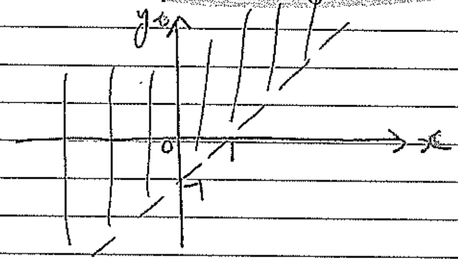
(ii) $\mathrm{Re}(z+iz) < 1$ Let $z = x+yi$

$\Rightarrow \mathrm{Re}(x+yi + i(x+yi))$

$= \mathrm{Re}(x-y + (x+y)i)$

$= x-y.$

$\therefore$ sketch $x-y<1$



2

(iii) $2|z| = z+\bar{z}+4$

Let $z = x+yi$

$2\sqrt{x^2+y^2} = x+yi+x-yi+4$

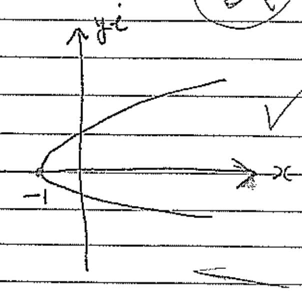
$2\sqrt{x^2+y^2} = 2x+4$

$\sqrt{x^2+y^2} = x+2$

$x^2+y^2 = x^2+4x+4$

$\Rightarrow y^2 = 4(x+1)$

Parabola $V(-1,0)$



2

SECTION B

(a) (i) (α) $\alpha\beta\gamma = -\frac{d}{a}$

$= -\frac{1}{2}$

(β) $(1-\alpha)(1-\beta)(1-\gamma)$

$= (1-\alpha)(1-\gamma-\beta+\beta\gamma)$

$= 1-\gamma-\beta+\beta\gamma-\alpha+\alpha\gamma+\alpha\beta-\alpha\beta\gamma.$

$= 1-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\alpha\gamma)-\alpha\beta\gamma$

$= 1-0-\frac{3}{2}+\frac{1}{2}.$

$= 0$

(γ) $(\alpha+\beta+\gamma)^2 = (\alpha^2+\beta^2+\gamma^2)+2(\alpha\beta+\beta\gamma+\alpha\gamma).$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= -2\left(-\frac{3}{2}\right)$

$= +3.$

(δ) $2x^4-3x^2+x=0$

$2(\alpha^4+\beta^4+\gamma^4)-3(\alpha^2+\beta^2+\gamma^2)+(\alpha+\beta+\gamma)=0$

$2(\alpha^4+\beta^4+\gamma^4) = 3(\alpha^2+\beta^2+\gamma^2) - (\alpha+\beta+\gamma)$

$= 3(+3) - 0$

$= +9$

$\alpha^4+\beta^4+\gamma^4 = +\frac{9}{2}.$

ii) $\frac{1}{2\alpha+\beta+\gamma}, \frac{1}{\alpha+2\beta+\gamma}, \frac{1}{\alpha+\beta+2\gamma}$.

$= \frac{1}{\alpha+(\alpha+\beta+\gamma)}, \frac{1}{\beta+(\alpha+\beta+\gamma)}, \frac{1}{\gamma+(\alpha+\beta+\gamma)}$

$= \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Let $X = \frac{1}{x}$.

$x = \frac{1}{X}$

$\frac{2}{X^3} - \frac{3}{X} + 1 = 0$

$2 - 3X^2 + X^3 = 0$.

$X^3 - 3X^2 + 2 = 0$.

(b) (i) $\frac{2x^2+x+5}{(x-3)(x^2+4)} \equiv \frac{A}{x-3} + \frac{Bx+C}{x^2+4}$.

$2x^2+x+5 \equiv A(x^2+4) + (Bx+C)(x-3)$.

when $x=3$.

$18+3+5 = 13A$.

$A = 2$.

