Sydney Boys High School

MOORE PARK, SURRY HILLS

2016

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 90 minutes
- Write using **black** pen.
- Board-approved calculators may be used.
- A reference sheet is provided with this paper.
- Reproduced diagram and answer paper is provided for Question 8 (d).
- In Questions 6 – 8, show relevant mathematical reasoning and/ or calculations.

Total marks – 70

Section I Pages 2 – 3

5 marks

- Attempt Questions 1 – 5
- Allow about 5 minutes for this section.

Section II Pages 4 – 8

65 marks

- Attempt Questions 6 – 8
- Allow about 1 hours and 25 minutes for this section.

Examiner: EC

Section I

5 marks
Attempt Questions 1 – 5
Allow about 5 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 5.

---

1. What are the solutions to the quadratic equation $2z^2+(1-i)z+(1-i)=0$?

(A) $z_1=-\frac{1}{2}-\frac{1}{2}i$ and $z_2=-2+2i$

(B) $z_1=i$ and $z_2=-2-2i$

(C) $z_1=-\frac{1}{2}-\frac{1}{2}i$ and $z_2=i$

(D) $z_1=-2+2i$ and $z_2=-i$

2. If $z=\cos\theta+i\sin\theta$, for $\pi<\theta<\frac{3\pi}{2}$. What is the value of the principal argument of z?

(A) θ

(B) $2\pi-\theta$

(C) $\theta-2\pi$

(D) $\theta+2\pi$

3. There are 10 English books and 10 Mathematics books.
From these 20 books Andrew must choose 4 books that cover both subjects. How many ways can this be done?

(A) $2\binom{20}{2}$

(B) $\binom{10}{2}\binom{10}{2}$

(C) $\binom{10}{1}\binom{10}{3}+\binom{10}{2}\binom{10}{2}$

(D) $2\binom{10}{1}\binom{10}{3}+\binom{10}{2}\binom{10}{2}$

4. It is known that the three roots of the cubic equation $2x^3 + 3x^2 + 6x + 16 = 0$ form a geometric progression. The second term in this geometric progression is:

(A) 2

(B) $-2i$

(C) -2

(D) $2i$

5. Given two complex numbers z_1 and z_2, $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$, also $|z_1| = r_1$ and $|z_2| = r_2$.

When $|z_1 - z_2|$ has a maximum value $r_1 + r_2$, what is the relation between θ_1 and θ_2?

(A) $|\theta_1 + \theta_2| = \pi$

(B) $|\theta_1 - \theta_2| = \pi$

(C) $|\theta_1 + \theta_2| = 2\pi$

(D) $|\theta_1 - \theta_2| = 2\pi$

Section II

65 marks
Attempt Questions 6 – 8
Allow about 1 hour and 25 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 6 – 8, your responses should include relevant mathematical reasoning and/ or calculations.

Question 6 (24 marks) Use a SEPARATE writing booklet

(a) If $z = 2 + 3i$ and $w = -4 - i$, find:

(i) $\dfrac{w}{z}$ **2**

(ii) $\left|\dfrac{\overline{w}}{z}\right|$ **2**

(b) Find the two square roots of $-7 + 24i$. **2**

(c) Sketch the following loci on an Argand diagram:

(i) $|z| < |z - 2 + i|$ **2**

(ii) $\left|\dfrac{z-2}{z-4i}\right| = 1$ **2**

(iii) $\arg\left(\dfrac{z-2}{z+i}\right) = -\dfrac{\pi}{3}$ **2**

(d) Given the equation $z^3 - 1 = 0$. If w is one of the non-real roots prove that:

(i) $1 + w = \dfrac{1}{1+w^2}$ **2**

(ii) $\left(1 + w - w^2\right)^3 - \left(1 - w + w^2\right)^3 = 0$ **2**

Question 6 continues on page 5

Question 6 (continued)

(e) If $z_1 = 6 + 8i$ and $|z_2| = 5$, find:

(i) the particular value of z_2 for which $|z_1 + z_2| = |z_1| + |z_2|$, **2**

(ii) the greatest possible value (to the nearest degree) of $\arg(z_1 + z_2)$. **2**

(f) Let z_1 and z_2 be complex numbers,

(i) By using the fact that $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$, prove that $\overline{\left(z_1 \overline{z_2}\right)} = \overline{z_1}\, z_2$. **2**

(ii) Hence prove that $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\left(z_1 \overline{z_2}\right)$. **2**

End of Question 6

Question 7 (22 marks) Use a SEPARATE writing booklet.

(a) It is given that $x = 3i$ is a root of the polynomial $P(x) = x^4 + 2x^3 + 14x^2 + 18x + 45$.

(i) Express $P(x)$ as a product of quadratic factors over $\mathbb{R}$. **2**

(ii) Express $P(x)$ as a product of linear factors over $\mathbb{C}$. **3**

(b) The polynomial equation $x^3 + 3px + q = 0$ has roots α, β and γ. If $\gamma = \alpha\beta$, show that $(3p - q)^2 + q = 0$. **3**

(c) The polynomial $f(x) = x^3 + 3x^2 - 5x + 7$ has zeros α, β and γ.

(i) Find $(\alpha + 1)(\beta + 1)(\gamma + 1)$ **2**

(ii) Write down a polynomial $g(x)$ with zeros $\alpha + 1$, $\beta + 1$ and $\gamma + 1$ (you do not have to simplify your answer). **3**

(iii) The polynomial $h(x) = f(x) + ax + b$ where $a, b \in \mathbb{R}$, has one triple zero ϕ. Find ϕ, a and b. **4**

(d) Prove by Mathematical Induction that for $n \geq 1$ **5**

$$\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \ldots + \frac{n}{(2n-1)!} \leq 2 - \frac{1}{(2n)!}.$$

End of Question 7

Question 8 (19 marks) Use a SEPARATE writing booklet.

