SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2015

HSC Task #1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 90 minutes.
- Write using black or blue pen.
  Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks – 62

Section I Pages 2–4

8 Marks

- Attempt Questions 1–8
- Allow about 12 minutes for this section.

Section II Pages 5–8

54 marks

- Attempt Questions 9–11
- Allow about 1 hour and 18 minutes for this section

Examiner: *P. Parker*

NAME: ____________________ **TEACHER** ____________________

NUMBER: ____________________

QUESTION	MARK
1 – 8	/8
9	/18
10	/18
11	/18
TOTAL	/62

Section I – Multiple Choice

8 Marks
Attempt question 1−8
Allow about 12 minutes for this section

Use the multiple-choice answer sheet for Questions 1−8

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1 If $w = 5 - 2i$ which of the following is $\dfrac{1}{2-w}$?

(A) $\dfrac{3}{13}-\dfrac{2}{13}i$

(B) $-\dfrac{3}{13}-\dfrac{2}{13}i$

(C) $-\dfrac{3}{13}+\dfrac{2}{13}i$

(D) $\dfrac{3}{13}+\dfrac{2}{13}i$

2 Which of the following are the solutions to the quadratic equation $w^2 - 4w + 5 = 0$?

(A) $z = \pm\,(2 - i)$

(B) $z = \pm\,(2 + i)$

(C) $z = -2 + i$ or $z = -2 - i$

(D) $z = 2 + i$ or $z = 2 - i$

3 One root of the quadratic equation $iz^2 + 3z + 3 - 11i = 0$ is $3 + 2i$.
What is the other root?

(A) $-6 - 2i$

(B) $-3 + i$

(C) $3 - 2i$

(D) $-3 - 5i$

4 Which of the following expressions is true?

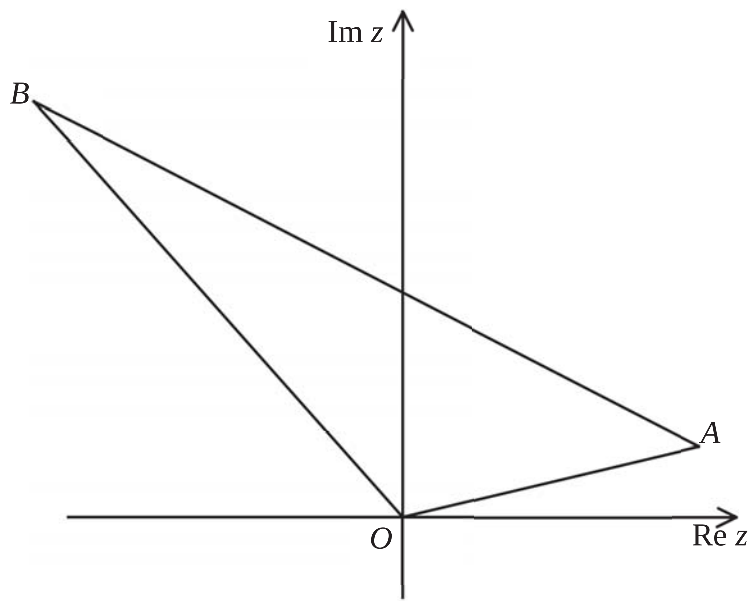
(A) $\tan^{-1} u = \sin^{-1}\left(\dfrac{1}{\sqrt{1-u^2}}\right)$

(B) $\tan^{-1} u = \sin^{-1}\left(\dfrac{1}{\sqrt{1+u^2}}\right)$

(C) $\tan^{-1} u = \sin^{-1}\left(\dfrac{u}{\sqrt{1-u^2}}\right)$

(D) $\tan^{-1} u = \sin^{-1}\left(\dfrac{u}{\sqrt{1+u^2}}\right)$

5 In the Argand diagram below, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers z_1 and z_2 respectively where $|z_2| = 4|z_1|$ and $\angle AOB = \dfrac{2\pi}{3}$.



Which of the following complex numbers is represented by vector AB?

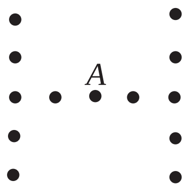
(A) $\left(-3+2\sqrt{3}\,i\right)z_1$

(B) $\left(3-2\sqrt{3}\,i\right)z_1$

(C) $\left(-2\sqrt{3}+2i\right)z_1$

(D) $\left(2\sqrt{3}-2i\right)z_1$

6 Angie draws the following diagram shows a shape made by 13 points.
She designs the diagram so that it has point symmetry about A.



How many triangles can she make with these points as vertices?

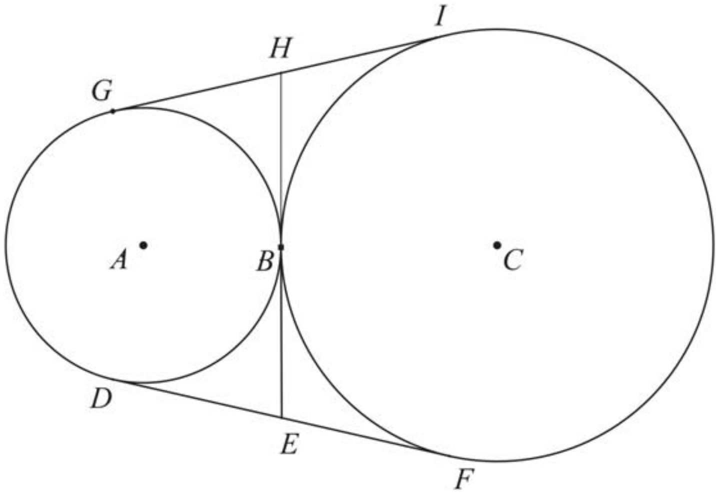
(A) ${}^{13}C_3 - 3\times {}^{5}C_3$

(B) ${}^{13}C_3 - 3\times {}^{5}C_3 - 1$

(C) ${}^{13}C_3 - 3\times {}^{5}C_3 - 2$

(D) ${}^{13}C_3 - 3\times {}^{5}C_3 - 4$

7 In the diagram below DF, GI and HE are common tangents to the two unequal circles that have centres at A and C.



Which of the following quadrilaterals are **NOT** cyclic?

(A) $ABED$

(B) $HICB$

(C) $GIFD$

(D) $ACFD$

8 Taryn is dealt a hand consisting of five cards from a SBHS Deck.
A SBHS deck has fifty-four cards: a numberless silver card, a numberless golden card, and a standard deck of fifty-two playing cards.
What is the probability that Taryn gets dealt two pairs?

(A) $\dfrac{52\times 48\times\left({}^{4}C_2\right)^2\times 46}{{}^{54}C_5}$

(B) $\dfrac{52\times 48\times\left({}^{4}C_2\right)^2\times 46\times {}^{5}C_2}{{}^{54}C_5}$

(C) $\dfrac{{}^{13}C_2\times\left({}^{4}C_2\right)^2\times 46}{{}^{54}C_5}$

(D) $\dfrac{{}^{5}C_4\times {}^{13}C_2\times 12^2\times 46}{{}^{54}C_5}$

Section II

54 marks
Attempt Questions 9–11
Allow about 1 hour and 18 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 9–11, your responses **should include** relevant mathematical reasoning and/or calculations.