$\underline{x^2}$ $2 = A + B$
$B = 0$

$\underline{x}$ $1 = -3B + C$
$C = 1$.

$$\frac{2}{x-3} + \frac{1}{x^2+4}$$

(ii) $\displaystyle\int \frac{2}{x-3} + \frac{1}{x^2+4}\,dx$

$$= 2\ln(x-3) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

(c)(i) Domain: $x \geqslant 1$

Range: $0 \leq y \leq \frac{\pi}{2}$

(ii) $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times -\frac{1}{x^2}$$

$$= -\frac{1}{\sqrt{x^4-x^2}}$$

$$= -\frac{1}{x\sqrt{x^2-1}}$$

(iii) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

$x = \sec\theta$

$\frac{dx}{d\theta} = \sec\theta\tan\theta.$

$dx = \sec\theta\tan\theta\,d\theta$

$$= \int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$$

$$= \theta + C.$$

$= \sec^{-1} x + C$ $\qquad$ $x = \sec\theta$

$\theta = \sec^{-1} x.$

(iv) $y = \sin^{-1}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$

$y = -\int \frac{1}{x\sqrt{x^2-1}}$

$y = -\sec^{-1} x + C$

$\sin^{-1}\left(\frac{1}{x}\right) = C - \sec^{-1} x.$

when $x = 1$

$\sin^{-1}(1) = C - \sec^{-1} 1$

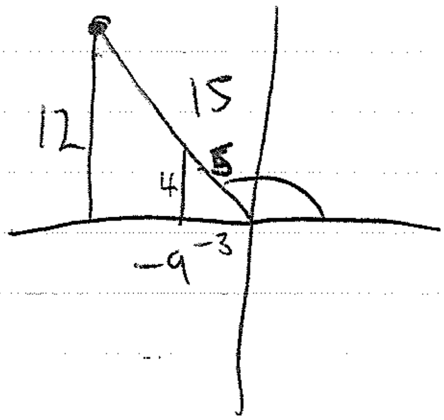
$\frac{\pi}{2} = C.$

So $\sec^{-1} x = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right).$

QUESTION 4

(a i) $|z_1+z_2| \le |z_1|+|z_2|$
$= 20.$

(ii)



max when $\arg(z_1)=\arg(z_2)$

$z_1 = -9+12i$

(b i) $\displaystyle\int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx$

$u=\cos x$
$-du = \sin x\,dx.$

$\displaystyle = \int_{-1}^{1} \frac{du}{\sqrt{1+u^2}}$

when $x=0$ $u=1$
$x=\pi$ $u=-1$

$\displaystyle = 2\int_0^1 \frac{du}{\sqrt{1+u^2}}$

$= 2\left[\ln\left(u+\sqrt{1+u^2}\right)\right]_0^1$

$= 2\ln\left(1+\sqrt{2}\right)$

(ii) $\displaystyle I = \int_0^{\pi} \frac{x\sin x}{\sqrt{1+\cos^2 x}}\,dx$

$\displaystyle = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{\sqrt{1+\cos^2(\pi-x)}}\,dx$

$$I = +\pi \int_0^{\pi} \frac{\sin x}{\sqrt{1+\cos^2 x}}\,dx - \int_0^{\pi} \frac{x \sin x}{\sqrt{1+\cos^2 x}}\,dx$$

$$2I = 2\pi \ln(1+\sqrt{2})$$

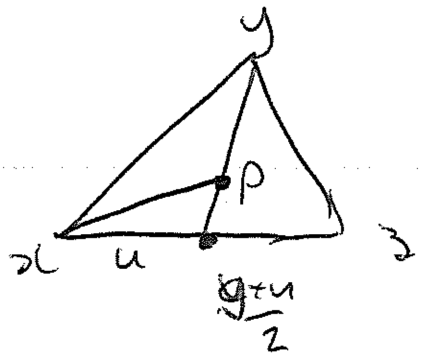
$$I = \pi \ln(1+\sqrt{2})$$

(c i) x

(ii) $z - x$

(iii) $y + z - x$

(iv)