(a) There are 14 girls in a squad of netballers who are trialling for the representative team. Only 7 of the girls will be selected.

(i) In how many different ways can the girls be divided into two teams of 7 for a trial game? **2**

(ii) The selectors eventually decide to choose 7 players plus an umpire. In how many ways can this be done? **2**

(b) Three men decide to have dinner together at Chatswood. They have agreed to meet at the "All U Can Eat" restaurant, Chatswood. Unknown to these men, there are three restaurants with this name in Chatswood. Assuming that each man is equally likely to choose any one of the "All U Can Eat" restaurants, what is the probability that:

(i) All three men go to different "All U Can Eat" restaurants. **2**

(ii) All three men go to the same "All U Can Eat" restaurant. **2**

(c) It is given that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

(i) By making the substitution $x = 2\cos\theta$, deduce that $2\cos\frac{\pi}{18}$ is one of the roots of the equation $x^6 - 6x^4 + 9x^2 - 3 = 0$. Find the other five roots of the equation. **4**

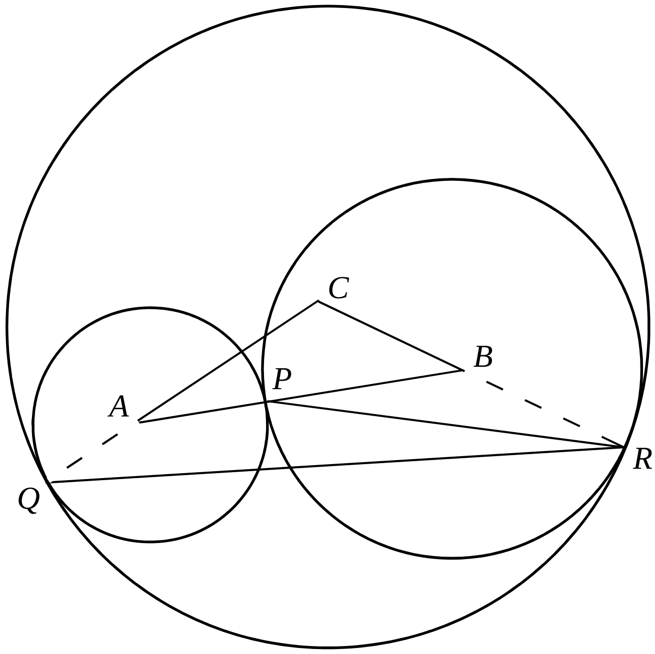
(ii) Hence, show the equation $x^3 - 6x^2 + 9x - 3 = 0$ has roots $2\left(1+\cos\left(\frac{\pi}{9}\right)\right)$, $2\left(1-\cos\left(\frac{2\pi}{9}\right)\right)$ and $2\left(1-\cos\left(\frac{4\pi}{9}\right)\right)$ **4**

Question 8 continues on page 8

Question 8 (continued)

(d) **You do NOT need to copy the diagram below. It has been reproduced on pages 1A and 2A.**

Two circles centre A, B touch externally at P, a third circle centre C, encloses both, touching the first circle at Q and the second circle at R as shown.



Your answer to Question 8 (d) needs to be provided on the additional pages 1A and 2A, then inserted into the Writing booklet for Question 8.

(i) Show that APB, CAQ and CBR are straight lines. **1**

(ii) Prove that $\angle BAC = 2\angle PRQ$. **2**

End of Paper

Sydney Boys High School

MOORE PARK, SURRY HILLS

2016

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK 1

Mathematics Extension 2

Suggested Solutions

Markers

Q6	AMG
Q7	PB
Q8	PSP

MC Answers

1. C 2. C 3. D 4. C 5. B

1. What are the solutions to the quadratic equation $2z^2+(1-i)z+(1-i)=0$?

(A) $z_1=-\frac{1}{2}-\frac{1}{2}i$ and $z_2=-2+2i$

(B) $z_1=i$ and $z_2=-2-2i$

(C) $z_1=-\frac{1}{2}-\frac{1}{2}i$ and $z_2=i$

(D) $z_1=-2+2i$ and $z_2=-i$

Method 1:
Look at the sums and products i.e. $\alpha+\beta=\frac{1}{2}(i-1)$ and $\alpha\beta=\frac{1}{2}(1-i)$

Method 2:
As $z=\pm i$ is so prevalent, test it. This means that B or C is the answer. Then use the sum of roots.

Method 3: Quadratic formula.

$$z=\frac{-1+i\pm\sqrt{(1-i)^2-8(1-i)}}{4}$$
$$=\frac{-1+i\pm\sqrt{-8+6i}}{4}$$
$$=\frac{-1+i\pm\sqrt{1^2-3^2+2\times3\times1i}}{4}$$
$$=\frac{-1+i\pm\sqrt{(1+3i)^2}}{4}$$
$$=\frac{-1+i\pm(1+3i)}{4}$$
$$=i,-\frac{1}{2}-\frac{1}{2}i$$

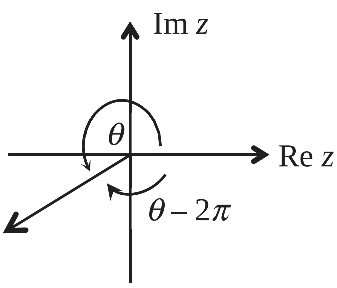
2. If $z=\cos\theta+i\sin\theta$, for $\pi<\theta<\frac{3\pi}{2}$. What is the value of the principal argument of z?

(A) θ

(B) $2\pi-\theta$

(C) $\theta-2\pi$

(D) $\theta+2\pi$



$-\pi<\arg z\leq\pi$

3. There are 10 English books and 10 Mathematics books.
From these 20 books Andrew must choose 4 books that cover both subjects. How many ways can this be done?

