SYDNEY GRAMMAR SCHOOL

2011 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 2

Monday 28th February 2011

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 78
- All six questions may be attempted.
- All six questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the six questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 6 per boy
- Candidature — 87 boys

Examiner

TCW

QUESTION ONE (13 marks) Use a separate writing booklet. **Marks**

(a) Solve $z^2 + 4z + 5 = 0$. **2**

(b) Given $z = 2 - i$ and $w = 3 - 4i$, express the following in the form $a + ib$, where a and b are real:

(i) iz **1**

(ii) $\overline{z + w}$ **1**

(iii) $\dfrac{1}{w}$ **1**

(iv) $(z - 1)^2$ **1**

(c) Given that $5 + 2i$ is a root of the equation $z^2 - 10z + n = 0$, find n. **2**

(d) (i) Find the complex square roots of $-24 + 10i$. **3**

(ii) Hence, or otherwise, solve: **2**

$$z^2 - (3 + i)z + 8 - i = 0$$

QUESTION TWO (13 marks) Use a separate writing booklet. **Marks**

(a) Find:

(i) $\displaystyle\int \frac{x^2}{(x^3 + 1)^2}\, dx$ **2**

(ii) $\displaystyle\int \frac{1}{\sqrt{16 - 9x^2}}\, dx$ **2**

(b) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x\, dx$ **2**

(ii) $\displaystyle\int_{\frac{\pi}{3}}^{\pi} \frac{\sin x}{1 - \cos x}\, dx$ **2**

(c) Use integration by parts to find $\displaystyle\int x \cos x\, dx$. **2**

(d) Use partial fractions to find $\displaystyle\int \frac{3x + 1}{x^2 - x - 6}\, dx$. **3**

Exam continues next page ...

QUESTION THREE (13 marks) Use a separate writing booklet. **Marks**

(a) Use implicit differentiation to find the gradient of the tangent to the curve $x^2 - y^2 = 1$ at the point $(2, \sqrt{3})$. **2**

(b) Let $z = p(\cos\alpha + i\sin\alpha)$ and $w = q(\cos\beta + i\sin\beta)$, where $p > 0$ and $q > 0$.

(i) Find zw in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Hence prove that $|zw| = |z||w|$ and $\arg(zw) = \arg z + \arg w$. **1**

(c) Let $z_1 = 1 - i$ and $z_2 = -\sqrt{6} + \sqrt{2}\,i$.

(i) Express z_1 and z_2 in modulus–argument form. **4**

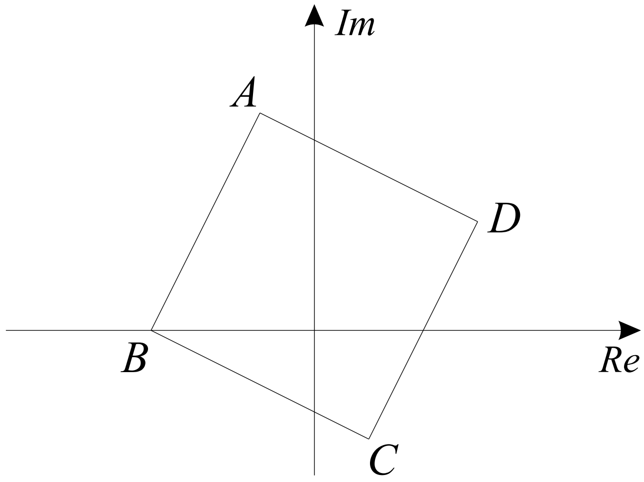
(ii) Show that $z_1 z_2 = 4\,\mathrm{cis}\,\frac{7\pi}{12}$. **1**

(iii) Plot the point representing $z_1 z_2$ on the Argand diagram and hence find the exact value of $\tan\frac{7\pi}{12}$. **3**

QUESTION FOUR (13 marks) Use a separate writing booklet. **Marks**

(a) Sketch the locus of z in the complex plane if $\arg(z-1) = \frac{\pi}{4}$. **2**

(b)



In the Argand diagram above, $ABCD$ is a square. A represents the number $z_1 = -1 + 4i$ and B represents the number $z_2 = -3 + 0i$.