Question 9 (18 Marks) Start a NEW Writing Booklet

(a) It is given that $z = x + iy$, where x and y are real numbers. **2**
Find the number z such that $i(z + 7) = 3(\overline{z} - i)$

(b) Let $a = 2 - 3i$ and $b = -1 - 2i$

(i) Find $\operatorname{Re}(ab)$. **1**

(ii) Evaluate $|b - a|^2$ **1**

(c) (i) Express the numbers $u = 1 - i$ and $v = 2\sqrt{3} - 2i$ in modulus-argument form. **2**

(ii) Hence, or otherwise, find $\arg\left(\frac{u}{v}\right)$. **1**

(d) (i) Find the two square roots of $6i - 8$. **2**

(ii) Solve $2z^2 - (3 + i)z + 2 = 0$ **2**

(e) Find the smallest positive integer values of p and q for which **3**

$$\frac{\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^q} = i$$

(f) The equation $x^3 + 3x^2 + 7x + 5 = 0$ has roots α, β and γ. **2**

(i) Find a simplified monic polynomial equation with roots
$\alpha + 1$, $\beta + 1$ and $\gamma + 1$

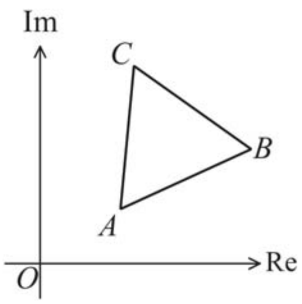
(ii) Hence, or otherwise, solve $x^3 + 3x^2 + 7x + 5 = 0$ **2**

Question 10 (18 Marks) Start a NEW Writing Booklet

(a) The cube roots of 1 are denoted by 1, ω and ω^2 where Im $\omega^2 < 0$

(i) Show that $1 + \omega + \omega^2 = 0$ **1**

In the diagram ABC is an equilateral triangle. The points A, B and C represent the numbers z_1, z_2 and z_3 respectively.



(ii) Explain why $z_1 - z_3 = \omega\left(z_3 - z_2\right)$ **1**

(iii) Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$ **2**

(b) The complex number z satisfies the relation $|\, z + 4 - 4i \,| = 4$

(i) Sketch, on an Argand diagram, the locus of z. **2**

(ii) Show that the greatest value of $|\, z \,|$ is $4\left(\sqrt{2}+1\right)$. **2**

(iii) Find the value of z for which $\arg(z + 4 - 4i) = \frac{1}{6}\pi$ **2**
leaving your answer in the form $a + ib$.

(c) It is given that α, β and γ are the roots of $x^3 - 4x + 3 = 0$ **2**
Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(d) The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where a and b are constants, has **3**
a multiple zero at $x = 2$.
Find the values of a and b.

(e) Prove that for $0 < x < 1$, $\tan^{-1}\dfrac{x}{1-x} - \tan^{-1}\dfrac{x}{1+x} = \tan^{-1} 2x^2$ **3**

Question 11 (18 Marks) Start a NEW Writing Booklet

(a) Consider the polynomial equation $x^5 - i = 0$

(i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$. **1**

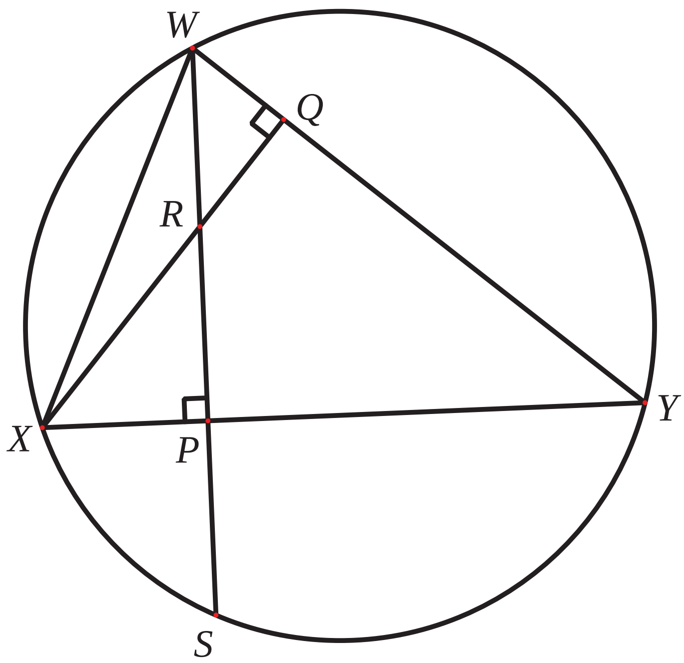
(ii) Show that $(x-i)\left(x^2 - 2ix\sin\frac{\pi}{10} - 1\right)\left(x^2 + 2ix\sin\frac{3\pi}{10} - 1\right) = 0$. **4**

(iii) Using the results in (i) and (ii) and using a suitable substitution, show that **2**

$$\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}.$$

(b) The altitudes *WP* and *XQ* in an acute-angled triangle *WXY* meet at *R*. **3**
WR produced cuts the circle through the vertices *W*, *X* and *Y* at *S*.

Prove that *PR* = *PS*.



Question 11 continues on page 8

Question 11 (continued)

(d) Twelve of the Maths Staff wanted to be photographed at the Extension 2 Big Day Out. **3**
This meant that, Paul, Bob, Mark, Andrew, Patrick, Brenda, Ross, Ron, Jacqui, Vicki, Jess and Doug all had to line up in a straight line.
There were a few rules that had to be followed:

- Teachers with the letter *a* in their name could not stand together in adjacent positions in the line.
- Since Paul and Ross get up to mischief, they can never stand in adjacent positions in the line.

The teachers randomly organise themselves in the line.
What is the probability that they arrange themselves in a correct order?

(e) (i) Let $u = a + ib$ be a fixed complex number. **2**
Show that as the complex number $z = x + iy$ varies, the locus defined by the equation

$$u\bar{z} + \bar{u}z = 2$$

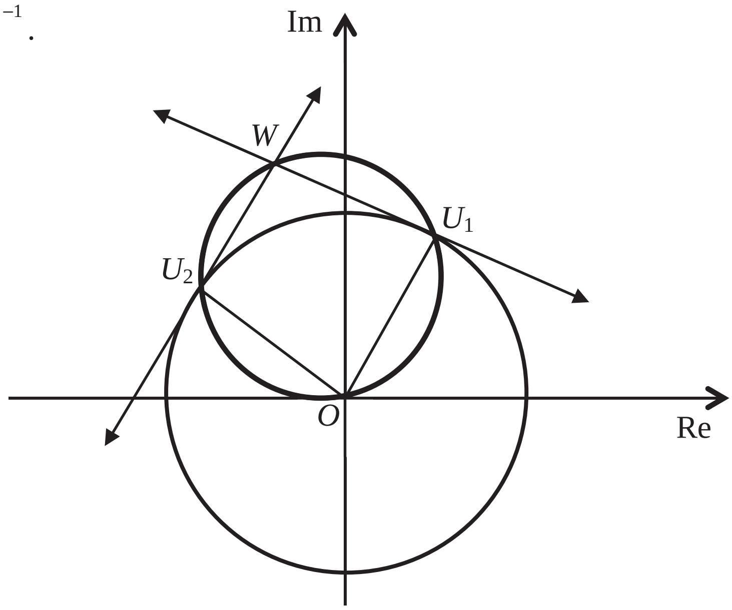
is a straight line in the Argand diagram.

(ii) Show that if $|u| = 1$, then $u\bar{z} + \bar{u}z = 2$ is a tangent to the unit circle $|z| = 1$. **1**

(iii) If $|u| = 1$ then the point of contact between the unit circle and the tangent $u\bar{z} + \bar{u}z = 2$ is u. (Do NOT prove) **2**

Let u_1 and u_2 be two distinct values of u, both of unit modulus, and suppose that $u_1 + u_2 \neq 0$.
In the diagram below, the tangents $u_1\bar{z} + \bar{u}_1 z = 2$ and $u_2\bar{z} + \bar{u}_2 z = 2$ touch $|z| = 1$ at U_1 and U_2 respectively, and intersect at W.

