SYDNEY GRAMMAR SCHOOL

2012 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 2

Monday 20th February 2012

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection

Section I Questions 1–10

- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.

Section II Questions 11–14

- Start each of these questions in a new booklet.
- Write your candidate number clearly on each booklet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Candidature — 88 boys

Examiner
KWM

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

Question One

Which of the following represents the complex number $1 - i$ expressed in modulus–argument form? **1**

(A) $\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$

(B) $\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

(C) $\sqrt{2}\,\text{cis}(-\frac{\pi}{4})$

(D) $2\,\text{cis}\,\frac{\pi}{4}$

Question Two

Which of the following is true for all complex numbers z? **1**

(A) $\text{Re}(z) = z + \overline{z}$

(B) $\text{Re}(z) = \dfrac{z - \overline{z}}{2}$

(C) $\text{Re}(z) = \dfrac{z + \overline{z}}{2}$

(D) $\text{Re}(z) = \dfrac{z\overline{z}}{2}$

Question Three

The primitive of $\dfrac{1}{\sqrt{4 - 9x^2}}$ is: **1**

(A) $\frac{1}{6}\sin^{-1}\frac{3x}{2} + C$

(B) $\frac{1}{3}\sin^{-1}\frac{3x}{2} + C$

(C) $\frac{2}{3}\sin^{-1}\frac{3x}{2} + C$

(D) $\frac{1}{3}\sin^{-1}\frac{2x}{3} + C$

Exam continues next page ...

Question Four

The solutions to the quadratic equation $z^2 - 2z + 2 = 0$ are: **1**

(A) $z = 1 + i$ or $1 - i$

(B) $z = -1 + i$ or $-1 - i$

(C) $z = \pm(1 + i)$

(D) $z = 1 \pm 2i$

Question Five

The expression $i^{19} + i^{20} + i^{21} + i^{22}$ is equal to: **1**

(A) i

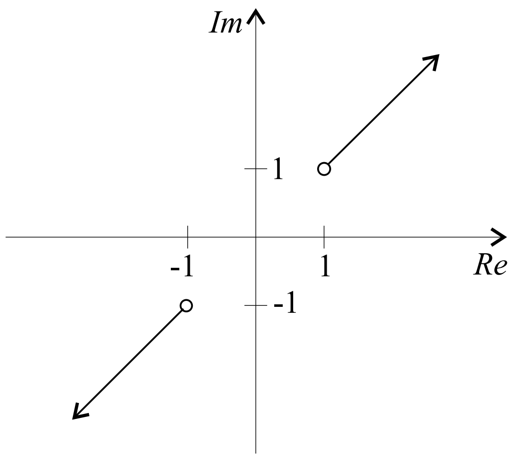
(B) 0

(C) $-i$

(D) -1

Question Six

1



Which of the following defines the locus of the complex number z sketched in the diagram above?

(A) $\arg\left(\dfrac{z - 1 + i}{z - 1 - i}\right) = \pi$

(B) $\arg(z - 1 + i) = \arg(z - 1 - i)$

(C) $\arg\left(\dfrac{z + 1 + i}{z - 1 - i}\right) = \dfrac{\pi}{2}$

(D) $\arg(z + 1 + i) = \arg(z - 1 - i)$

Exam continues overleaf ...

Question Seven

What is the minimum value of $2\sin x - 3\cos x$? **1**

(A) -1

(B) $-\sqrt{13}$

(C) -5

(D) $-\sqrt{2} - \sqrt{3}$

Question Eight

What is the primitive of $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$? **1**

(A) $\sin\sqrt{x} + C$

(B) $-\sin\sqrt{x} + C$

(C) $\frac{1}{2}\cos^2\sqrt{x} + C$

(D) $2\sin\sqrt{x} + C$

Question Nine

What is the range of $g(x) = \tan(\frac{1}{2}\cos^{-1} x)$? **1**

(A) $y \le 0$

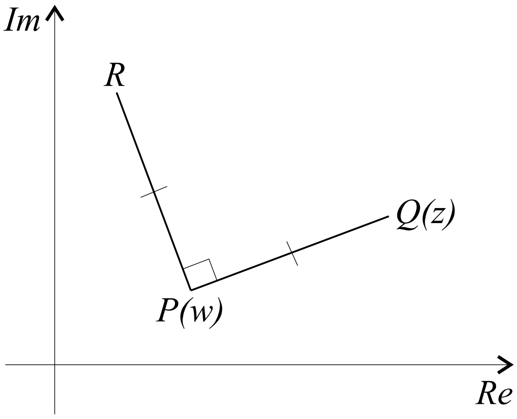
(B) $y \in \mathbf{R}$

(C) $0 \le y < 1$

(D) $y \ge 0$

Exam continues next page ...