(i) Find the complex number represented by the vector $\overrightarrow{BA}$. **1**

(ii) Hence find the complex number represented by the vector $\overrightarrow{BC}$. **1**

(iii) Find the complex number represented by the point C. **1**

(iv) Find the complex number represented by the point M, the intersection of the diagonals of $ABCD$. **1**

(v) Let $w = \frac{3}{2} i z_1$ and let P represent the complex number w. Copy the diagram above and clearly show the point P. **1**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 2x \, dx$. **3**

(d) Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{1}{(9+x)\sqrt{x}} \, dx$. **3**

Exam continues next page ...

QUESTION FIVE (13 marks) Use a separate writing booklet. **Marks**

(a) Show that $\cos^{-1}\frac{3}{4} - \sin^{-1}\frac{9}{16} = \sin^{-1}\frac{1}{8}$. **3**

(b) A complex number z satisfies $|z - 6i| = 3$. Use a diagram to find the maximum possible values of

(i) $|z|$, **1**

(ii) $\arg z$, where $-\pi < \arg z \leq \pi$. **2**

(c) The complex numbers z_1, z_2, z_3 and z_4 correspond to the distinct points A, B, C and D respectively in the complex plane. If $z_1 - z_2 + z_3 - z_4 = 0$, what type of quadrilateral is $ABCD$? Justify your answer. **2**

(d) Prove by mathematical induction that for all positive integers n, **5**

$$\tan x + 2\tan 2x + 4\tan 4x + \cdots + 2^{n-1}\tan(2^{n-1}x) = \cot x - 2^n\cot(2^n x).$$

The Examination Continues over the Page

Exam continues overleaf ...

QUESTION SIX (13 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = \tan^{-1}(\frac{1}{x})$.

(i) Describe the symmetry of the graph of $y = f(x)$. $\boxed{1}$

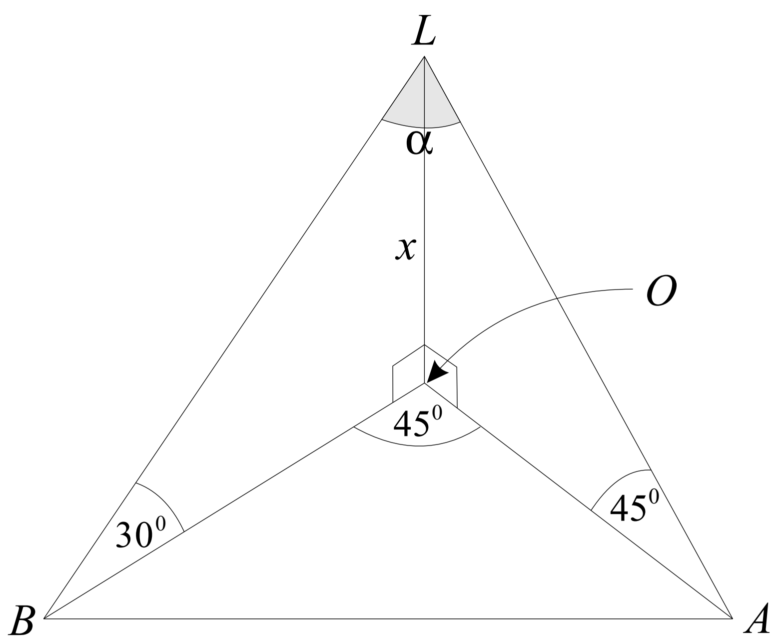
(ii) Find $f'(x)$. $\boxed{1}$

(iii) Describe the behaviour of $f(x)$ as $x \to 0$ from above and from below. $\boxed{1}$

(iv) Find the equation of the horizontal asymptote. $\boxed{1}$

(v) Sketch the graph of $y = f(x)$. $\boxed{1}$

(b)



The diagram above shows a vertical light tower OL of height x metres. The angle of elevation of L from A is $45°$, the angle of elevation of L from B is $30°$ and the angle AOB is $45°$. Let $\angle ALB = \alpha$.

(i) Show that $AB^2 = (4 - \sqrt{6})x^2$. $\boxed{2}$

(ii) Show that $\cos\alpha = \dfrac{2 + \sqrt{6}}{4\sqrt{2}}$. $\boxed{1}$

(iii) Suppose that Pablo walks in a straight line from B to O. Let the variable point P represent his position and let the angle of elevation of L from P be θ. $\boxed{2}$

Let $\angle ALP = \beta$ and show that $\cos\beta = \dfrac{1}{\sqrt{2}}\sin\theta + \dfrac{1}{2}\cos\theta$.