Show that the cyclic quadrilateral OU_1WU_2 is circumscribed by a circle of radius $\left| u_1 + u_2 \right|^{-1}$.



You may assume that the sum of the vectors OU_1 and OU_2 lies on OW.

End of paper

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HSC Task #1

Mathematics Extension 2

Suggested Solutions
&
Markers' Comments

QUESTION	Marker
1 – 8	–
9	DH
10	PB
11	AMG

Multiple Choice Answers

1. B
2. D
3. B
4. D
5. A
6. D
7. D
8. C

Section I – Multiple Choice

1 If $w = 5 - 2i$ which of the following is $\dfrac{1}{2-w}$?

(A) $\dfrac{3}{13} - \dfrac{2}{13}i$

(B) $-\dfrac{3}{13} - \dfrac{2}{13}i$

(C) $-\dfrac{3}{13} + \dfrac{2}{13}i$

(D) $\dfrac{3}{13} + \dfrac{2}{13}i$

$$\frac{1}{2-w} = \frac{1}{2-(5-2i)}$$
$$= \frac{1}{-3+2i}$$
$$= \frac{-3-2i}{13}$$

$$\left[\frac{1}{z} = \frac{\overline{z}}{|z|^2}\right]$$

2 Which of the following are the solutions to the quadratic equation $w^2 - 4w + 5 = 0$?

(A) $z = \pm(2 - i)$

(B) $z = \pm(2 + i)$

(C) $z = -2 + i$ or $z = -2 - i$

(D) $z = 2 + i$ or $z = 2 - i$

$w^2 - 4w + 5 = (w - 2)^2 + 1$

3 One root of the quadratic equation $iz^2 + 3z + 3 - 11i = 0$ is $3 + 2i$.
What is the other root?

(A) $-2i$

(B) $-3 + i$

(C) $-6 - 2i$

(D) $-3 - 5i$

Sum of roots $= -\dfrac{3}{i} = 3i$

$\therefore \alpha + 3 + 2i = 3i$

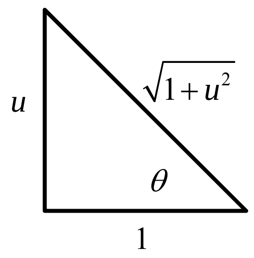
4 Which of the following expressions is true?

(A) $\tan^{-1} u = \sin^{-1}\left(\dfrac{1}{\sqrt{1-u^2}}\right)$

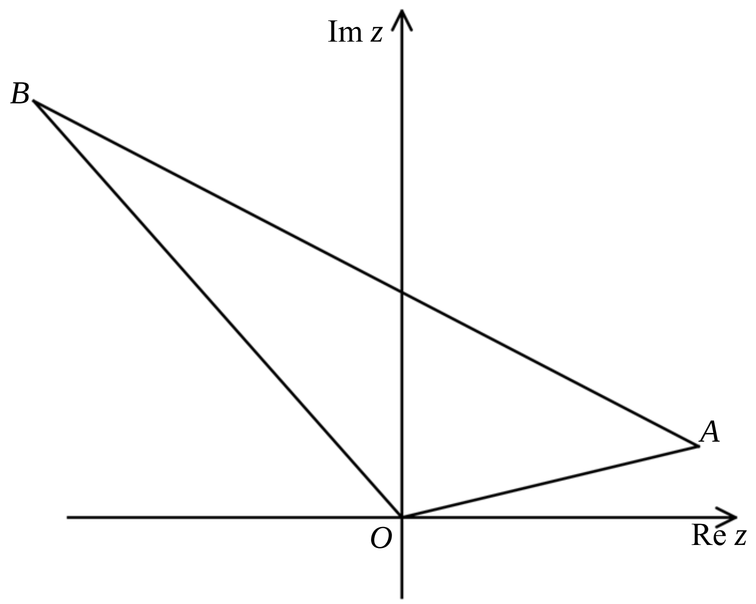
(B) $\tan^{-1} u = \sin^{-1}\left(\dfrac{1}{\sqrt{1+u^2}}\right)$

(C) $\tan^{-1} u = \sin^{-1}\left(\dfrac{u}{\sqrt{1-u^2}}\right)$

(D) $\tan^{-1} u = \sin^{-1}\left(\dfrac{u}{\sqrt{1+u^2}}\right)$



5 In the Argand diagram below, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers z_1 and z_2 respectively where $|z_2| = 4|z_1|$ and $\angle AOB = \dfrac{2\pi}{3}$.



Which of the following complex numbers is represented by vector AB?

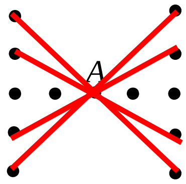
(A) $\left(-3+2\sqrt{3}\,i\right)z_1$ (B) $\left(3-2\sqrt{3}\,i\right)z_1$

(C) $\left(-2\sqrt{3}+2i\right)z_1$ (D) $\left(2\sqrt{3}-2i\right)z_1$

Vector AB is represented by $4z_1\text{cis}\frac{2\pi}{3} - z_1$

$$\begin{aligned}4z_1\text{cis}\tfrac{2\pi}{3} - z_1 &= z_1\left(4\text{cis}\tfrac{2\pi}{3} - 1\right)\\ &= z_1\left(-2+2i\sqrt{3}-1\right)\\ &= z_1\left(-3+2i\sqrt{3}\right)\end{aligned}$$

6 Angie draws the following diagram shows a shape made by 13 points.
She designs the diagram so that it has point symmetry about A.



There are $^{13}C_3$ ways of picking any 3 points.
However some of these points are collinear.
If 3 of the points are taken from the set of 5 horizontal or vertical then no triangle can be formed.

How many triangles can she make with these points as vertices?

(A) $^{13}C_3 - 3\times {^5C_3}$

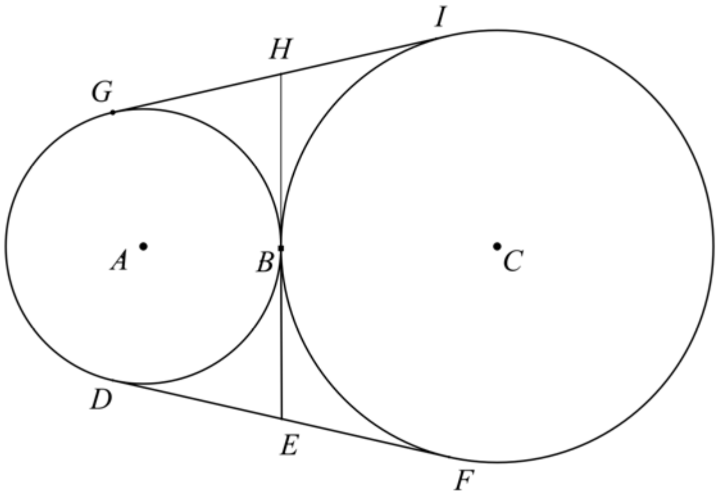
(B) $^{13}C_3 - 3\times {^5C_3} - 1$

(C) $^{13}C_3 - 3\times {^5C_3} - 2$

(D) $^{13}C_3 - 3\times {^5C_3} - 4$

In addition, if the 3 points are any of the lines drawn above then no triangle can be formed.

7 In the diagram below DF, GI and HE are common tangents to the two unequal circles that have centres at A and C.



Which of the following quadrilaterals are **NOT** cyclic?

(A) $ABED$

(B) $HICB$

(C) $GIFD$

(D) $ACFD$

$\angle ADF = \angle CFD = 90°$
$\therefore CF \parallel AD$
If $ACFD$ was cyclic then $ACFD$ is a rectangle and $AD = FC$, but the circles are unequal.