Question Ten **1**



In the diagram, $\angle QPR = 90^\circ$ and $PQ = PR$. The points P and Q represent the complex numbers w and z respectively. Which complex number represents the point R in the complex plane?

(A) $i(w - z)$

(B) $w + i(z - w)$

(C) $w + i(w - z)$

(D) $i(z - w)$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

Question Eleven (15 marks) Use a separate writing booklet. **Marks**

(a) Let $u = 2 + i$ and $v = -1 - i$.

(i) Find $\text{Im}(uv)$. **1**

(ii) Find $\dfrac{u}{i}$. **1**

(iii) Evaluate $|u - v|^2$. **1**

(iv) Express $\dfrac{u}{v}$ in the form $a + ib$, where a and b are real numbers. **2**

(b) (i) Express the complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ in modulus-argument form. **1**

(ii) Hence, or otherwise, find $\arg\left(\dfrac{{z_1}^2}{z_2}\right)$. **1**

(c) (i) Find the two square roots of $16 + 30i$. **2**

(ii) Hence solve $z^2 + (1 + i)z - (4 + 7i) = 0$. **2**

(d) (i) Sketch the region in the complex plane which simultaneously satisfies $-\frac{\pi}{4} \le \arg(z + i) \le \frac{\pi}{4}$ and $|z - 2| \le 1$. **2**

(ii) Find the maximum value of $|z|$. **1**

(iii) Find the minimum value of $\arg(z)$, given $-\pi < \arg(z) \le \pi$. **1**

Exam continues next page ...

Question Twelve (15 marks) Use a separate writing booklet. **Marks**

(a) Find the following indefinite integrals.

(i) $\displaystyle\int \frac{1}{\sqrt{9x^2+2}}\,dx$ (Use the substitution $u = 3x$.) **2**

(ii) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$ (Use the substitution $u = \sqrt{x}$.) **2**

(b) Evaluate:

(i) $\displaystyle\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$ **2**

(ii) $\displaystyle\int_0^1 \frac{x^2-1}{x^2+1}\,dx$ **2**

(c) Use integration by parts to find $\displaystyle\int x\ln x\,dx$. **2**

(d) (i) Express $\dfrac{4x+2}{(x+1)(x^2+1)}$ in the form $\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$ for real constants a, b and c. **2**

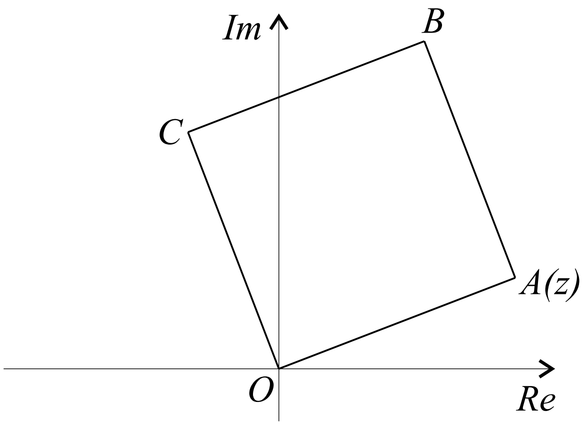
(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{4x+2}{(x+1)(x^2+1)}\,dx$. **3**

Exam continues overleaf ...

Question Thirteen (15 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $x^3+y^3-5y-3=0$ at the point $(1,-2)$. **3**

(b)



In the diagram $OABC$ is a square where O is the origin. The point A represents the complex number z.

(i) Find the complex number represented by B in terms of z. **1**

(ii) The square is rotated about O anti-clockwise through 45° to $OA'B'C'$. **2**
Show that the point C' represents the complex number $\frac{\sqrt{2}}{2}(-1+i)z$.