(A) $2\binom{20}{2}$

(B) $\binom{10}{2}\binom{10}{2}$

(C) $\binom{10}{1}\binom{10}{3}+\binom{10}{2}\binom{10}{2}$

(D) $2\binom{10}{1}\binom{10}{3}+\binom{10}{2}\binom{10}{2}$

With 4 books and having at least one of each, then there are two cases:

Case 1: 1 and 3 i.e. 1 English and 3 Mathematics (and vice versa): $2\binom{10}{1}\binom{10}{3}$

Case 2: 2 of each $\binom{10}{2}\binom{10}{2}$

Total $= 2\binom{10}{1}\binom{10}{3}+\binom{10}{2}\binom{10}{2}$

4. It is known that the three roots of the cubic equation $2x^3 + 3x^2 + 6x + 16 = 0$ form a geometric progression. The second term in this geometric progression is:

(A) 2

(B) $-2i$

(C) -2

(D) $2i$

Let the roots be $\frac{a}{r}, a, ar$.

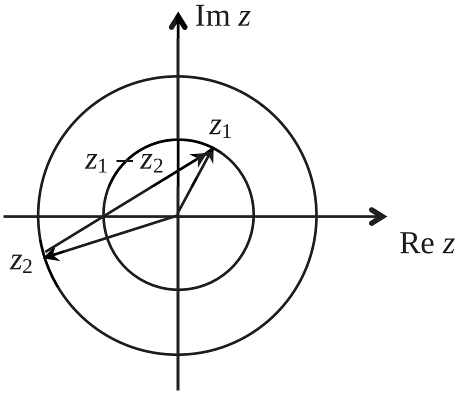
Using the product of the roots: $\frac{a}{r} \times a \times ar = -\frac{16}{2} = -8$

$\therefore a^3 = -8$

$\therefore a = -2$

5. Given two complex numbers z_1 and z_2, $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$, also $|z_1| = r_1$ and $|z_2| = r_2$.

When $|z_1 - z_2|$ has a maximum value $r_1 + r_2$, what is the relation between θ_1 and θ_2?



(A) $|\theta_1 + \theta_2| = \pi$

(B) $|\theta_1 - \theta_2| = \pi$

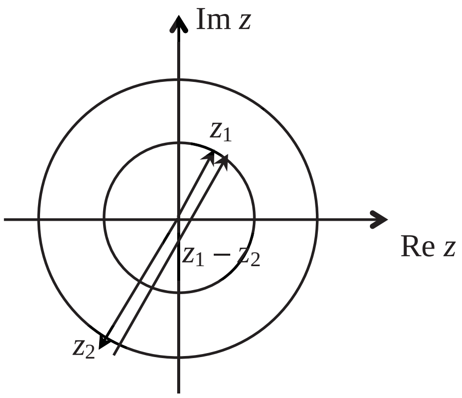
(C) $|\theta_1 + \theta_2| = 2\pi$

(D) $|\theta_1 - \theta_2| = 2\pi$

In the diagram above, $|z_1 - z_2| \neq r_1 + r_2$

The maximum value of $|z_1 - z_2|$ is when z_1 and z_2 are collinear with the origin and in opposite quadrants.

From the diagram $\theta_1 - \theta_2 = \pi$, but if z_1 and z_2 are switched then $\theta_1 - \theta_2 = -\pi$



Question 6 Solutions

(a) $z = 2 + 3i$, $w = -4 - i$

(i) $$\frac{w}{z} = \frac{-4-i}{2+3i} \times \frac{2-3i}{2-3i}$$

$$= \frac{-8+12i-2i-3}{4+9}$$

$$= \frac{-11+10i}{13}$$

This question was very well answered. The only errors were careless, or transcription.

(ii) $$\left|\frac{\overline{w}}{z}\right| = \frac{|\overline{w}|}{|z|}$$

$$= \frac{|w|}{\sqrt{13}}$$

$$= \frac{\sqrt{17}}{\sqrt{13}}$$

$$= \sqrt{\frac{17}{13}}$$

Alternatively

$$\left|\frac{\overline{w}}{z}\right| = \left|\frac{-4+i}{2+3i} \times \frac{2-3i}{2-3i}\right|$$

$$= \left|\frac{-8+12i+2i+3}{4+9}\right|$$

$$= \left|\frac{-5+14i}{13}\right|$$

$$= \sqrt{\frac{221}{169}}$$

$$= \sqrt{\frac{17}{13}}$$

Whilst a few lost sight of what was required, most answered the part well.

(b) Let $-7+24i=(a+ib)^2$

$\therefore \quad a^2-b^2+2abi=-7+24i$

Equating real and imaginary parts:

Real: $a^2-b^2=-7$

Imaginary: $2ab=24$

$$b=\frac{12}{a}$$

Thus $a^2-\dfrac{144}{a^2}=-7$

$$a^4+7a^2-144=0$$

$$\therefore \quad a^2=\frac{-7\pm\sqrt{49+576}}{2}$$

$$=-16,9$$

Now a is real, so

$$a=3,\ -3,\ \ b=4,\ -4$$

$\therefore$ Square roots are $3+4i$ and $-3-4i$.

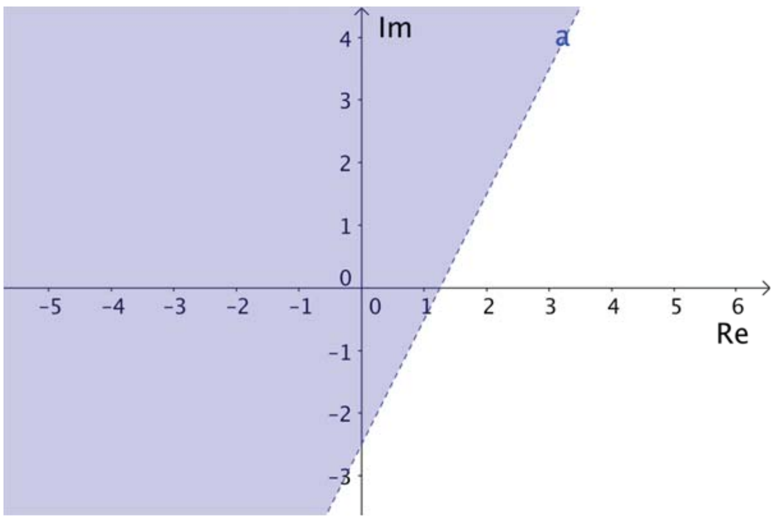
Again this was very well answered, often by inspection.