8 Taryn is dealt a hand consisting of five cards from a SBHS Deck.
A SBHS deck has fifty-four cards: a numberless silver card, a numberless golden card, and a standard deck of fifty-two playing cards.
What is the probability that Taryn gets dealt two pairs?

(A) $\dfrac{52 \times 48 \times \left({}^{4}C_{2}\right)^{2} \times 46}{{}^{54}C_{5}}$

(B) $\dfrac{52 \times 48 \times \left({}^{4}C_{2}\right)^{2} \times 46 \times {}^{5}C_{2}}{{}^{54}C_{5}}$

(C) $\dfrac{{}^{13}C_{2} \times \left({}^{4}C_{2}\right)^{2} \times 46}{{}^{54}C_{5}}$

(D) $\dfrac{{}^{5}C_{4} \times {}^{13}C_{2} \times 12^{2} \times 46}{{}^{54}C_{5}}$

To get two pairs, first get the number of ways of choosing two ranks from A, 2, 3, …, K i.e. ${}^{13}C_{2}$ ways

Now for each pair there are 4 different suits and 2 need to be chosen in ${}^{4}C_{2}$ ways.

So with two pairs this can be done in $\left({}^{4}C_{2}\right)^{2}$ ways.

The fifth card must be chosen from the remaining cards possible i.e. 54 – 8 = 46.

2015 Extension 2 Mathematics Task 1:
Solutions— Question 9

9. (a) It is given that $z = x + iy$, where x and y are real numbers. Find the number z such that $i(z + 7) = 3(\overline{z} - i)$. $\boxed{2}$

Solution:
$$\begin{aligned} ix - y + 7i &= 3x - 3iy - 3i, \\ (3x + y) - i(x + 3y + 10) &= 0, \\ 3x + y &= 0, \ldots\ldots\ldots\ldots\ldots\ldots \boxed{1} \\ x + 3y &= -10, \ldots\ldots\ldots\ldots\ldots \boxed{2} \\ y &= -3x, \text{ from } \boxed{1} \\ \text{sub. in } \boxed{2}: \ x - 9x &= -10, \\ 8x &= 10, \\ x &= \tfrac{5}{4}, \\ y &= -3\tfrac{3}{4}, \\ \therefore z &= \frac{5 - 15i}{4}. \end{aligned}$$

Comment: This part (along with many others) was marred by a multitude of careless errors— sometimes caused by the candidate's inability to read his own writing.
If not answered explicitly, *i.e.* $z = \ldots$,
not just $x = \ldots$, $y = \ldots$, full marks were not awarded.

(b) Let $a = 2 - 3i$ and $b = -1 - 2i$.

(i) Find $\mathfrak{Re}(ab)$. $\boxed{1}$

Solution:
$$\begin{aligned} ab &= -2 - 4i + 3i - 6, \\ &= -8 - i, \\ \therefore \mathfrak{Re}(ab) &= -8. \end{aligned}$$

Comment: Well done.

(ii) Evaluate $|b - a|^2$. $\boxed{1}$

Solution:
$$\begin{aligned} b - a &= -1 - 2i - 2 + 3i, \\ &= -3 + i, \\ |b - a|^2 &= 9 + 1, \\ &= 10. \end{aligned}$$

Comment: Well done.

(c) (i) Express the numbers $u = 1 - i$ and $v = 2\sqrt{3} - 2i$ in modulus-argument form. $\boxed{2}$

Solution:

$$\begin{aligned} u &= 1 - i, & r &= \sqrt{1^2 + (-1)^2}, & \theta &= \tan^{-1}\left(\tfrac{-1}{1}\right), \\ &= \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right). & &= \sqrt{2}. & &= -\frac{\pi}{4}. \end{aligned}$$

$$\begin{aligned} v &= 2\sqrt{3} - 2i, & r &= 2\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}, & \theta &= \tan^{-1}\left(\tfrac{-1}{\sqrt{3}}\right), \\ &= 4\operatorname{cis}\left(-\frac{\pi}{6}\right). & &= 4. & &= -\frac{\pi}{6}. \end{aligned}$$

Comment: Arguments should be expressed in terms of their principal values, *i.e.* $[-\pi,\ \pi]$.

(ii) Hence, or otherwise, find $\arg\left(\dfrac{u}{v}\right)$. $\boxed{1}$

Solution:

$$\begin{aligned} \arg\left(\frac{u}{v}\right) &= \arg(u) - \arg(v), \\ &= -\frac{\pi}{4} - \frac{-\pi}{6}, \\ &= -\frac{\pi}{12}. \end{aligned}$$

Comment: As above.

(d) (i) Find the two square roots of $6i - 8$. $\boxed{2}$

Solution: Method 1—

$$\begin{aligned} -8 + 6i &= (1 - 9) + 2 \times 1 \times 3i, \\ &= (1 + 3i)^2. \end{aligned}$$

$\therefore$ Square roots of $6i - 8$ are $\pm(1 + 31)$.

Method 2—

$$\begin{aligned} (a + ib)^2 &= -8 + 6i, \\ a^2 + 2iab - b^2 &= -8 + 6i, \\ a^2 - b^2 &= -8, \\ ab &= 3, \\ a^2 + b^2 &= 10, \\ 2a^2 &= 2, \\ a &= \pm 1, \quad b = \pm 3. \end{aligned}$$

$\therefore$ Roots are $\pm(1 + 3i)$

Comment: Both methods were well done, although the second had many more ways of making errors.

(ii) Solve $2z^2 - (3+i)z + 2 = 0$. $\boxed{2}$

Solution:
$$\begin{aligned} z &= \frac{(3+i) \pm \sqrt{(3+i)^2 - 4 \times 2 \times 2}}{2 \times 2}, \\ &= \frac{3+i \pm \sqrt{9+6i-1-16}}{4}, \\ &= \frac{3+i \pm \sqrt{6i-8}}{4}, \\ &= \frac{3+i \pm (1+3i)}{4}, \\ &= \frac{4+4i}{4}, \frac{2-2i}{4}, \\ &= 1+i, \frac{1-i}{2}. \end{aligned}$$

Or:
$$\begin{aligned} 2\left(z^2 - \left(\frac{3+i}{2}\right)z + \left(\frac{3+i}{4}\right)^2\right) &= -2 + \frac{9-1+6i}{8}, \\ \left(z - \frac{3+i}{4}\right)^2 &= -1 + \frac{8+6i}{16}, \\ &= \frac{6i-8}{16}, \\ z &= \frac{3+i}{4} \pm \frac{1+3i}{4}, \\ &= \frac{4+4i}{4}, \frac{2-2i}{4}, \\ &= 1+i, \frac{1-i}{2}. \end{aligned}$$

Comment: Only the first, easier method was used.

(e) Find the smallest positive integer values of p and q for which $\boxed{3}$

$$\frac{\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^p}{\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^q} = i.$$

Solution: Method 1—
$$\begin{aligned} \operatorname{cis}\left(p.\frac{\pi}{8} - q.\frac{-\pi}{12}\right) &= i, \\ \operatorname{cis}\left(\frac{3\pi p + 2\pi q}{24}\right) &= \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right),\ k \in \mathbb{Z}. \\ 3p + 2q &= 12,\ (\text{taking } k = 0 \text{ for smallest } p,\ q) \\ p &= \frac{12-2q}{3},\ (\text{so need factor of 3 in numerator}) \\ \therefore q &= 3, \quad p = 2. \end{aligned}$$

Comment: The most common error was failing to realise that $\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)^q = \left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)^q$.