(iv) If the light tower is 10 metres high, find the exact distance Pablo has to walk along BO so that his position P will minimise β. Justify your answer. $\boxed{3}$

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\;\; n \neq -1;\;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\;\; a > 0,\;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\ln x = \log_e x,\; x > 0$$

Question 1

(a) $z^2 + 4z + 5 = 0$

$(z+2)^2 = -1$

$z = -2+i$ OR $-2-i$

(b) $z = 2-i$, $w = 3-4i$

(i) $iz = i(2-i)$
$= 1+2i$

(ii) $\overline{z+w} = \overline{5-5i}$
$= 5+5i$

(iii) $\frac{1}{w} = \frac{1}{3-4i} \times \frac{3+4i}{3+4i}$
$= \frac{3+4i}{9+16}$
$= \frac{3}{25} + \frac{4}{25}i$

(iv) $(z-1)^2 = (1-i)^2$
$= 1-2i-1$
$= -2i$

(c) substituting $z = 5+2i$, $25+20i-4-50-20i+n = 0$
$n = 29$

(d) (i) Let $(x+iy)^2 = -24+10i$
$x^2-y^2+2ixy = -24+10i$
Equating real and imaginary parts:

$$\left.\begin{array}{l} x^2-y^2 = -24 \\ xy = 5 \end{array}\right\} \begin{array}{l} x = 1,\ y = 5 \\ \text{OR } x = -1,\ y = -5. \end{array}$$

The complex square roots are $1+5i$ and $-1-5i$.

(ii) $z^2-(3+i)z+8-i = 0$

$\Delta = 9+6i-1-32+4i$
$= -24+10i$

$z = \frac{3+i \pm (1+5i)}{2}$ from (i)

$z = 2+3i$ OR $1-2i$

13

Question 2

(a) (i) $\int \frac{x^2}{(x^3+1)^2}\,dx = \frac{1}{3}\int 3x^2(x^3+1)^{-2}\,dx$

$= -\frac{1}{3(x^3+1)} + c$ ✓✓

(ii) $\int \frac{1}{\sqrt{16-9x^2}}\,dx = \int \frac{1}{3\sqrt{(\frac{4}{3})^2-x^2}}\,dx$

$= \frac{1}{3}\sin^{-1}\frac{3x}{4} + c$ ✓✓

(b) (i) $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x\,dx = \frac{1}{2}\left[\sec 2x\right]_0^{\frac{\pi}{6}}$ ✓

$= \frac{1}{2}\left(\sec\frac{\pi}{3} - \sec 0\right)$

$= \frac{1}{2}(2-1)$

$= \frac{1}{2}$ ✓

(ii) $\int_{\frac{\pi}{3}}^{\pi} \frac{\sin x}{1-\cos x}\,dx = \left[\ln|1-\cos x|\right]_{\frac{\pi}{3}}^{\pi}$ ✓