(c) The origin O and the points P, Q and R, representing the complex numbers z, $z+\dfrac{1}{z}$ and $\dfrac{1}{z}$ respectively, are joined to form a quadrilateral. Suppose that $0<\arg(z)<\frac{\pi}{2}$ and $|z|\geq 1$.

(i) Sketch the quadrilateral $OPQR$ on the complex plane. **2**

(ii) Find the complex number z for $OPQR$ to be a square. **1**

(d) (i) Show that $(k+1)^2(k+4)=k^3+6k^2+9k+4$. **1**

(ii) Use mathematical induction to prove that for all integers $n\geq 1$, **4**

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}.$$

(iii) Hence find **1**

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}.$$

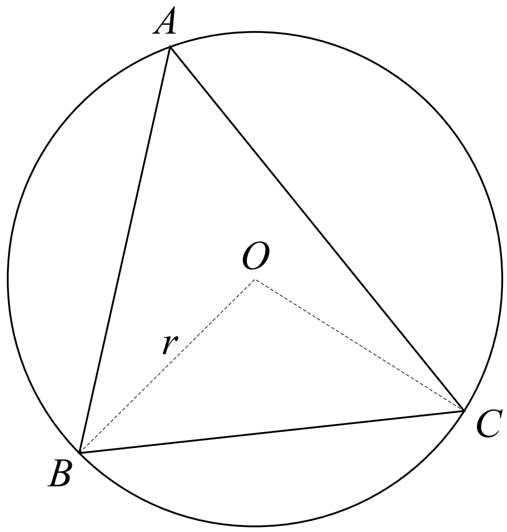
Exam continues next page ...

Question Fourteen (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Find a general solution to the equation $\cos 3x = \sin 2x$. **3**

(ii) Hence, or otherwise, find the smallest positive solution of the equation $\cos 3x - \sin 2x = 0$. **1**

(b)



The circle drawn through the vertices of triangle ABC has centre O and radius r.

(i) Show that $BC = 2r \sin A$. **2**

(ii) Use the fact that the sum of the areas of triangles OBC, OCA and OAB is equal to the area of triangle ABC, to show that **2**

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

(c) (i) If $z = \cos\theta + i\sin\theta$, show that $1 + z = 2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})$. **1**

(ii) Suppose that z_1 and z_2 are complex numbers such that $|z_1| = |z_2| = 1$. If z_1 and z_2 have arguments α and β respectively, where $-\pi < \alpha < \pi$ and $-\pi < \beta < \pi$, show that $\dfrac{z_1 + z_1 z_2}{z_1 + 1}$ has modulus $\dfrac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}}$ and argument $\dfrac{\alpha + \beta}{2}$. **2**

(iii) Given that $|z_1| = |z_2| = 1$ and $\dfrac{z_1 + z_1 z_2}{z_1 + 1} = 2i$, find z_1 and z_2 in the form $x + iy$, where x and y are real. **4**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE : } \ln x = \log_e x, \; x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

QUESTION 1

$r\cos\theta = 1$, $r\sin\theta = -1$ } $r = \sqrt{2}$, $\theta = -\frac{\pi}{4}$

$\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

C

QUESTION 2

let $z = a + ib$

$\bar{z} = a - ib$

$\frac{z+\bar{z}}{2} = \frac{a+ib+a-ib}{2}$

$= a$

$= Re(z)$ C

QUESTION 3

$\int \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{1}{3}\sin^{-1}\frac{3x}{2} + c$

B

QUESTION 4

$z^2 - 2z + 2 = 0$

$(z-1)^2 + 1 = 0$

$(z-1)^2 = -1$

$z - 1 = \pm i$

$z = 1 + i$ or $z = 1 - i$

A

QUESTION 5

$i^{19} + i^{20} + i^{21} + i^{22}$

$= i^{19}(1 + i + i^2 + i^3)$

$= i^{19}(1 + i - 1 - i)$

$= 0$

B

QUESTION 6

$\arg(z - (1+i)) = \arg(z - (-1-i))$

$\arg(z - 1 - i) = \arg(z + 1 + i)$

$\arg(z + 1 + i) = \arg(z - 1 - i)$

D

QUESTION 7

let $2\sin x - 3\cos x = R\sin(x - \alpha)$

$R\cos\alpha = 2$, $R\sin\alpha = 3$ } $R = \sqrt{13}$

minimum value is $-\sqrt{13}$ B

QUESTION 8

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$= 2\int \cos u\,du$

$= 2\sin u + c$

$= 2\sin\sqrt{x} + c$ D

QUESTION 9

$\cos^{-1}x$ range $[0, \pi]$

$\frac{1}{2}\cos^{-1}x$ range $[0, \frac{\pi}{2}]$

$\tan\left(\frac{1}{2}\cos^{-1}x\right)$ (graph)