Method 2—

$$\begin{aligned}
\cos\left(p.\frac{\pi}{8}\right) + i\sin\left(p.\frac{\pi}{8}\right) &= i\left(\cos\left(q.\frac{\pi}{12}\right) - i\sin\left(q.\frac{\pi}{12}\right)\right), \\
\cos\frac{p\pi}{8} + i\sin\frac{p\pi}{8} &= i\cos\frac{q\pi}{12} + \sin\frac{q\pi}{12},
\end{aligned}$$

Now, equating real and imaginary parts,

$$\begin{aligned}
\cos\frac{p\pi}{8} &= \sin\frac{q\pi}{12} = \cos\left(\frac{\pi}{2} - \frac{q\pi}{8}\right), \\
\sin\frac{p\pi}{8} &= \cos\frac{q\pi}{12} = \sin\left(\frac{\pi}{2} - \frac{q\pi}{8}\right), \\
\therefore \frac{p\pi}{8} &= \frac{\pi}{2} - \frac{q\pi}{8}, \\
3p + 2q &= 12, \\
p &= \frac{12-2q}{3}, \text{ (so need factor of 3 in numerator)} \\
\therefore q &= 3, \quad p = 2.
\end{aligned}$$

Comment: After equating real and imaginary parts, candidates often forgot the complementary relationship between sine and cosine.

(f) The equation $x^3 + 3x^2 + 7x + 5 = 0$ has roots α, β, and γ.

(i) Find a simplified monic equation with roots $\alpha + 1$, $\beta + 1$, and $\gamma + 1$. 2

Solution: Method 1—

$$\begin{aligned}
\text{Put } y = x + 1, \text{ then } x &= y - 1. \\
(y-1)^3 + 3(y-1)^2 + 7(y-1) + 5 &= 0, \\
y^3 - 3y^2 + 3y - 1 + 3y^2 - 6y + 3 + 7y - 7 + 5 &= 0, \\
y^3 + 4y &= 0, \\
x^3 + 4x &= 0 \text{ is the new equation.}
\end{aligned}$$

Comment: Candidates who used this method frequently had trouble with the algebraic expansion.

Method 2—

$$\begin{aligned}
S_1 :\ & (\alpha+1) + (\beta+1) + (\gamma+1) \\
&= \alpha + \beta + \gamma + 3, \\
&= -3 + 3, \\
&= 0. \\
S_2 :\ & (\alpha+1)(\beta+1) + (\beta+1)(\gamma+1) + (\gamma+1)(\alpha+1) \\
&= \alpha\beta + \beta\gamma + \gamma\alpha + 2(\alpha+\beta+\gamma) + 3, \\
&= 7 + 2(-3) + 3, \\
&= 4. \\
S_3 :\ & (\alpha+1)(\beta+1)(\gamma+1) \\
&= \alpha\beta\gamma + \alpha\beta + \beta\gamma + \gamma\alpha + \alpha + \beta + \gamma + 1, \\
&= -5 + 7 - 3 + 1, \\
&= 0.
\end{aligned}$$

$\therefore x^3 + 4x = 0$ is the new equation.

Comment: The expansions in this method caused many problems for candidates.

(ii) Hence, or otherwise, solve $x^3 + 3x^2 + 7x + 5 = 0$. $\boxed{2}$

Solution: (Hence—)

$$\begin{aligned} y\big(y^2+4\big) &= 0, \\ y &= 0, \pm 2i, \\ \text{so } x &= -1, -1 \pm 2i. \end{aligned}$$

Comment: Although quite elegant, this technique depended on part (i) being correct. Candidates who used it were generally successful.

(Otherwise—)

$$\begin{aligned} P(-1) &= -1+3-7+5, \\ &= 0. \end{aligned}$$

$$\begin{array}{r} x^2+2x+5 \\ x+1\overline{\big)\; x^3+3x^2+7x+5} \\ \underline{-x^3 \;\; -x^2 \qquad\qquad} \\ 2x^2+7x \quad\; \\ \underline{-2x^2-2x \quad\;} \\ 5x+5 \\ \underline{-5x-5} \\ 0 \end{array} \quad \text{or} \quad \begin{array}{r|rrrr} & 1 & 3 & 7 & 5 \\ -1 & & -1 & -2 & -5 \\ \hline & 1 & 2 & 5 & 0 \end{array}$$

$$\begin{aligned} \text{So } P(x) &= (x+1)(x^2+2x+5), \\ &= (x+1)(x^2+2x+1+4), \\ &= (x+1)\big((x+1)^2-(2i)^2\big), \\ &= (x+1)(x+1-2i)(x+1+2i), \\ \therefore x &= -1,\ -1+21,\ -1-21. \end{aligned}$$

Comment: This technique, although tedious, was independent of part (i) being correct. It was well done by those who used it.

Question 10 (18 Marks)

(a) The cube roots of 1 are denoted by 1, ω and ω^2 where Im $\omega^2 < 0$

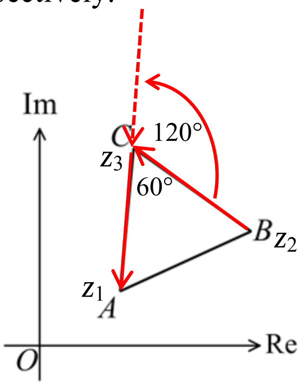
(i) Show that $1 + \omega + \omega^2 = 0$ **1**

$z^3 - 1 = 0$

$\therefore 1 + \omega + \omega^2 = 0$ (sum of roots)

NB $\omega = \text{cis}\,\frac{2\pi}{3}$ and $\omega^2 = \text{cis}\left(-\frac{2\pi}{3}\right)$

In the diagram ABC is an equilateral triangle. The points A, B and C represent the numbers z_1, z_2 and z_3 respectively.



(ii) Explain why $z_1 - z_3 = \omega\left(z_3 - z_2\right)$ **1**

$z_3 - z_2$ rotated 120° anti-clockwise is $z_1 - z_3$

From (i) $\omega = \text{cis}\,\frac{2\pi}{3}$

$\therefore z_1 - z_3 = \left(z_3 - z_2\right)\text{cis}\,\frac{2\pi}{3}$

$= \omega\left(z_3 - z_2\right)$

(iii) Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$ **2**

$\therefore z_1 - z_3 = \omega z_3 - \omega z_2$

$\therefore z_1 + \omega z_2 - z_3\left(1 + \omega\right) = 0$

$\therefore \therefore z_1 + \omega z_2 + \omega^2 z_3 = 0$ $\left[1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega = -\omega^2\right]$

Markers comments

(i) Most students were able to answer this part.

The most common solution was: $\omega^3 - 1 = \left(\omega - 1\right)\left(1 + \omega + \omega^2\right)$

$\therefore 1 + \omega + \omega^2 = 0$ $\left(\omega \neq 1\right)$

As was $1 + \omega + \omega^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 0$

Some students did attempt to use geometric series as well

(ii) Many were unable to explain properly, not seeing that $\omega = \text{cis}\,\frac{2\pi}{3}$.

(iii) Quite well done.

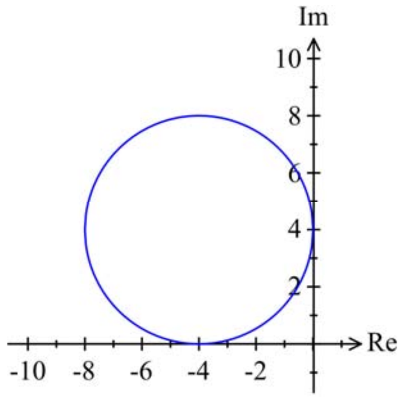
Question 10 (continued)

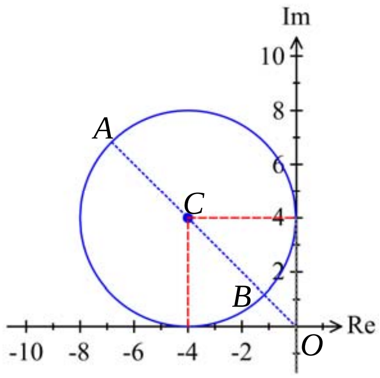
(b) The complex number z satisfies the relation $|z + 4 - 4i| = 4$

(i) Sketch, on an Argand diagram, the locus of z. **2**

$|z+4-4i| = 4 \Rightarrow |z-(-4+4i)| = 4$.