$$\cancel{u = \frac{c+a}{2}}$$

$$\cancel{p = \frac{1}{3}(b-u) + u}$$

$$\cancel{= \frac{b}{3} - \frac{u}{3} + \frac{3u}{3}}$$

$$\cancel{= \frac{b}{3} + \frac{c+a}{3}}$$

$$u = \frac{z+x}{2}$$

$$p = \frac{1}{3}(y-u) + u$$

$$= \frac{y}{3} - \frac{u}{3} + \frac{3u}{3}$$

$$= \frac{y}{3} + \frac{z+x}{3}$$

$$= \frac{x+y+z}{3}.$$

Q5 (a) (i) $z^7 = 1$

$\therefore z^7 - 1 = 0$

$\therefore (z-1)(z^6+z^5+z^4+z^3+z^2+z+1) = 0$

now α is a root. $\therefore \boxed{\alpha^7 = 1}$

$\therefore (\alpha-1)(\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1) = 0$

$\alpha \neq 1.$

$\therefore \boxed{\alpha^6+\alpha^5+\alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0.}$ (A)

(ii)

$\theta + \phi = \alpha + \alpha^2 + \alpha^4 + \alpha^3 + \alpha^5 + \alpha^6$

$= -1$ from (A).

$\theta\phi = (\alpha+\alpha^2+\alpha^4)(\alpha^3+\alpha^5+\alpha^6)$

$= \alpha^4+\alpha^6+\alpha^7+\alpha^5+\alpha^7+\alpha^8+\alpha^7+\alpha^9+\alpha^{10}$

$= \alpha^4+\alpha^6+1+\alpha^5+1+\alpha+1+\alpha^2+\alpha^3$

$= 3 + (\alpha+\alpha^2+\alpha^3+\alpha^5+\alpha^5+\alpha^6)$

$= 3 + (-1)$

$= 2.$

(iii) Form an equation (quadratic) with roots θ & ϕ where $\theta+\phi = -1$ & $\theta\phi = 2$

$x^2 - (\theta+\phi)x + \theta\phi = 0.$

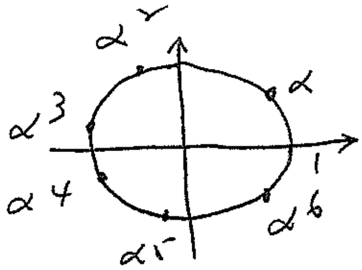
$x^2 + x + 2 = 0.$

$x = \dfrac{-1 \pm \sqrt{1-8}}{2}$

(CONTD) ②

$\therefore x = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$

From



Clearly $\theta = \alpha + \alpha^2 + \alpha^4$ has a positive imaginary part.

& $\phi = \alpha^3 + \alpha^5 + \alpha^6$ has a negative imaginary part.

$\therefore \theta = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$ & $\phi = -\frac{1}{2} - i\frac{\sqrt{7}}{2}.$

(iv) Now $\alpha = \text{cis}\,\frac{2\pi}{7}.$

$\theta = \alpha + \alpha^2 + \alpha^4 = -\frac{1}{2} + i\frac{\sqrt{7}}{2}.$

ie $\text{cis}\,\frac{2\pi}{7} + \text{cis}\,\frac{4\pi}{7} + \text{cis}\,\frac{8\pi}{7} = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$

Taking real parts.