range $y \geq 0$ D

QUESTION 10

$PQ = z - w$

$PR = i(z - w)$

$OR = OP + PR$

$OR = w + i(z - w)$

B

QUESTION 11

(a) $u = 2+i \quad v = -1-i$

(i) $uv = (2+i)(-1-i)$
$= -2-2i-i+1$
$= -1-3i$
$\text{Im}(-1-3i) = -3$ ✓

(ii) $\frac{u}{i} = \frac{2+i}{i} \times \frac{i}{i}$
$= \frac{2i-1}{-1}$
$= 1-2i$ ✓

(iii) $|u-v|^2 = |3+2i|^2$
$= 9+4$
$= 13$ ✓

(iv) $\frac{u}{v} = \frac{2+i}{-1-i} \times \frac{-1+i}{-1+i}$
$= \frac{-2+2i-i-1}{1+1}$ ✓
$= \frac{i-3}{2}$
$= -\frac{3}{2} + \frac{1}{2}i$ ✓

(b) (i) $z_1 = 1+i \quad z_2 = \sqrt{3}-i$
$z_1 = \sqrt{2}\,\text{cis}\,\frac{\pi}{4} \quad z_2 = 2\,\text{cis}\left(\frac{-\pi}{6}\right)$ ✓

(ii) $\arg\left(\frac{z_1^2}{z_2}\right) =$
$= 2\arg(z_1) - \arg(z_2)$
$= 2 \times \frac{\pi}{4} - -\frac{\pi}{6}$
$= \frac{\pi}{2} + \frac{\pi}{6}$
$= \frac{2\pi}{3}$ ✓

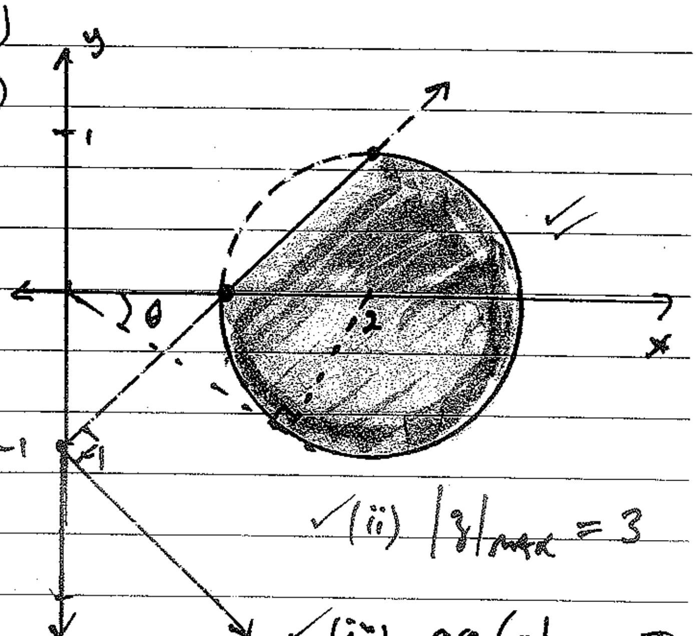
(15)

(c) $(a+bi)^2 = 16+30i$

(i) $a^2-b^2+2abi = 16+30i$
$a^2-b^2 = 16$
$2ab = 30$ ✓
$a=5$ and $b=3$ or
$a=-5$ and $b=-3$
the square roots of $16+30i$ are $\pm(5+3i)$. ✓

(ii) $z^2+(1+i)z-(4+7i) = 0$
$z = \frac{-(1+i) \pm \sqrt{(1+i)^2+4(4+7i)}}{2}$
$z = \frac{-(1+i) \pm \sqrt{16+30i}}{2}$
$z = \frac{-1-i \pm (5+3i)}{2}$
$z = \frac{4+2i}{2}$ ✓ or $z = \frac{-6-4i}{2}$ ✓
$z = 2+i$ or $z = -3-2i$