This is a circle with centre $(-4, 4)$ and radius 4





(ii) Show that the greatest value of $|z|$ is $4\left(\sqrt{2}+1\right)$. **2**

The maximum value of $|z| = OA$.

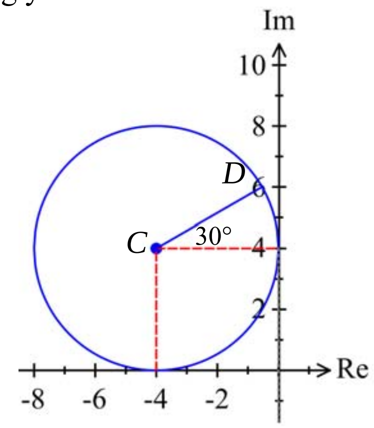
By Pythagoras' Theorem, $OC^2 = 4^2 + 4^2 = 32$ i.e. $OC = 4\sqrt{2}$

$AC = 4$

$|z|_{\max} = OA =$

$$= 4 + 4\sqrt{2} = 4\left(1+\sqrt{2}\right)$$

(iii) Find the value of z for which $\arg(z + 4 - 4i) = \frac{1}{6}\pi$ leaving your answer in the form $a + ib$. **2**



D is the point satisfying $\arg(z + 4 - 4i) = 30°$

As $CD = 4$, then the vector CD represents the number $4\text{cis}30°$.

$\therefore$ D represents $-4 + 4i + 4\text{cis}30°$

i.e. $2\sqrt{3} - 4 + 6i$

Markers comments

(i) Most students gained full marks

(ii) Well done

(iii) This part was poorly answered by most students.

Question 10 (continued)

(c) It is given that α, β and γ are the roots of $x^3 - 4x + 3 = 0$ **2**

Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

$\alpha + \beta + \gamma = 0$ (sum of roots)

Substitute α, β and γ i.e.

$\alpha^3 - 4\alpha + 3 = 0$ – (1)

$\beta^3 - 4\beta + 3 = 0$ – (2)

$\gamma^3 - 4\gamma + 3 = 0$ – (3)

(1) + (2) + (3):$\Rightarrow$ $(\alpha^3 + \beta^3 + \gamma^3) - 4(\alpha + \beta + \gamma) + 9 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = -9$

Markers comments

This question was well done. Most students used the suggested method.

A few formed a cubic with roots α^3, β^3 and γ^3 and found the sum of the rots of this equation.

(d) The polynomial $x^5 + ax^4 + 3x^3 + bx^2 + a$, where a and b are constants, has a multiple zero at $x = 2$. **3**

Find the values of a and b.

Let $P(x) = x^5 + ax^4 + 3x^3 + bx^2 + a$

$P'(x) = 5x^4 + 4ax^3 + 9x^2 + 2bx$.

$\therefore P(2) = P'(2) = 0$

$P(2) = 32 + 16a + 24 + 4b + a = 0$

$\therefore 17a + 4b = -56$ – (1)

$P'(2) = 80 + 32a + 36 + 4b = 0$

$\therefore 32a + 4b = -116$ – (2)

(1) – (2): $-15a = 60$

$\therefore a = -4$

Substitute into (1): $b = 3$

$\therefore a = -4, b = 3$

Markers comments

This was disappointingly done.

Many failed to properly answer a straight forward bookwork problem

(e) Prove that for $0 < x < 1$, $\tan^{-1}\dfrac{x}{1-x} - \tan^{-1}\dfrac{x}{1+x} = \tan^{-1} 2x^2$ **3**

$$\tan\left(\tan^{-1}\frac{x}{1-x} - \tan^{-1}\frac{x}{1+x}\right) = \frac{\frac{x}{1-x} - \frac{x}{1+x}}{1+\frac{x}{1-x}\times\frac{x}{1+x}} \qquad \left[\times\frac{(1-x)(1+x)}{(1-x)(1+x)}\right]$$

$$= \frac{x(1+x) - x(1-x)}{(1-x)(1+x)+x^2}$$

$$= \frac{x+x^2-x+x^2}{1-x^2+x^2}$$

$$= 2x^2$$

$$\therefore \tan^{-1}\frac{x}{1-x} - \tan^{-1}\frac{x}{1+x} = \tan^{-1} 2x^2$$

Why $0 < x < 1$?

As $\tan^{-1} 2x^2 \geq 0$ then $\tan^{-1}\left(\dfrac{x}{1-x}\right) \geq \tan^{-1}\left(\dfrac{x}{1+x}\right)$ as $\tan^{-1} x$ is an increasing function.

$\therefore \dfrac{x}{1-x} \geq \dfrac{x}{1+x}$

$\therefore \dfrac{x}{1-x} - \dfrac{x}{1+x} \geq 0$

$\therefore 2x^2(1-x)(1+x) \geq 0$

$\therefore -1 < x < 1$

$\therefore$ true for $0 < x < 1$

Markers comments

Most students were able to establish the ersult but virtually no one addressed the limitation i.e. $0 < x < 1$

Question 11 (18 Marks)

(a) Consider the polynomial equation $x^5 - i = 0$

(i) Show that $1 - ix - x^2 + ix^3 + x^4 = 0$ for $x \neq i$. **1**

Method 1: $i^5 = i$

$$x^5 - i^5 = (x-i)\left(x^4 + ix^3 + i^2x^2 + i^3x + i^4\right)$$

$$= (x-i)\left(x^4 + ix^3 - x^2 - ix + 1\right)$$

$$x^5 - i^5 = 0 \Rightarrow (x-i)\left(x^4 + ix^3 - x^2 - ix + 1\right) = 0$$

$$x \neq i \Rightarrow x^4 + ix^3 - x^2 - ix + 1 = 0$$

Method 2: Geometric series with $a = 1$, $r = -ix$, $n = 5$

$$S_5 = 1 - ix - x^2 + ix^3 + x^4$$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

$$S_5 = \frac{(-ix)^5 - 1}{-ix - 1}$$

$$= \frac{ix^5 + 1}{ix + 1}$$

$$= \frac{i\left(x^5 - i\right)}{ix + 1}$$

$$= 0 \qquad \left(x^5 = i, x \neq i\right)$$

(ii) Show that $(x-i)\left(x^2 - 2ix\sin\dfrac{\pi}{10} - 1\right)\left(x^2 + 2ix\sin\dfrac{3\pi}{10} - 1\right) = 0$. **4**

$x^5 = i^5 \Rightarrow x = i$ is a root and the others are equally distributed around the unit circle by $\frac{2\pi}{5}$.