$= \ln(1-\cos\pi) - \ln\left(1-\cos\frac{\pi}{3}\right)$

$= \ln 2 - \ln\frac{1}{2}$

$= 2\ln 2$ ✓

(c) $\int x\cos x\,dx = x\sin x - \int \sin x \times 1\,dx$ ✓

$= x\sin x + \cos x + c$ ✓

(d) Let $\frac{3x+1}{(x-3)(x+2)} \equiv \frac{A}{x+2} + \frac{B}{x-3}$

$3x+1 \equiv A(x-3) + B(x+2)$ ✓

when $x=-2$, $-5 = -5A$

$A = 1$

when $x = 3$, $10 = 5B$

$B = 2$ ✓

$\int \frac{3x+1}{(x-3)(x+2)}\,dx = \int \frac{1}{x+2} + \frac{2}{x-3}\,dx$

$= \ln|x+2| + 2\ln|x-3| + c$ ✓

$= \ln|x+2|(x-3)^2 + c$

$\boxed{13}$

Question 5

(a) $x^2 - y^2 = 1$

$2x - 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{x}{y}$

At $(2, \sqrt{3})$

gradient of tangent $= \dfrac{2}{\sqrt{3}}$

(b) $z = p(\cos\alpha + i\sin\alpha)$, $w = q(\cos\beta + i\sin\beta)$

(i) $zw = pq(\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta)$

$= pq\big((\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)\big)$

$= pq\,(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$

(ii) From (i), $|zw| = pq$

$= |z||w|$

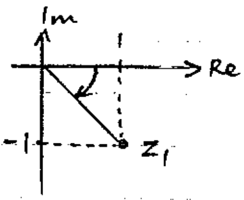
and $\arg(zw) = \alpha + \beta$

$= \arg z + \arg w$

(c) (i) $z_1 = 1 - i$

$|z_1| = \sqrt{1+1}$

$= \sqrt{2}$



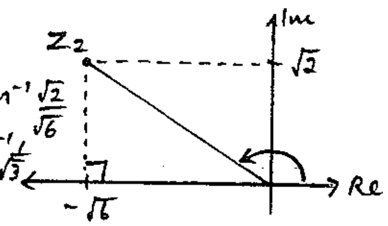
$\arg z_1 = -\dfrac{\pi}{4}$

$z_1 = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right)$ (accept $\frac{7\pi}{4}$)

$z_2 = -\sqrt{6} + \sqrt{2}\,i$

$|z_2| = \sqrt{6+2}$

$= 2\sqrt{2}$



$\arg z_2 = \pi - \tan^{-1}\dfrac{\sqrt{2}}{\sqrt{6}}$

$= \pi - \tan^{-1}\dfrac{1}{\sqrt{3}}$

$= \dfrac{5\pi}{6}$

$z_2 = 2\sqrt{2}\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

(ii) $z_1 z_2 = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right) \times 2\sqrt{2}\,\text{cis}\left(\dfrac{5\pi}{6}\right)$

$= (\sqrt{2} \times 2\sqrt{2})\,\text{cis}\left(-\dfrac{\pi}{4} + \dfrac{5\pi}{6}\right)$

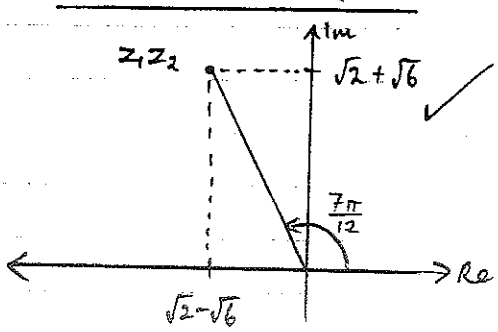
$= 4\,\text{cis}\,\dfrac{7\pi}{12}$

(iii) $z_1 z_2 = (1-i)(-\sqrt{6} + \sqrt{2}\,i)$

$= -\sqrt{6} + \sqrt{2}\,i + \sqrt{6}\,i - \sqrt{2}\,i^2$

$= (\sqrt{2} - \sqrt{6}) + (\sqrt{2} + \sqrt{6})\,i$

NOTE: $\sqrt{2} - \sqrt{6} < 0$

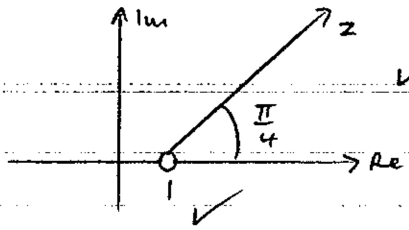


$\tan\dfrac{7\pi}{12} = \dfrac{\sqrt{2}+\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$= -2 - \sqrt{3}$

Question 4

(a)



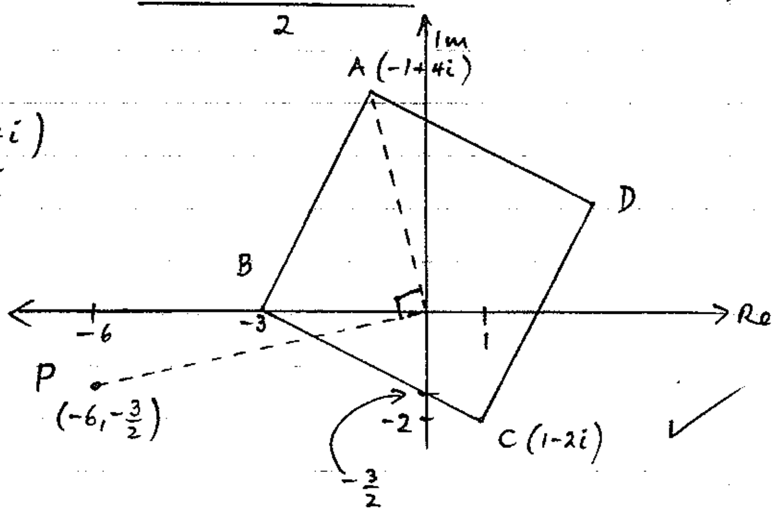
(b) (i) $\overrightarrow{BA}$ represents $z_1 - z_2 = -1+4i+3$
$= 2+4i$