(c) (i) $|\,z\,|<|\,z-(2-i)\,|$

Let $z=a+ib$ and square both sides:

$$x^2+y^2<(x-2)^2+(y+1)^2$$

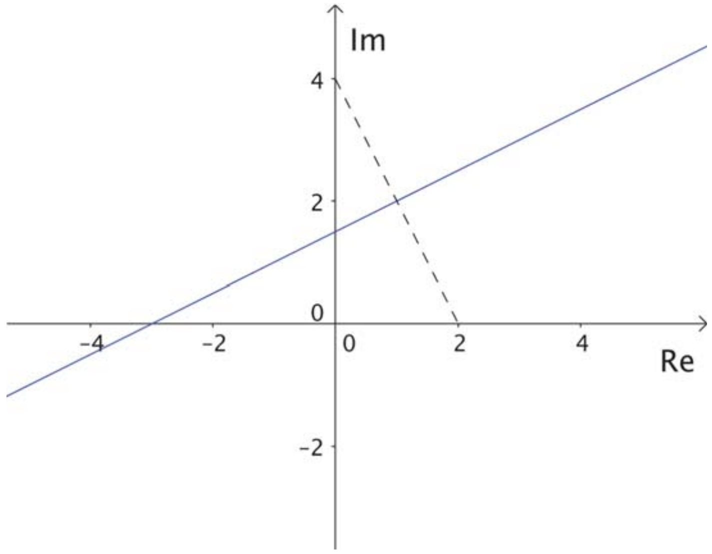
$$0<-4x+2y+5$$



Quite well answered, though some omitted the dotted boundary.

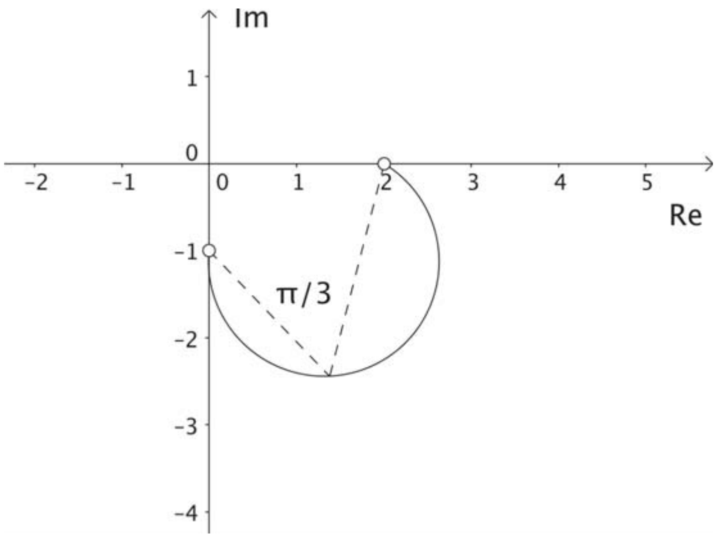
(c) (ii) $\left|\dfrac{z-2}{z-4i}\right|=1$

That is $|z-2|=|z-4i|$



Very well answered.

(iii) $\arg\left(\dfrac{z-2}{z+i}\right)=-\dfrac{\pi}{3}$



Many candidates placed the curve on the wrong side, and many others failed to exclude the end points. Otherwise the question seemed to be well understood.

(d) $z^3 - 1 = 0$ has roots are $1, w, w^2$

(i) Consider $1 + w - \left(\dfrac{1}{1+w^2}\right)$

$$\begin{aligned}
1 + w - \left(\frac{1}{1+w^2}\right) &= \frac{(1+w)(1+w^2) - 1}{1+w^2} \\
&= \frac{1 + w^2 + w + w^3 - 1}{1+w^2} \\
&= \frac{0 + 1 - 1}{1+w^2} \\
&= 0
\end{aligned}$$

$\therefore\ 1 + w = \dfrac{1}{1+w^2}$ as required.

This was quite well answered.
Most used the "LHS = … = RHS" approach.
A common error was "begging the question", that is, starting with that which must be proved.

(ii) RTP: $(1 + w - w^2)^3 - (1 - w + w^2)^3 = 0$

$$\begin{aligned}
\text{LHS} &= (1 + w + w^2 - 2w^2)^3 - (1 + w + w^2 - 2w)^3 \\
&= (-2w^2)^3 - (-2w)^3 \\
&= -8w^6 + 8w^3 \\
&= -8 + 8 \qquad [\because w^3 = 1] \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

Whilst many answered this question well, about a third got lost along the way.
Some used the actual roots, with limited success.

(e) $z_1 = 6+8i$, $|z_2| = 5$

(i) For equality in the triangle inequality, the longest side is the sum of the other two. In the Argand diagram this means the arguments are all the same, that is the same as z_1.