So the other roots are $x = \text{cis}\left(\frac{\pi}{2} \pm \frac{2\pi}{5}\right), \text{cis}\left(\frac{\pi}{2} \pm \frac{4\pi}{5}\right)$.

i.e. $x = \text{cis}\left(\frac{\pi}{10}\right), \text{cis}\left(\frac{9\pi}{10}\right), \text{cis}\left(-\frac{3\pi}{10}\right), \text{cis}\left(-\frac{7\pi}{10}\right)$

$$\left[x - \text{cis}\left(\tfrac{\pi}{10}\right)\right]\left[x - \text{cis}\left(\tfrac{9\pi}{10}\right)\right] = x^2 - \left[\text{cis}\left(\tfrac{\pi}{10}\right) + \text{cis}\left(\tfrac{9\pi}{10}\right)\right] + \text{cis}\left(\tfrac{\pi}{10}\right)\text{cis}\left(\tfrac{9\pi}{10}\right)$$

$$= x^2 - \left[\text{cis}\left(\tfrac{\pi}{10}\right) + \text{cis}\left(\tfrac{9\pi}{10}\right)\right] + \text{cis}(\pi)\text{cis}\left(\tfrac{9\pi}{10}\right)$$

$$= x^2 - 2i\sin\tfrac{\pi}{10}x + \text{cis}\pi \qquad \left[\sin\tfrac{\pi}{10} = \sin\tfrac{9\pi}{10}, \cos\tfrac{9\pi}{10} = -\cos\tfrac{\pi}{10}\right]$$

$$= x^2 - 2i\sin\tfrac{\pi}{10}x - 1$$

Similarly,

$$\left[x - \text{cis}\left(-\tfrac{3\pi}{10}\right)\right]\left[x - \text{cis}\left(-\tfrac{7\pi}{10}\right)\right] = x^2 - 2i\sin\left(-\tfrac{3\pi}{10}\right)x + \text{cis}(-\pi)$$

$$= x^2 + 2i\sin\tfrac{3\pi}{10}x - 1 \qquad \left[\sin\tfrac{3\pi}{10} = \sin\tfrac{7\pi}{10}, \cos\tfrac{7\pi}{10} = -\cos\tfrac{3\pi}{10}\right]$$

$$x^5 - i = (x-i)\left(x^2 - 2i\sin\tfrac{\pi}{10}x - 1\right)\left(x^2 + 2i\sin\tfrac{3\pi}{10}x - 1\right)$$

(iii) Using the results in (i) and (ii) and using a suitable substitution, show that **2**

$$\sin\frac{\pi}{10}\sin\frac{3\pi}{10}=\frac{1}{4}.$$

From (i) and (ii): $1-ix-x^2+ix^3+x^4=\left(x^2-2i\sin\tfrac{\pi}{10}x-1\right)\left(x^2+2i\sin\tfrac{3\pi}{10}x-1\right)$

Method 1: Substitute $x=1$

$1-i-1+i+1=\left(1-2i\sin\tfrac{\pi}{10}-1\right)\left(1+2\sin\tfrac{3\pi}{10}i-1\right)$

$\therefore 1=\left(-2i\sin\tfrac{\pi}{10}\right)\left(2i\sin\tfrac{3\pi}{10}\right)$

$\therefore 1=-4i^2\left(\sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}\right)$

$\therefore \sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}=\tfrac{1}{4}$

Method 2: Coefficient of x^2

The coefficient of x from $\left(x^2-2i\sin\tfrac{\pi}{10}x-1\right)\left(x^2+2i\sin\tfrac{3\pi}{10}x-1\right)$ is

$-1-1-4i^2\sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}=-2+4\sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}$

$\therefore -2+4\sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}=-1$

$\therefore \sin\tfrac{\pi}{10}\sin\tfrac{3\pi}{10}=\tfrac{1}{4}$

Markers comments

(i) Candidates should be more explanatory. For instance "$i=i^5$, now factorise …"

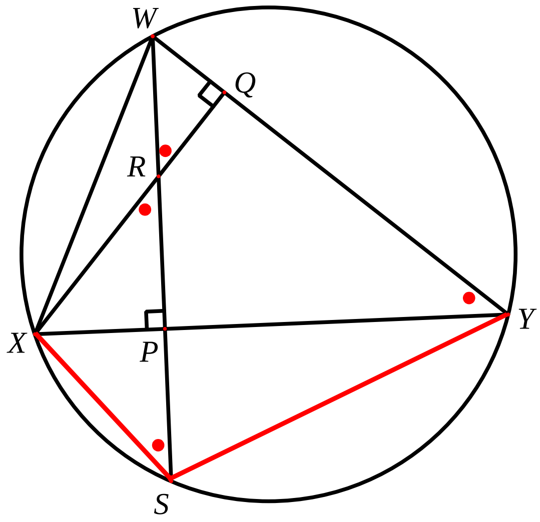
(ii) Too many candidates did not draw a diagram, and assumed that the roots behaved as with roots of unity. They are not successive powers of ω, nor are they in simple conjugate pairs, but rather $a+ib\rightarrow -a+ib$, which reverses the sum/product relationships.

(iii) Generally well answered, mainly by one of the two methods above.

Question 11 (continued)

(b) The altitudes WP and XQ in an acute-angled triangle WXY meet at R. WR produced cuts the circle through the vertices W, X and Y at S. **3**

Prove that $PR = PS$.



Let $\angle XRP = x°$	
$\therefore \angle WRQ = x°$	(vert. opp. $\angle$s)
$\Delta WQR \parallel\!\mid \Delta YPW$	(equiangular)
$\therefore \angle WXP = x°$	(matching $\angle$s of similar Δs)
$\therefore \angle XSP = x°$	(angles in same segment)
$\therefore \Delta XPR \equiv \Delta XPS$	(RHS)
$\therefore PR = PS$	(matching sides of cong. Δs)

Markers comments

Candidates should always copy the diagram into their answer booklet, **especially** if they are going to add points.

Several exhibited a lack of understanding of the necessary conditions for a kite, and some confused similarity with congruency.

(d) Twelve of the Maths Staff wanted to be photographed at the Extension 2 Big Day Out. **3**
This meant that, Paul, Bob, Mark, Andrew, Patrick, Brenda, Ross, Ron, Jacqui, Vicki, Jess and Doug all had to line up in a straight line.
There were a few rules that had to be followed:

- Teachers with the letter *a* in their name could not stand together in adjacent positions in the line.
- Since Paul and Ross get up to mischief, they can never stand in adjacent positions in the line.

The teachers randomly organise themselves in the line.
What is the probability that they arrange themselves in a correct order?

How many ways to correctly line up?
Put the teachers that don't have an "*a*" in their name to line up first and leave space between themselves for the others to be put in

$\_\ \boxed{6}\ \_\ \boxed{5}\ \_\ \boxed{4}\ \_\ \boxed{3}\ \_\ \boxed{2}\ \_\ \boxed{1}\ \_$

There are 6! ways for these teachers to line up.

$\_\ \boxed{}\ \_\ \boxed{}\ \_\ \boxed{}\ \cancel{P}\ \boxed{R}\ \cancel{P}\ \boxed{}\ \_\ \boxed{}\ \_$

There are 7 positions that can now be filled with the "*a*" teachers.
Remembering that Paul and Ross can't stand next to each other, so no matter where Ross (R) is, there are two positions either side that Paul cannot stand in.
So Paul has a choice of 5 positions.

Now the remaining 5 teachers can be arranged in ${}^6P_5$ ways.

$\therefore$ there are $6! \times 5 \times {}^6P_5 =$ ways for the teachers to line up correctly.

$\therefore$ the probability is $\dfrac{5\times(6!)^2}{12!} = \dfrac{5}{924}$

Markers comments

One candidate achieved full marks, another was very close.
One mark was awarded for the correct denominator.
The problem arises because of the double conditions – either alone would have been better answered.