(ii) $\overrightarrow{BC}$ represents $(2+4i) \times -i = 4-2i$

(iii) $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$
$= -3 + 4 - 2i$ So C represents $1-2i$.
$= 1-2i$

(iv) M is the midpoint of AC :
M represents $\dfrac{-1+4i+1-2i}{2} = i$

(v) $w = \frac{3}{2} i z_1$
$= \frac{3}{2} i(-1+4i)$
$= -6 - \frac{3}{2} i$



(c)

$$\int_0^{\frac{\pi}{2}} \sin^3 2x \, dx = \int_0^{\frac{\pi}{2}} \sin 2x (1 - \cos^2 2x) \, dx$$
$$= \int_0^{\frac{\pi}{2}} \sin 2x - \sin 2x \cos^2 2x \, dx$$
$$= \left[-\frac{1}{2} \cos 2x + \frac{1}{6} \cos^3 2x \right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2} - \frac{1}{6} - \left(-\frac{1}{2} + \frac{1}{6} \right)$$
$$= \frac{2}{3}$$

(d) Let $u = \sqrt{x}$
$du = \dfrac{1}{2\sqrt{x}} dx$
$2\,du = \dfrac{1}{\sqrt{x}} dx$

$$\int \frac{1}{(9+x)\sqrt{x}} dx = 2 \int \frac{1}{3^2+u^2} du$$
$$= \frac{2}{3} \tan^{-1} \frac{u}{3} + c$$
$$= \frac{2}{3} \tan^{-1} \frac{\sqrt{x}}{3} + c$$

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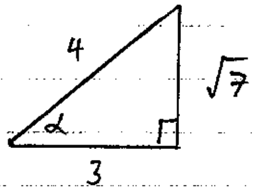
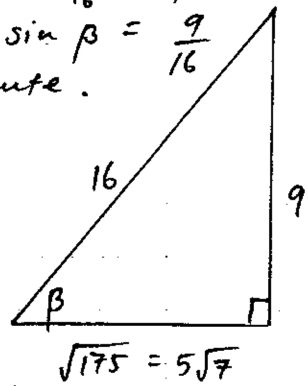
Question 5

(a) Let $\cos^{-1}\frac{3}{4} = \alpha$ and $\sin^{-1}\frac{9}{16} = \beta$

$\cos\alpha = \frac{3}{4}$ $\qquad$ $\sin\beta = \frac{9}{16}$

where α, β are acute.

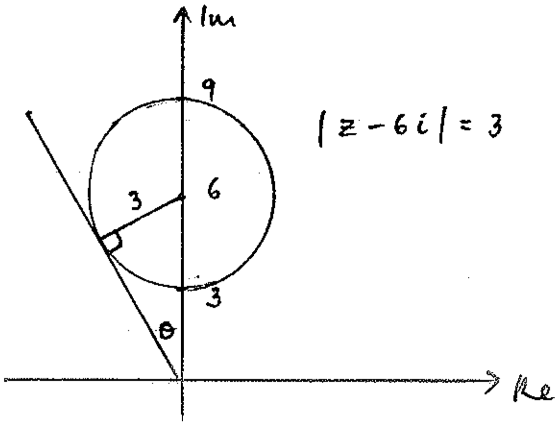




$$\sin\left(\cos^{-1}\frac{3}{4} - \sin^{-1}\frac{9}{16}\right) = \sin(\alpha - \beta)$$
$$= \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$= \frac{\sqrt{7}}{4} \times \frac{5\sqrt{7}}{16} - \frac{9}{16} \times \frac{3}{4}$$
$$= \frac{35 - 27}{64}$$
$$= \frac{1}{8}$$

$\therefore \cos^{-1}\frac{3}{4} - \sin^{-1}\frac{9}{16} = \sin^{-1}\frac{1}{8}$, since $(\alpha - \beta)$ is acute.