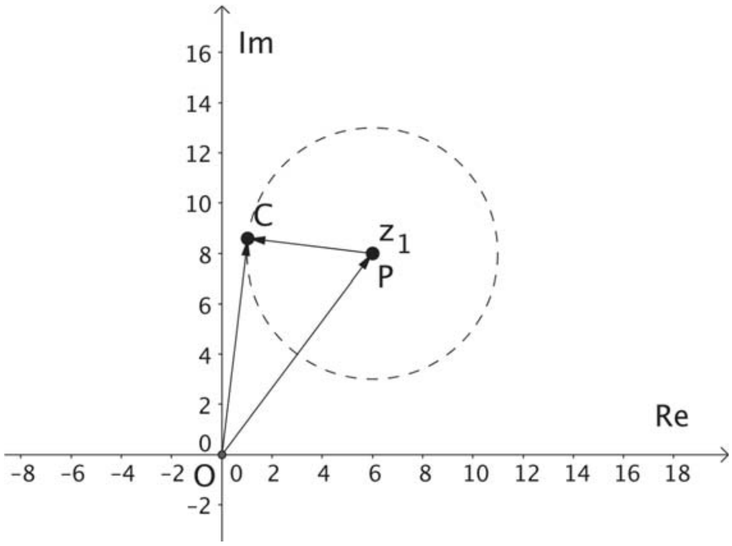
$$\text{Thus } z_2 = z_1 \times \frac{|z_2|}{|z_1|}$$

$$= \frac{5(6+8i)}{10}$$

$$\therefore \quad z_2 = 3+4i$$

This was generally answered correctly, but many took a long way round to get to the simple result.
Some erroneously thought – z_2 would lead to the result. Some found the sum, which was not required.

(ii) Using triangle of vectors addition, $z_1 + z_2$ lies on the circle shown:



$OP = z_1$ $\quad PC = z_2$ $\quad OC = z_1 + z_2$

Now $\quad \arg(z_1 + z_2) = \arg(z_1) + \angle POC$

Clearly this argument has maximum value when *OC* is tangent to the circle.
In this situation there is a right angle at *C*, and

$$\arg(z_1 + z_2) = \arg(z_1) + \sin^{-1}\left(\frac{|z_2|}{|z_1|}\right)$$

$$= \arg(z_1) + \sin^{-1}\left(\tfrac{1}{2}\right)$$

$$\doteqdot 83.13°$$

$$\doteqdot 83°$$

This proved to be the most difficult question. Very few (1 or 2) got full marks, as most failed to see the need for the tangent, whether the circle was placed as above, or at the origin.
Unfortunately many answers obtained by wrong methods were $\doteqdot$ 83°, but had to be marked wrong.

(f) (i) RTP: $\overline{(z_1\overline{z_2})} = \overline{z_1}z_2$

$$
\begin{aligned}
\text{LHS} &= \overline{(z_1\overline{z_2})} \\
&= \overline{z_1}\,\overline{\overline{z_2}} \\
&= \overline{z_1}z_2 \\
&= \text{RHS}
\end{aligned}
$$

QED

Very well answered – only careless errors. Some used a Cartesian approach, unnecessarily.

(ii) RTP: $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2})$

$$
\begin{aligned}
\text{LHS} &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + |z_2|^2 + \left(z_1\overline{z_2} + \overline{z_1\overline{z_2}}\right) \\
&= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \\
&= \text{RHS}
\end{aligned}
$$

QED

Answered well by only perhaps 40% of candidates.
Many who succeeded used a Cartesian approach, whilst the greater number proceeded as above.
Those who failed often began $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1z_2|$, and went downhill from there.

QUESTION 7. (X2)

(a) (i) Since $x = 3i$ is a zero of $P(x)$
by the "Conjugate root" theorem $x = -3i$
is also a zero.
$\therefore$ $x^2 + 9$ is a factor of $P(x)$

let $P(x) = (x^2+9)(x^2+ax+b)$

now $9b = 45$
$\therefore$ $\underline{b = 5}$
& $9a = 18$
$\therefore$ $\underline{a = 2}$.

$\therefore$ $P(x) = (x^2+2x+5)(x^2+9)$

(ii) $x^2+2x+5 = (x+1)^2+4$.
$= (x+1+2i)(x+1-2i)$

$\therefore$ $P(x) = (x-3i)(x+3i)(x+1+2i)(x+1-2i)$

<u>COMMENT</u> * many students used "long division" in (i).

* generally well answered.

(b) Given $x^3 + 3px + q = 0$ with roots α, β & γ.

now. $\alpha + \beta + \gamma = 0$ — (1)

$\alpha\beta + \alpha\gamma + \beta\gamma = 3p$. — (2)

$\alpha\beta\gamma = -q$. — (3)

& $\gamma = \alpha\beta$ (4) (given)

$\therefore \gamma^2 = -q$ from (3) & (4)

Also $(\alpha\beta)^2 = -q$.

Now from (2).

$\alpha\beta + \gamma(\alpha + \beta) = 3p$.

$\therefore \gamma - \gamma^2 = 3p$ (from (1) $\alpha + \beta = -\gamma$)

$\therefore \gamma + q = 3p$

$\gamma = 3p - q$.

$\gamma^2 = (3p-q)^2$

$\therefore (3p-q)^2 = -q$.

$(3p-q)^2 + q = 0$.

<u>COMMENT</u>: * There were a variety of different methods.

*. Well done.

(c) $f(x) = x^3 + 3x^2 - 5x + 7$ has roots α, β & γ

$\therefore \alpha+\beta+\gamma = -3$

$\alpha\beta+\alpha\gamma+\beta\gamma = -5.$

$\alpha\beta\gamma = -7.$

(i) $(\alpha+1)(\beta+1)(\gamma+1) = \alpha\beta\gamma + \alpha\beta+\alpha\gamma+\beta\gamma + \alpha+\beta+\gamma + 1.$

$= -7 - 5 - 3 + 1.$

$= -14.$

(ii) let $X = x+1 \quad \therefore x = X-1.$

$\therefore g(x) = (X-1)^3 + 3(X-1)^2 - 5(X-1) + 7.$

(iii) $h(x) = f(x) + ax + b$ has a triple root.