Question 11 (continued)

(e) (i) Let $u = a + ib$ be a fixed complex number. **2**

Show that as the complex number $z = x + iy$ varies, the locus defined by the equation

$$u\bar{z} + \bar{u}z = 2$$

is a straight line in the Argand diagram.

$$\begin{aligned} u\bar{z} + \bar{u}z &= 2\,\mathrm{Re}\,u\bar{z} \\ &= 2\,\mathrm{Re}(a+ib)(x-iy) \\ &= 2ax + 2by \end{aligned}$$

$u\bar{z} + \bar{u}z = 2 \Rightarrow 2ax + 2by = 2$

$\therefore ax + by = 1$

$\therefore$ $u\bar{z} + \bar{u}z = 2$ is a straight line in the Argand diagram

(ii) Show that if $|u| = 1$, then $u\bar{z} + \bar{u}z = 2$ is a tangent to the unit circle $|z| = 1$. **1**

The distance, d, from the centre of the circle to the line is given by:

$$\begin{aligned} d &= \frac{|a \times 0 + b \times 0 - 1|}{\sqrt{a^2 + b^2}} \\ &= \frac{1}{|u|} \\ &= 1 \qquad [|u| = 1] \end{aligned}$$

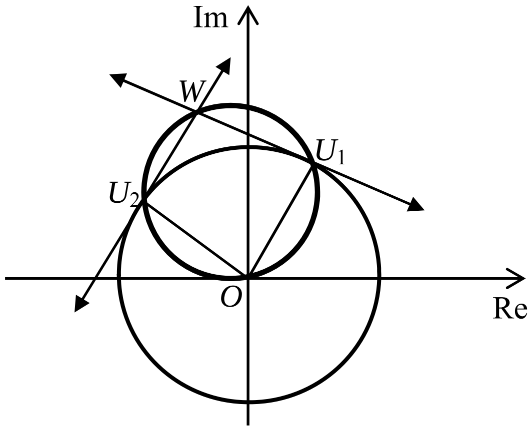
The distance is equal to the radius and so $u\bar{z} + \bar{u}z = 2$ is a tangent.

Question 11 (continued)

(e) (iii) If $|u| = 1$ then the point of contact between the unit circle and the tangent $u\overline{z} + \overline{u}z = 2$ is u. (Do NOT prove) **2**

Let u_1 and u_2 be two distinct values of u, both of unit modulus, and suppose that $u_1 + u_2 \neq 0$.
In the diagram below, the tangents $u_1\overline{z} + \overline{u}_1 z = 2$ and $u_2\overline{z} + \overline{u}_2 z = 2$ touch $|z| = 1$ at U_1 and U_2 respectively, and intersect at W.

Show that the cyclic quadrilateral OU_1WU_2 is circumscribed by a circle of radius $\left| u_1 + u_2 \right|^{-1}$.

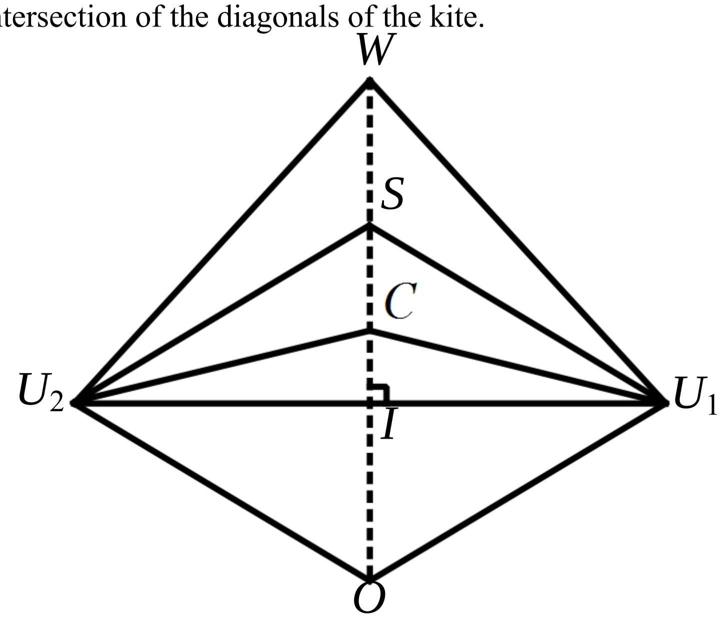


You may assume that the sum of the vectors OU_1 and OU_2 lies on OW.

Let C be the centre of the circle and S the endpoint of the sum of vectors OU_1 and OU_2.

Let r be the radius of the circle i.e. $OW = 2r$.

By construction SU_1OU_2 is a rhombus with $OU_1 = U_1S = SU_2 = U_2O = 1$
OU_1WU_2 is a kite.
$IO = SI = \frac{1}{2}| u_1 + u_2 |$
Let I be the intersection of the diagonals of the kite.



Question 11 (continued)

(e) (iii) **Method 1**

$\angle OU_1W = 90°$	(radius perpendicular to tangent)
$\angle OIU_1 = 90°$	(diagonals of rhombus are perpendicular)
$\angle WOU_1 = \angle IOU_1$	(shared angle)
$\therefore \Delta OWU_1 \text{ ||| } \Delta OIU_1$	(equiangular)
$\therefore \dfrac{OW}{OU_1} = \dfrac{OU_1}{IO}$	(matching sides of similar Δs in same ratio)

$$\therefore \frac{2r}{1} = \frac{1}{\frac{1}{2}\left| u_1 + u_2 \right|}$$

$$\therefore r = \frac{1}{\left| u_1 + u_2 \right|}$$

Method 2

$OS = | u_1 + u_2 |$

$U_1U_2 = | u_1 - u_2 |$

$SI = \frac{1}{2}| u_1 + u_2 |$ and $IU_1 = \frac{1}{2}| u_1 - u_2 |$

Let $x = CI$ and so $WI = r + x$

$r = x + \frac{1}{2}| u_1 + u_2 | \Rightarrow r - x = \frac{1}{2}| u_1 + u_2 |$

Applying Pythagoras' Theorem to ΔISU_1:

$U_1I^2 + SI^2 = U_1S^2$

$\therefore \frac{1}{4}| u_1 - u_2 |^2 + \frac{1}{4}| u_1 + u_2 |^2 = 1$

$\therefore | u_1 - u_2 |^2 + | u_1 + u_2 |^2 = 4$

Similarly, in ΔCSU_2:

$x^2 + \frac{1}{4}| u_1 - u_2 |^2 = r^2$

$\therefore | u_1 - u_2 |^2 = 4\left(r^2 - x^2\right)$

$\therefore 4 - | u_1 + u_2 |^2 = 4\left(r^2 - x^2\right)$

$4(r - x)(r + x) = 4 - | u_1 + u_2 |^2$

$\therefore 4 \times \frac{1}{2}| u_1 + u_2 | \times (r + x) = 4 - | u_1 + u_2 |^2 \qquad \left[r - x = \frac{1}{2}| u_1 + u_2 |\right]$

$$\therefore r + x = \frac{4 - | u_1 + u_2 |^2}{2| u_1 + u_2 |} \qquad -(1)$$

$$r - x = \frac{1}{2}| u_1 + u_2 | \qquad -(2)$$

Question 11 (continued)

(e) (iii) (1) + (2):

$$2r = \frac{4 - |u_1 + u_2|^2}{2|u_1 + u_2|} + \frac{|u_1 + u_2|}{2}$$

$$\therefore 2r = \frac{4}{2|u_1 + u_2|}$$

$$\therefore r = \frac{1}{|u_1 + u_2|}$$

Markers comments

(i) Quite well answered, generally by multiplication and addition.

(ii) Very few of those who attempted this part got anywhere with it.
Some succeeded by forming a quadratic, and showing that the discriminant was zero.
One only used the method suggested above.

(iii) Very few even began to attempt this part – none was successful.

End of solutions and comments