(d) (i)



✓ (ii) $|z|_{max} = 3$

✓ (iii) $\arg(z) = \frac{-\pi}{6}$

QUESTION 12

(a) (i)

$$\int \frac{1}{\sqrt{9x^2+2}}\,dx$$

let $u = 3x$

$du = 3\,dx$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u^2+2}}\,du \checkmark$$

$$= \frac{1}{3}\ln\left(u + \sqrt{u^2+2}\right) + c$$

$$= \frac{1}{3}\ln\left(3x + \sqrt{9x^2+2}\right) + c \checkmark$$

(ii)

$$\int \frac{1}{x+\sqrt{x}}\,dx$$

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2u\,du = dx$

$$= \int \frac{1}{u^2+u} \times 2u\,du \checkmark$$

$$= 2\int \frac{1}{u+1}\,du$$

$$= 2\ln(u+1) + c$$

$$= 2\ln(\sqrt{x}+1) + c \checkmark$$

(b) (i) $$\int_0^1 \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \left[\sin^{-1}x - \sqrt{1-x^2} \checkmark\right]_0^1$$

$$= \frac{\pi}{2} + 1 \checkmark$$

(ii)

$$\int_0^1 \frac{x^2-1}{x^2+1}\,dx$$

$$= \int_0^1 \frac{x^2+1-2}{x^2+1}\,dx$$

$$= \int_0^1 1 - \frac{2}{1+x^2}\,dx \checkmark$$

$$= \left[x - 2\tan^{-1}x\right]_0^1$$

$$= \left(1 - \frac{\pi}{2}\right) - 0$$

$$= \frac{1}{2}(2-\pi) \checkmark$$

(c)

$$\int x \ln x\,dx$$

let $u = \ln x$ $\quad v' = x$

$u' = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

... (c)

$$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{1}{x} \times \frac{x^2}{2}\, dx \quad ✓$$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{x^2}{4} + c \quad ✓$$

(d)

(i) let

$$\frac{4x+2}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

$$4x+2 \equiv a(x^2+1) + (x+1)(bx+c)$$

put $x = -1$:

$-2 = 2a$, $\underline{a = -1}$

$$4x+2 = (a+b)x^2 + (b+c)x + (a+c)$$

equating powers of x:

$a+b = 0$
$b+c = 4$
$a+c = 2$

$\underline{b = 1}$ ✓✓

$\underline{c = 3}$

(ii)

$$\int_0^{\sqrt{3}} \frac{4x+2}{(x+1)(x^2+1)}\, dx$$

$$= \int_0^{\sqrt{3}} \frac{-1}{x+1} + \frac{x+3}{x^2+1}\, dx$$

$$= \int_0^{\sqrt{3}} \frac{x}{x^2+1} + \frac{3}{x^2+1} - \frac{1}{x+1}\, dx \quad ✓$$

$$= \left[\frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x - \ln(x+1)\right] ✓$$

$$= \frac{1}{2}\ln 4 + \pi - \ln(\sqrt{3}+1) \quad ✓$$

$$= \ln\left(\frac{2}{\sqrt{3}+1}\right) + \pi$$

$$= \ln\left(\frac{2}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}\right) + \pi$$

$$= \ln\left(\frac{2\sqrt{3}-2}{3-1}\right) + \pi$$

$$= \ln(\sqrt{3}-1) + \pi$$

(15)

QUESTION 13

(a)

$x^3 + y^3 - 5y - 3 = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 5\frac{dy}{dx} = 0$ ✓

$\frac{dy}{dx}(3y^2 - 5) = -3x^2$

$\frac{dy}{dx} = \frac{3x^2}{5 - 3y^2}$ ✓

at $(1, -2)$

gradient $= \frac{3}{-7}$

equat^n of the tangent:

$y - y_1 = m(x - x_1)$

$y + 2 = \frac{-3}{7}(x - 1)$ ✓

$7y + 14 = -3x + 3$

$3x + 7y + 11 = 0$.