(b)



$|z - 6i| = 3$

(i) $\max|z| = 9$

(ii) $\sin\theta = \frac{3}{6}$

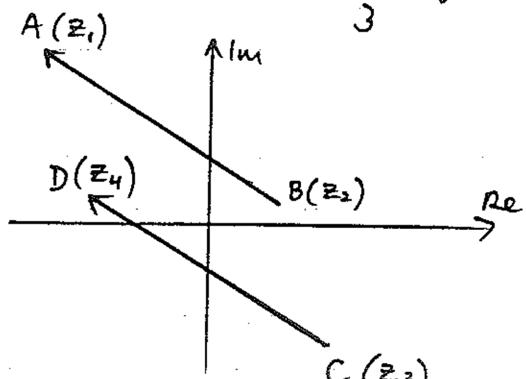
$\theta = \frac{\pi}{6}$

$\max \arg z = \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3}$

(c) $z_1 - z_2 + z_3 - z_4 = 0$

$z_1 - z_2 = z_4 - z_3$



So $z_1 - z_2$ represents $\overrightarrow{BA}$
and $z_4 - z_3$ represents $\overrightarrow{CD}$.

Equal vectors ($\overrightarrow{BA} = \overrightarrow{CD}$) have the same magnitude and direction,
So ABCD is a <u>parallelogram</u>
(2 opposite sides equal ad parallel).

(d) $\tan x + 2\tan 2x + 4\tan 4x + \ldots + 2^{n-1}\tan(2^{n-1}x) = \cot x - 2^n \cot(2^n x)$

A. when $n = 1$,

$$LHS = \tan x$$

$$RHS = \cot x - 2\cot 2x$$

$$= \frac{1}{\tan x} - \cancel{2} \times \frac{1 - \tan^2 x}{\cancel{2}\tan x}$$

$$= \frac{1 - 1 + \tan^2 x}{\tan x}$$

$$= \tan x$$

$$= LHS$$

so the statement is true for $n = 1$.

B. Assume the statement is true for some integer $n = k$.

ie Assume that

$$\tan x + 2\tan 2x + 4\tan 4x + \ldots + 2^{k-1}\tan(2^{k-1}x) = \cot x - 2^k \cot(2^k x) \quad \text{— (1)}$$

Let us prove the statement is true for $n = k+1$.

ie Prove

$$\tan x + 2\tan 2x + 4\tan 4x + \ldots + 2^{k-1}\tan(2^{k-1}x) + 2^k \tan(2^k x) = \cot x - 2^{k+1}\cot(2^{k+1}x)$$

$$LHS = \cot x - 2^k \cot(2^k x) + 2^k \tan(2^k x) \quad \text{— from (1)}$$

$$= \cot x - 2^k\left(\cot 2^k x - \tan(2^k x)\right)$$

$$= \cot x - 2^k\left(\frac{1}{\tan(2^k x)} - \tan(2^k x)\right)$$

$$= \cot x - 2^k\left(\frac{1 - \tan^2(2^k x)}{\tan(2^k x)}\right)$$

$$= \cot x - 2^k \times 2\left(\frac{1 - \tan^2(2^k x)}{2\tan(2^k x)}\right)$$

$$= \cot x - 2^{k+1}\left(\frac{1}{\tan(2 \times 2^k x)}\right)$$

$$= \cot x - 2^{k+1}\cot(2^{k+1}x)$$

$$= RHS \quad \text{as required}$$

C. It follows, from parts A and B, by mathematical induction that the statement is true for all positive integers n.

Question 6

(a) $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$

(i) $f(-x) = \tan^{-1}\left(-\frac{1}{x}\right)$

$= -\tan^{-1}\left(\frac{1}{x}\right)$

$= -f(x)$

ie $f(x)$ is an odd function.

So the graph of $y = f(x)$ has odd symmetry. ✓

ie Rotational symmetry of $180°$ about the origin.

(ii) $f'(x) = \dfrac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}$

$= -\dfrac{1}{1+x^2}$ ✓

(iii) Discontinuity at $x = 0$.

as $x \to 0^+$, $\frac{1}{x} \to \infty$, so $f(x) \to \frac{\pi}{2}$.