$h'(x) = 3x^2 + 6x - 5 + a.$

$h''(x) = 6x + 6$

now $h''(x) = 0.$

$\therefore 6x+6 = 0. \quad \therefore \boxed{x = \phi = -1.}$

$\therefore h'(-1) = 0$

$\therefore 3 - 6 - 5 + a = 0$

$\boxed{a = 8}$

&. $h(-1) = -1 + 3 + 5 + 7 - 8 + b = 0$

$\boxed{b = -6.}$

COMMENT ON (c)

* Some students found $g(x)$ to answer (i).
* . Well done.

(d) Aim to prove.

$$\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \cdots + \frac{n}{(2n-1)!} \leq 2 - \frac{1}{(2n)!}$$

for $n \in \mathbb{Z}^+$.

STEP I. Let $n = 1$

i.e. $\frac{1}{1!} \leq 2 - \frac{1}{2} = 1\frac{1}{2}$

$\therefore$ true when $n = 1$.

STEP II. Assume true when $n = k$.

i.e. $\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \cdots + \frac{k}{(2k-1)!} \leq 2 - \frac{1}{(2k)!}$

STEP III R.T.P. true when $n = k+1$
(using assumption)

i.e. $\frac{1}{1!} + \frac{2}{3!} + \cdots + \frac{k}{(2k-1)!} + \frac{k+1}{(2k+1)!} \leq 2 - \frac{1}{(2k+2)!}$

$$\begin{aligned} LHS &\leq 2 - \frac{1}{(2k)!} + \frac{k+1}{(2k+1)!} \\ &= 2 - \frac{(2k+1) - (k+1)}{(2k+1)!} \end{aligned}$$

$\boxed{*.*}$

$$= 2 - \frac{k}{(2k+1)!}$$

$$= 2 - \frac{k(2k+2)}{(2k+2)!}$$

$$\leq 2 - \frac{1}{(2k+2)!}$$

as required

REASON

$k(2k+2) \geq 1 \quad ; k \in \mathbb{Z}^+$

$\therefore \frac{k(2k+2)}{(2k+2)!} \geq \frac{1}{(2k+2)!}$

$-\frac{k(2k+2)}{(2k+2)!} \leq \frac{-1}{(2k+2)!}$

$\therefore 2 - \frac{k(2k+2)}{(2k+2)!} \leq 2 \; \frac{-1}{(2k+2)!}$

CONCLUSION

By the Principle of Mathematical Induction it has been proven for $n \in \mathbb{Z}^+$.

COMMENTS

* many students made an algebraic error in the step marked $\boxed{**.}$

*. a wide variety approaches were used in STEP III.

*. a significant number of students obtained full marks.

Question 8 Solutions

(a) There are 14 girls in a squad of netballers who are trialling for the representative team. Only 7 of the girls will be selected.

(i) In how many different ways can the girls be divided into two teams of 7 for a trial game? **2**

Pick 7 girls in $\binom{14}{7}$ ways. The remaining 7 are now chosen automatically to play against them.
So if ABCDEFG are chosen as one team, then HIJKLMN are playing against them. However, this means that when HIJKLMN is chosen and they play against ABCDEFG, this situation will be double counted.

$\therefore \frac{1}{2}\times\binom{14}{7}=1716$

Comment

A straightforward textbook question, but many students' answers were $\binom{14}{7}$ as they didn't realise the notion of indistinguishable groups.

(ii) The selectors eventually decide to choose 7 players plus an umpire. In how many ways can this be done? **2**

Pick the umpire in $\binom{14}{1}=14$ ways. This leaves 7 players to be chosen from 13 i.e. $\binom{13}{7}=1716$. So there are $\binom{14}{1}\times\binom{13}{7}=24\,024$ ways.

Alternative 1

The 7 players could be picked first in $\binom{14}{7}$ ways. As they are not playing against the remaining 7, there is NO double counting. The umpire can be chosen from the remaining 7 players in $\binom{7}{1}$ ways i.e. $\binom{14}{7}\times\binom{7}{1}=24\,024$.

Alternative 2

Eight players can be selected in $\binom{14}{8}$ ways. Then the umpire can be chosen in $\binom{8}{1}$ ways from these players, giving a total of $\binom{14}{8}\times\binom{8}{1}=24\,024$ ways.

Comment

Generally done well, though many students did leave their answer as $\binom{14}{8}$.

(b) Three men decide to have dinner together at Chatswood. They have agreed to meet at the “All U Can Eat” restaurant, Chatswood. Unknown to these men, there are three restaurants with this name in Chatswood. Assuming that each man is equally likely to choose any one of the “All U Can Eat” restaurants, what is the probability that:

As each man has 3 choices as to where to dine, then there are $3^3 = 27$ choices with no restrictions.

(i) All three men go to different “All U Can Eat” restaurants. **2**

Method 1: Counting

The first man has 3 choices, the 2nd man now has only 2 choices and the last is stuck! This is 3!.

Probability = $\frac{3!}{27} = \frac{2}{9}$

Method 2: Probability

The first man goes to a restaurant. The second man has a $\frac{2}{3}$ chance of picking a different restaurant to the first man. This leaves the last man with a $\frac{1}{3}$ chance of picking a completely different restaurant.

Probability = $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

(ii) All three men go to the same “All U Can Eat” restaurant. **2**

Method 1: Counting

There are only three ways that they can go to the same restaurant.

Probability = $\frac{3}{27} = \frac{1}{9}$

Method 2: Probability

The first man goes to a restaurant. The second man has a $\frac{1}{3}$ chance of picking the same restaurant as the first man. This leaves the last man with a $\frac{1}{3}$ chance of also picking the same restaurant.

Probability = $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

Comment

Both parts were generally well done, using either method.