(b)

(i) B:

$OC = AB = iz$

$OB = OA + AB$

$= z + iz$ ✓

(ii)

$OC' = OC \times cis\frac{\pi}{4}$

$= iz\, cis\frac{\pi}{4}$ ✓

$= iz\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

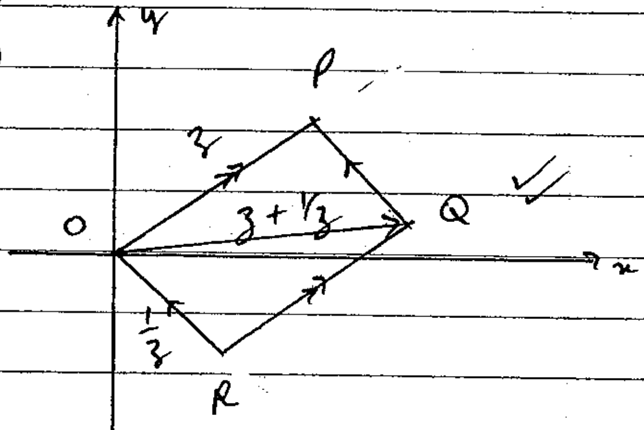
$= iz\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

$= \frac{\sqrt{2}}{2}iz - \frac{\sqrt{2}}{2}z$ ✓

$= \frac{\sqrt{2}}{2}(-1 + i)z$

(c)

(i)



✓✓

note: $\arg(z) = -\arg\left(\frac{1}{z}\right)$

(ii) $|z| = 1$ and $\arg(z) = \frac{\pi}{4}$

Hence $z = cis\frac{\pi}{4}$ ✓ for OPQR to be a square.

(d)

(i)

$LHS = (k+1)^2(k+4)$

$= (k^2 + 2k + 1)(k + 4)$

$= k^3 + 4k^2 + 2k^2 + 8k + k + 4$

$= k^3 + 6k^2 + 9k + 4$

$= RHS$. ✓

(ii)

A/ Show the statement is true for $n = 1$.

put $n = 1$:

$LHS = \frac{1}{1 \times 2 \times 3}$

$= \frac{1}{6}$ ✓

$$RHS = \frac{1(1+3)}{4(1+1)(1+2)}$$

$$= \frac{4}{4 \times 2 \times 3}$$

$$= \frac{1}{6}$$

Hence the statement is true for $n=1$.

B/ Assume true for $n=k$. (k an integer.)

ie/

$$\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad (*)$$

To prove for $n=k+1$.

To prove:

$$\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

Using the assumption (*)

$$LHS = \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$ (from part (i))

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$= RHS.$

C/ It follows from parts A and B by mathematical induction that the statement is true for all integers $n \geq 1$. ✓

(iii) $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

$= \lim_{n \to \infty} \frac{n(n+3)}{4(n+1)(n+2)}$

Divide by n^2:

$= \lim_{n \to \infty} \frac{1(1+3/n)}{4(1+\frac{1}{n})(1+\frac{2}{n})}$ ✓

(as $n \to \infty$, $\frac{1}{n} \to 0$)

$= \frac{1}{4}$ //

(15)

QUESTION 14

(a)

(i) $\cos 3x = \sin 2x$

$\cos 3x = \cos\left(\frac{\pi}{2} - 2x\right)$ ✓

$3x = 2\pi n + \left(\frac{\pi}{2} - 2x\right)$

$5x = 2\pi n + \frac{\pi}{2}$

$x = \frac{2\pi n}{5} + \frac{\pi}{10}$ ✓

OR $3x = 2\pi n - \left(\frac{\pi}{2} - 2x\right)$

$x = 2\pi n - \frac{\pi}{2}$ ✓

$(n \in \mathbb{Z})$

(ii) $\cos 3x - \sin 2x = 0$

$x = \frac{2\pi n}{5} + \frac{\pi}{10}$ or

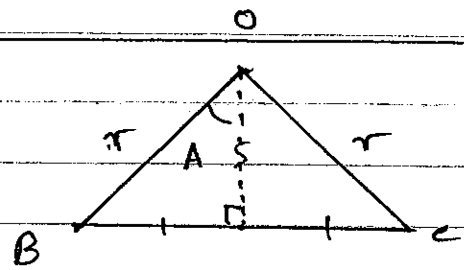
$x = 2\pi n - \frac{\pi}{2}$

put $n = 0$: $x = \frac{\pi}{10}$ ✓

is the smallest positive solution.