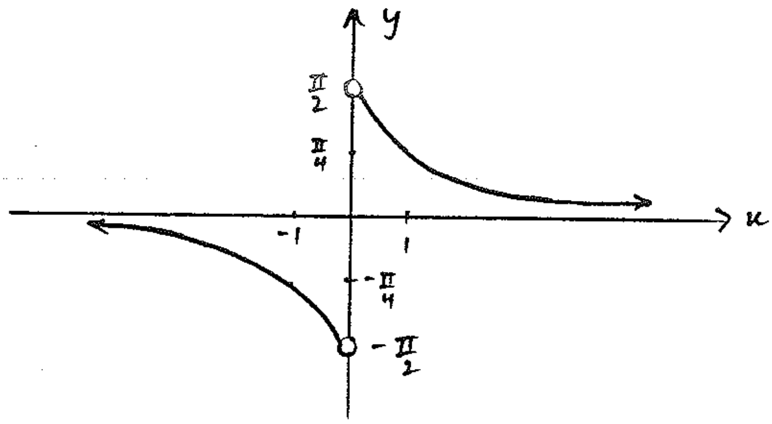
as $x \to 0^-$, $\frac{1}{x} \to -\infty$, so $f(x) \to -\frac{\pi}{2}$. ✓

(iv) as $x \to \infty$, $\frac{1}{x} \to 0$, so $f(x) \to 0^+$

as $x \to -\infty$, $\frac{1}{x} \to 0$, so $f(x) \to 0^-$

The horizontal asymptote has equation $y = 0$. ✓

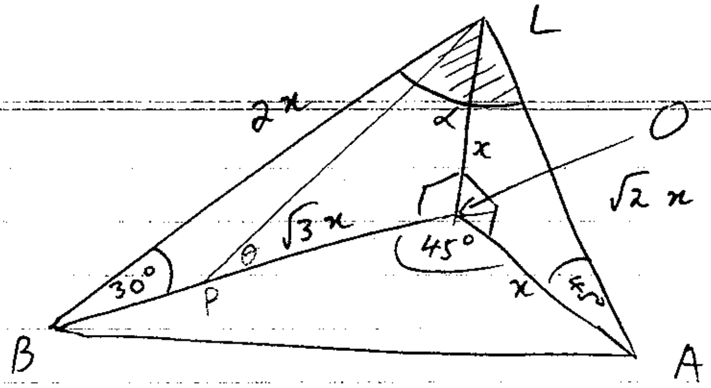
(v)



✓

NOTE: $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{d}{dx}\left(-\tan^{-1}x\right)$

6 (b)



Right Triangle Trig to solve for OA, OB, AL and BL.

(i) In ΔAOB, $AB^2 = 3x^2 + x^2 - 2.\sqrt{3}x^2 \times \frac{1}{\sqrt{2}}$

$$= 4x^2 - \sqrt{6}\,x^2$$

$$= (4-\sqrt{6})\,x^2$$

(ii) From ΔALB,

AB^2: $\quad 4x^2 + 2x^2 - 2.2\sqrt{2}x^2 \cos\alpha = (4-\sqrt{6})\,x^2$

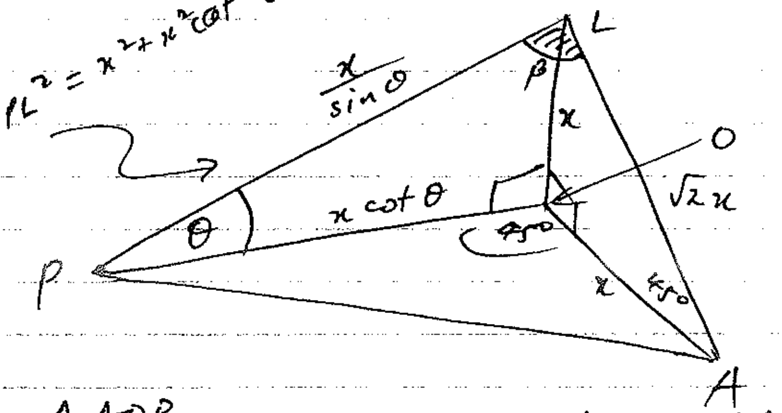
$$6 - 4\sqrt{2}\cos\alpha = 4 - \sqrt{6}$$

$$\cos\alpha = \frac{2+\sqrt{6}}{4\sqrt{2}}$$

$$(\alpha \doteqdot 38^\circ)$$

(iii) $PL^2 = x^2 + x^2\cot^2\theta$

Pablo



as per (ii)

cos rule in ΔAOP cos rule in ΔALP

AP^2: $\quad x^2\cancel{\cot^2\theta} + x^2 - 2x^2\cot\theta\left(\frac{1}{\sqrt{2}}\right) = 2x^2 + x^2 + \cancel{x^2\cot^2\theta} - \frac{2\sqrt{2}\,x^2}{\sin\theta}\cos\beta$