(c) It is given that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

(i) By making the substitution $x = 2\cos\theta$, deduce that $2\cos\dfrac{\pi}{18}$ is one of the roots of the equation $x^6 - 6x^4 + 9x^2 - 3 = 0$. Find the other five roots of the equation. **4**

Substitute $x = 2\cos\theta$ into $x^6 - 6x^4 + 9x^2 - 3 = 0$

$\therefore (2\cos\theta)^6 - 6(2\cos\theta)^4 + 9(2\cos\theta)^2 - 3 = 0$

$\therefore 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0$

$\therefore 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2 = 1$

$\therefore 2(32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1) = 1$

$\therefore 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 = \frac{1}{2}$

$\therefore \cos 6\theta = \frac{1}{2}$

Now substitute $\theta = \frac{\pi}{18}$ i.e. $\cos(6\times\frac{\pi}{18}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$

$\therefore$ $2\cos\frac{\pi}{18}$ is a root.

So what are the remaining 5 solutions?

$\cos 6\theta = \frac{1}{2} \Rightarrow 6\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \ldots$

$\therefore \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \ldots$

$\therefore x = 2\cos\theta = 2\cos\frac{\pi}{18}, 2\cos\frac{5\pi}{18}, 2\cos\frac{7\pi}{18}, 2\cos\frac{11\pi}{18}, 2\cos\frac{13\pi}{18}, 2\cos\frac{17\pi}{18}$

As these 6 angles are between 0 and π, then the roots are all distinct.

However, as the indices are all even then $x = \pm 2\cos\frac{\pi}{18}, \pm 2\cos\frac{5\pi}{18}, \pm 2\cos\frac{7\pi}{18}$.

Alternative

$\cos 6\theta = \frac{1}{2}(2\cos\theta)^6 - 3(2\cos\theta)^4 + \frac{9}{2}(2\cos\theta)^2 - 1$

$\cos 6\theta = \frac{1}{2}x^6 - 3x^4 + \frac{9}{2}x^2 - 1$

$\therefore 2\cos 6\theta = x^6 - 6x^4 + 9x^2 - 2$

$\therefore 2\cos 6\theta + 1 = x^6 - 6x^4 + 9x^2 - 1$

As $x^6 - 6x^4 + 9x^2 - 1 = 0$ then $2\cos 6\theta + 1 = 0$

The rest is the same above.

Comment

This was a relatively straightforward textbook question.

Students need to follow instructions: "By making the substitution …". Students who didn't do this found themselves unable to solve the equation.

Also the identity for $\cos 6\theta$ was given to them, yet too many students either tried to prove this or thought that by trying to prove it that this would help them.

Some students failed to recognise that $\cos\theta$ is an even function and so didn't realise that the roots $2\cos\frac{\pi}{18}$ and $2\cos(-\frac{\pi}{18})$ were the same root.

(c) (ii) Hence, show the equation $x^3-6x^2+9x-3=0$ has roots $2\left(1+\cos\frac{\pi}{9}\right)$, $2\left(1-\cos\frac{2\pi}{9}\right)$ and $2\left(1-\cos\frac{4\pi}{9}\right)$ **4**

Let $x = u^2$ then $x^3-6x^2+9x-3=0$ becomes $u^6-6u^4+9u^2-3=0$.
So the roots of $x^3-6x^2+9x-3=0$ are the squares of the answers in (i).
i.e. $x=4\cos^2\frac{\pi}{18}, 4\cos^2\frac{5\pi}{18}, 4\cos^2\frac{7\pi}{18}$ and so there are 3 distinct roots

NB only 3 distinct roots present, since it was shown above in (i) that the roots of the degree 6 equation are $x=\pm 2\cos\frac{\pi}{18}, \pm 2\cos\frac{5\pi}{18}, \pm 2\cos\frac{7\pi}{18}$.

Now using the fact that $2\cos^2\theta = 1+\cos 2\theta$:

$$\begin{aligned} 4\cos^2\tfrac{\pi}{18} &= 2(2\cos^2\tfrac{\pi}{18}) \\ &= 2[1+\cos(2\times\tfrac{\pi}{18})] \\ &= 2(1+\cos\tfrac{\pi}{9}) \end{aligned}$$

$$\begin{aligned} 4\cos^2\tfrac{5\pi}{18} &= 2(2\cos^2\tfrac{5\pi}{18}) \\ &= 2[1+\cos(2\times\tfrac{5\pi}{18})] \\ &= 2(1+\cos\tfrac{5\pi}{9}) \\ &= 2(1-\cos\tfrac{4\pi}{9}) \qquad \left[\cos\tfrac{5\pi}{9} = \cos(\pi-\tfrac{5\pi}{9}) = -\cos\tfrac{4\pi}{9}\right] \end{aligned}$$

$$\begin{aligned} 4\cos^2\tfrac{7\pi}{18} &= 2(2\cos^2\tfrac{7\pi}{18}) \\ &= 2[1+\cos(2\times\tfrac{7\pi}{18})] \\ &= 2(1+\cos\tfrac{7\pi}{9}) \\ &= 2(1-\cos\tfrac{2\pi}{9}) \qquad \left[\cos\tfrac{7\pi}{9} = \cos(\pi-\tfrac{2\pi}{9}) = -\cos\tfrac{2\pi}{9}\right] \end{aligned}$$

Comment

This was not generally well done.

Too many students did not show what they were asked to show.

There was a penalty if students just took the first three solutions from (i) and didn't justify why they were the three to take. There needed to be some evidence that the roots in (i) were of the form $x=\pm 2\cos\frac{\pi}{18}, \pm 2\cos\frac{5\pi}{18}, \pm 2\cos\frac{7\pi}{18}$ or equivalent.

(d) Two circles centre *A*, *B* touch externally at *P*, a third circle centre *C*, encloses both, touching the first circle at *Q* and the second circle at *R* as shown.