(b) $\angle BOC = 2A$

(the angle subtended by an arc at the centre is twice the angle subtended by the arc at the circumference)



$\sin A = \frac{\frac{1}{2} BC}{r}$ ✓

$\frac{1}{2} BC = r \sin A$

$\therefore BC = 2r \sin A.$

(ii) $|\triangle OBC| = \frac{1}{2} r^2 \sin 2A$

$|\triangle OCA| = \frac{1}{2} r^2 \sin 2B$

$|\triangle OAB| = \frac{1}{2} r^2 \sin 2C$

$|\triangle ABC| = \frac{1}{2} r^2 (\sin 2A + \sin 2B + \sin 2C)$ — (1) ✓

Also:

$|\triangle ABC| = \frac{1}{2} BC \times AC \times \sin C$

($BC = 2r \sin A$ and $AC = 2r \sin B$ from (i))

$|\triangle ABC| = \frac{1}{2} \times 2r\sin A \times 2r \sin B \times \sin C$ ✓

$= 2r^2 \sin A \sin B \sin C$ — (2)

equate (1) and (2)

$\frac{1}{2} r^2 (\sin 2A + \sin 2B + \sin 2C) = 2r^2 \sin A \sin B \sin C$

and

$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(as required.)

QUESTION 14 ....

(c)

(i) $z = \cos\theta + i\sin\theta$

$z+1 = 1 + \cos\theta + i\sin\theta$

$z+1 = 1 + \left(2\cos^2\frac{\theta}{2} - 1\right) + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ ✓

$z+1 = 2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$z+1 = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

(ii) $\dfrac{z_1 + z_1 z_2}{z_1 + 1}$

$= z_1 \dfrac{(1+z_2)}{(1+z_1)}$

using the result from (i) :-

$= \dfrac{(\cos\alpha + i\sin\alpha)\, 2\cos\beta/2\, (\cos\beta/2 + i\sin\beta/2)}{2\cos\alpha/2\, (\cos\alpha/2 + i\sin\alpha/2)}$ ✓

$= \dfrac{\cos\beta/2}{\cos\alpha/2}\, \text{cis}\left(\alpha + \beta/2 - \alpha/2\right)$ ✓

$= \dfrac{\cos\beta/2}{\cos\alpha/2}\, \text{cis}\left(\dfrac{\alpha+\beta}{2}\right)$

(iii) $\dfrac{z_1 + z_1 z_2}{z_1 + 1} = 2i$

$\dfrac{\cos\beta/2}{\cos\alpha/2}\, \text{cis}\left(\dfrac{\alpha+\beta}{2}\right) = 2i$

① $\dfrac{\cos\beta/2}{\cos\alpha/2} = 2$ ✓

② $\cos\left(\dfrac{\alpha+\beta}{2}\right) = 0$

$\alpha + \beta = \pi$

$\alpha = \pi - \beta$ ✓

sub. into ①

① $\cos\beta/2 = 2\cos\alpha/2$

$\cos\beta/2 = 2\cos\left(\frac{\pi}{2} - \beta/2\right)$

$\cos\beta/2 = 2\sin\beta/2$

$\tan\beta/2 = \frac{1}{2} = t$

Now $\cos\beta = \dfrac{1-t^2}{1+t^2}$

$= \dfrac{1 - 1/4}{1 + 1/4}$

$= \dfrac{3}{5}$

and $\sin\beta = \dfrac{2t}{1+t^2}$

$= \dfrac{1}{1 + 1/4}$

$= 4/5$

$\therefore z_2 = \frac{3}{5} + \frac{4}{5} i$. ✓

(i) $2\cos\frac{\alpha}{2} = \cos\beta/_2$

$2\cos\frac{\alpha}{2} = \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$

$2\cos\frac{\alpha}{2} = \sin\alpha/_2$

$\tan\frac{\alpha}{2} = 2 = t$

$\therefore \sin\alpha = \frac{2t}{1+t^2}$

$= \frac{4}{5}$

$\cos\alpha = \frac{1-t^2}{1+t^2}$

$= \frac{1-4}{5}$

$= -\frac{3}{5}$

$\therefore z_1 = -\frac{3}{5} + \frac{4}{5} i$ ✓