$\boxed{\times\sin\theta}$

$$1 - \sqrt{2}\cot\theta = 3 - \frac{2\sqrt{2}\cos\beta}{\sin\theta}$$

$$-\sqrt{2}\cos\theta = 2\sin\theta - 2\sqrt{2}\cos\beta$$

$$\cos\beta = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$$

as required

(iv)

$$\cos\beta = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$$

Let $a\cos(\theta - u) = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$

$a\cos\theta\cos u + a\sin\theta\sin u = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$

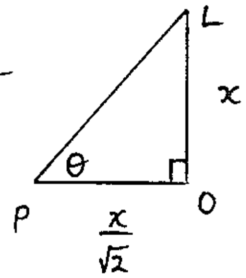
$\therefore$ $\left.\begin{aligned} a\cos u &= \frac{1}{2} \\ a\sin u &= \frac{1}{\sqrt{2}} \end{aligned}\right\}$ $\quad a^2 = \frac{1}{4} + \frac{1}{2}$, $\quad a = \frac{\sqrt{3}}{2}$ $\qquad \tan u = \frac{1}{\sqrt{2}} \times \frac{2}{1}$, $\quad u = \tan^{-1}\sqrt{2}$

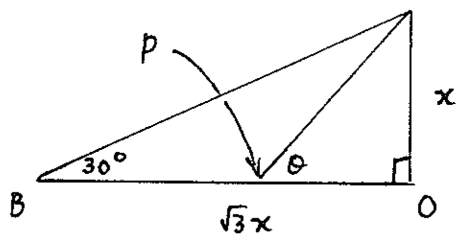
$$\cos\beta = \frac{\sqrt{3}}{2}\cos\left(\theta - \tan^{-1}\sqrt{2}\right)$$

$$\beta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\cos\left(\theta - \tan^{-1}\sqrt{2}\right)\right)$$

β is minimised when $\cos\left(\theta - \tan^{-1}\sqrt{2}\right)$ is maximised.
ie when $\theta = \tan^{-1}\sqrt{2}$

$\tan\theta = \sqrt{2} = \dfrac{x}{\frac{x}{\sqrt{2}}}$





When $x = 10$, $OP = \frac{10}{\sqrt{2}} = 5\sqrt{2}$

So when Pablo has walked $(10\sqrt{3} - 5\sqrt{2})$ m, β is at a minimum.

NOTE: Minimum β is $30°$ when $\theta = \tan^{-1}\sqrt{2} \doteqdot 55°$.
When $\theta = 30°$, $\beta \doteqdot 38°$
When $\theta \to 90°$, $\beta \to 45°$.

OR Using Calculus :

(iv) $\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta\right)$

$$\frac{d\beta}{d\theta} = -\frac{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{2}\sin\theta}{\sqrt{1-\left(\frac{1}{2}\sin\theta+\frac{1}{2}\cos\theta\right)^2}}$$

when $\frac{d\beta}{d\theta} = 0$, $\frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}\cos\theta$

$\tan\theta = \sqrt{2}$

$\theta = \tan^{-1}\sqrt{2} \doteq 55°$

Implicit Differentiation

$\cos\beta = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$

$-\sin\beta \frac{d\beta}{d\theta} = \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{2}\sin\theta$

So $\frac{d\beta}{d\theta} = 0$ when

$\frac{1}{2}\sin\theta = \frac{1}{\sqrt{2}}\cos\theta$

since $\sin\beta > 0$

ie $\tan\theta = \sqrt{2}$

θ	$50°$	$\tan^{-1}\sqrt{2}$	$60°$
$\frac{d\beta}{d\theta}$	-0.101 (3dp)	0	0.109 (3dp)

So we have minimum β when $\theta = \tan^{-1}\sqrt{2}$.

The result follows...

OR To minimise β we want to maximise $\cos\beta$.

Let $\psi = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{2}\cos\theta$

$\frac{d\psi}{d\theta} = \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{2}\sin\theta$

when $\frac{d\psi}{d\theta} = 0$, $\frac{1}{\sqrt{2}}\cos\theta = \frac{1}{2}\sin\theta$

$\tan\theta = \sqrt{2}$

$\frac{d^2\psi}{d\theta^2} = -\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{2}\cos\theta$

$\frac{d^2\psi}{d\theta^2} < 0$ for $\theta = \tan^{-1}\sqrt{2}$

So $\cos\beta$ is maximised for $\theta = \tan^{-1}\sqrt{2}$.

The result follows...