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$

$\therefore \boxed{-\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} = -\frac{1}{2}.}$

(b) (i) $6^4 = 1296$

(ii) $6 \times 5 \times 4 \times 3 = 1170.$

(iii) $6^4 - (6 + {}^4C_3 \times 6 \times 5) = 1170.$

[OR $6 \times 5 \times 4 \times 3 + {}^4C_2 \times 6 \times 5 \times 4 + \dfrac{{}^4C_2 \times 6 \times 5}{2} = 1170$]

(iv) $6^4 - 6^3 = 1080$

[OR $6 \times 5 \times 6 \times 6 = 1080$]

Q6. (a)

$$I_n = \int_0^1 x^n \sqrt{1-x}\, dx.$$

$$= \int_0^1 x^n \frac{d}{dx}\left(-\frac{2}{3}\right)(1-x)^{3/2} dx.$$

$$= \left[-\frac{2}{3}x^n(1-x)^{3/2}\right]_0^1 + \frac{2}{3}\int_0^1 n x^{n-1}(1-x)^{3/2} dx$$

$$= [0-0] + \frac{2}{3}n\int_0^1 x^{n-1}(1-x)(1-x)^{\frac{1}{2}} dx.$$

$$= \frac{2n}{3}\int_0^1 x^{n-1}\sqrt{1-x}\,dx - \frac{2n}{3}\int_0^1 x^n\sqrt{1-x}\,dx$$

$$\therefore I_n = \frac{2n}{3} I_{n-1} - \frac{2n}{3} I_n.$$

$$I_n\left(1+\frac{2n}{3}\right) = \frac{2n}{3} I_{n-1}.$$

$$I_n \frac{2n+3}{3} = \frac{2n}{3} I_{n-1}$$

$$\boxed{I_n = \frac{2n}{2n+3} I_{n-1}}$$

(b) Prove $I_n = \dfrac{n!\,(n+1)!}{(2n+3)!}\, 4^{n+1}$

When $n = 1$.

$$LHS = I_1 = \frac{2}{5} I_0$$
$$= \frac{2}{5} \cdot \left[-\frac{2}{3} (1-x)^{3/2} \right]_0^1$$
$$= \frac{2}{5} \times \frac{2}{3}$$
$$= \frac{4}{15}$$
$$RHS = \frac{1! \times 2!}{5!} \times 4^2$$
$$= \frac{32}{5 \times 4 \times 3 \times 2 \times 1}$$
$$= \frac{4}{15}$$

$\therefore$ True when $n = 1$.

Assume the statement to be true when $n = k$.

i.e. $$I_k = \frac{k!\,(k+1)!\,.\,4^{k+1}}{(2k+3)!}$$

R.T.P. statement is true when $n = k+1$ (using the assumption)

i.e. $$I_{k+1} = \frac{(k+1)!\,(k+2)!\,4^{k+2}}{(2k+5)!}$$

now $$I_{k+1} = \frac{2(k+1)}{(2k+5)} \cdot I_k.$$

$$= \frac{2(k+1)}{2k+5} \times \frac{k!\,(k+1)!}{(2k+3)!} \times 4^{k+1}$$

$$= \frac{2(k+1)}{2k+5} \times \frac{k! \times (k+1)!}{(2k+3)!} \times \frac{2k+4}{2k+4} \times 4^{k+1}$$

$$= \frac{4(k+1)\,k!\,(k+1)!\,(k+2)}{(2k+5)!} \times 4^{k+1}$$

$$= \frac{(k+1)!\,(k+2)!}{(2k+5)!} \times 4^{k+2}$$

$= \underline{RHS}.$

$\therefore$ By the Principle of mathematical induction the statement is true for all positive integers.

(iii) $$I_3 = \frac{3!\,(3+1)!\,4^4}{9!}$$

$$= \frac{3! \times 4! \times 4^4}{9!}$$

$$= \frac{32}{315}.$$

(5) given $P(x) = ax^3 + 3bx^2 + 3cx + d$.

let $a = a$

$b = ar$

$c = ar^2$

$d = ar^3$

$\therefore P(x) = ax^3 + 3arx^2 + 3ar^2x + ar^3$

$= a(x^3 + 3rx^2 + 3r^2x + r^3)$

$= a(x+r)^3$ which has a triple root of $-r$.

(There are many other ways of doing this question)