(i) Show that *APB*, *CAQ* and *CBR* are straight lines. **1**

When circles touch, the line of centres passes through the point of contact.

Comment

There was only 1 mark allotted for this problem, so quoting the relevant theorem would be a first start rather than trying to prove all three.

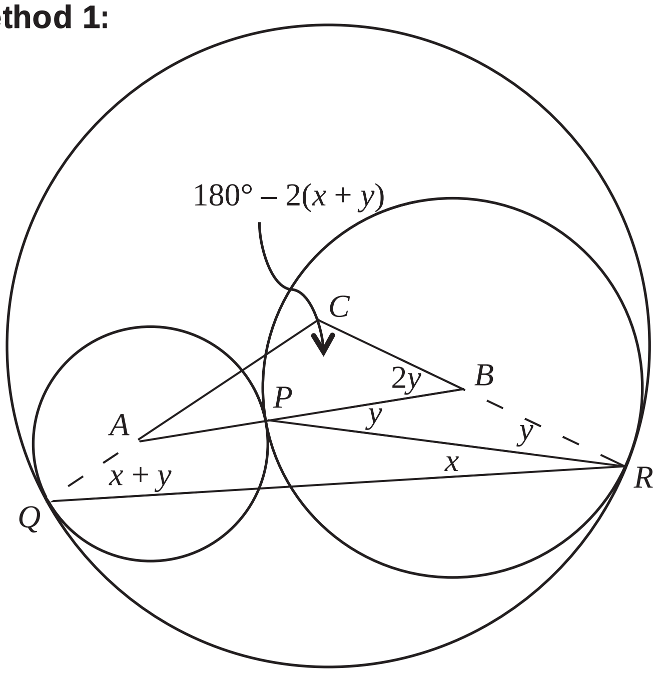
Misquoting the theorem was penalised, especially if it referred to a "point of intersection" rather than "point of contact".
An alternative approach that was successful was to prove one and then use "Similarly" for the other two. For this, the student had to at least draw or indicate that there were common tangents.

When referring to a circle it is customary to refer to it by three points on its circumference, or referring to it like "the circle with centre *A*".

Students who decided to only prove one of the lines could only score $\frac{1}{2}$ mark.

(ii) Prove that $\angle BAC = 2\angle PRQ$. **2**

Method 1:



The following is only possible since part (i) is true i.e. *APB*, *CAQ* and *CBR* are straight lines

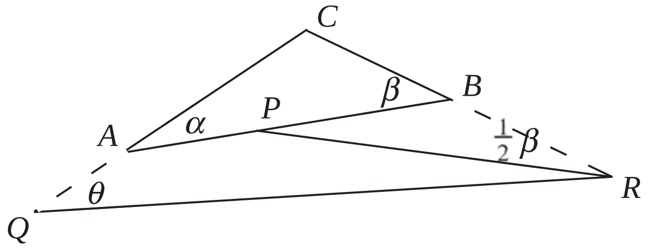
Let $\angle PRQ = x$ and $\angle CRP = y$.
As ΔBPR is isosceles then $\angle CBP = 2y$ (Exterior $\angle$ of ΔBPR)
$\therefore \angle CRQ = x + y$
$\therefore \angle CQR = x + y$ (ΔCQR isosceles)
$\therefore \angle QCR = 180° - 2(x + y)$ ($\angle$ sum ΔQCR)
$\therefore \angle BAC = 2x$ ($\angle$ sum ΔCAB)

$\therefore \angle BAC = 2 \times \angle PRQ$

(d) (ii) continued

Method 2: PB

Since C is the centre of the larger circle then $\Delta\, CQR$ is isosceles.
Let $\angle CAB = \alpha$, $\angle CBA = \beta$ and $\angle CQR = \theta$.
$\therefore \angle CRQ = \theta$ ($\Delta\, CQR$ is isosceles)



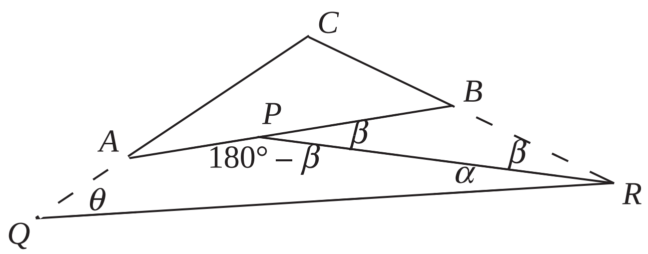
As B is the centre of a smaller circle then $\Delta\, PRB$ is isosceles and so
$\angle PRB = \frac{1}{2}\beta$ (Exterior $\angle$ of $\Delta\, PRB$)

By considering the angle sums of $\Delta\, CQR$ and $\Delta\, CAB$ then $2\theta = \alpha + \beta$.
$2\theta = \alpha + \beta \Rightarrow \theta = \frac{1}{2}(\alpha + \beta)$ i.e. $\angle PRQ = \theta - \frac{1}{2}\beta = \frac{1}{2}\alpha$.

Also $\alpha = 2\theta - \beta \Rightarrow \alpha = 2(\theta - \frac{1}{2}\beta)$ i.e. $\angle BAC = 2 \times \angle PRQ$

Method 3: EC

Let $\angle PRQ = \alpha$ and $\angle CRP = \beta$.
As $\Delta\, RBP$ is isosceles then $\angle BPR = \beta$.
By considering straight line APB then $\angle BPR = 180° - \beta$.



$\angle QAP = 180° - 2\alpha$ (angle sum of quad. $APRQ$)
$\therefore \angle BAC = 2\alpha$ (straight angle CAQ)

$\therefore\ \angle BAC = 2 \times \angle PRQ$

Comment

This question was either not attempted or poorly done.
The purpose of giving the insert sheet was that students would use it to try (i.e. write on it) and solve the problem without the need to try and draw it themselves.

End of solutions