SYDNEY GRAMMAR SCHOOL

2013 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 2

Thursday 21st February 2013

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 71 boys

Examiner
REP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which of the following represents the complex number $-1-\sqrt{3}\,i$ expressed in modulus–argument form? **1**

(A) $2\,\mathrm{cis}\,\frac{5\pi}{6}$

(B) $2\,\mathrm{cis}\left(-\frac{5\pi}{6}\right)$

(C) $2\,\mathrm{cis}\,\frac{2\pi}{3}$

(D) $2\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$

QUESTION TWO

Which of the following represents the complex number iz where $|z|=r$ and $\arg(z)=\theta$. **1**

(A) $-r\,\mathrm{cis}\,\theta$

(B) $-r\,\mathrm{cis}(\theta+\frac{\pi}{2})$

(C) $r\,\mathrm{cis}(\theta+\frac{\pi}{2})$

(D) $r\,\mathrm{cis}(\theta-\frac{\pi}{2})$

QUESTION THREE

The primitive of $\dfrac{x}{1+x^2}$ is: **1**

(A) $\log_e\sqrt{(1+x^2)}+C$

(B) $\log_e(1+x^2)+C$

(C) $x\tan^{-1}x+C$

(D) $\frac{1}{2}x^2\tan^{-1}x+C$

Exam continues next page ...

QUESTION FOUR

The solutions to the quadratic equation $z^2 - 2iz + 3 = 0$ are: **1**

(A) $z = -2 + i$ or $-2 - i$

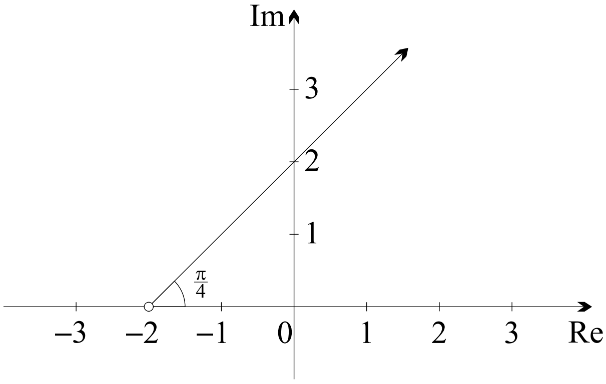
(B) $z = 2 - i$ or $-2 - i$

(C) $z = i$ or $-3i$

(D) $z = -i$ or $3i$

QUESTION FIVE

1



The Argand diagram above shows the locus of the complex number z, given by the equation $\arg(z + 2) = \frac{\pi}{4}$. The minimum value of $|z|$ is:

(A) $2 - \sqrt{2}$

(B) $2 + \sqrt{2}$

(C) $\frac{1}{\sqrt{2}}$

(D) $\sqrt{2}$

QUESTION SIX

The derivative of $x \cos e^x$ is: **1**

(A) $\cos e^x - xe^x \sin e^x$

(B) $-e^x \sin e^x$

(C) $\cos e^x - x \sin e^x$

(D) $e^x (\cos e^x - x \sin e^x)$

QUESTION SEVEN

The expression $\dfrac{3x+11}{(x-3)(x+1)}$ can be expressed in partial fractions as: **1**

(A) $\dfrac{-1}{x-3} - \dfrac{4}{x+1}$

(B) $\dfrac{5}{x-3} - \dfrac{2}{x+1}$

(C) $\dfrac{5}{x-3} + \dfrac{2}{x+1}$

(D) $\dfrac{4}{x+1} - \dfrac{1}{x-3}$

QUESTION EIGHT

The gradient of the tangent to the curve $\sin x + 2\sin y = 1$ at the point $(\pi, \frac{\pi}{6})$ is: **1**

(A) $\frac{1}{\sqrt{3}}$

(B) $-\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

QUESTION NINE

If $z_1 = 1 + i$, $z_2 = 1 + \sqrt{3}\,i$ and $z_3 = \sqrt{3} - i$, then $\arg(z_1 z_2 z_3)$ equals: **1**

(A) $\frac{3\pi}{4}$

(B) $\frac{11\pi}{12}$

(C) $\frac{5\pi}{12}$

(D) $-\frac{\pi}{12}$

QUESTION TEN

If $z = \dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$ then z can be expressed as: **1**

(A) $\cos 2\theta + i\sin 2\theta$

(B) $\cos 2\theta - i\sin 2\theta$

(C) $\sin 2\theta + i\cos 2\theta$

(D) $\sin 2\theta - i\cos 2\theta$

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $u = 3 - i$ and $w = 4 + 3i$.

(i) Find $\mathrm{Im}(uw)$. $\boxed{1}$

(ii) Find $-iw$. $\boxed{1}$

(iii) Evaluate $|u + w|^2$. $\boxed{1}$

(iv) Express $\dfrac{u}{w}$ in the form $a + ib$, where a and b are real numbers. $\boxed{2}$

(b) (i) Sketch the region in the complex plane which simultaneously satisfies $\boxed{2}$

$$\mathrm{Im}(z) \geq 1 \text{ and } \arg\left(\frac{z+2i}{z-2i}\right) = \pm\frac{\pi}{2}.$$

(ii) Find the particular z in part (i) that gives the maximum value of $\arg(z)$, given $-\pi < \arg(z) \leq \pi$. $\boxed{2}$

(c) (i) Let $z_1 = a + ib$ and $z_2 = x + iy$. Prove that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$. $\boxed{2}$

(ii) (α) Express $(8 + 7i)(5 + 4i)$ in the form $a + ib$. $\boxed{1}$

(β) Use part (i) to write down $(8 - 7i)(5 - 4i)$ in the form $a + ib$. $\boxed{1}$

(iii) Hence find the prime factorisation of $12^2 + 67^2$. $\boxed{2}$

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\sin x}{\cos x + x}\,dx$ **1**

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^3 2x\,dx$ **2**

(b) Find the following indefinite integrals.

(i) $\displaystyle\int \frac{1}{4x^2+25}\,dx$ **2**

(ii) $\displaystyle\int \frac{x^2}{(1+x^2)^2}\,dx$ (Use the substitution $x = \tan u$.) **3**

(c) Use integration by parts to find $\displaystyle\int \sin^{-1} x\,dx$. **2**

(d) Solve the equation $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \sin^{-1} x$. **2**

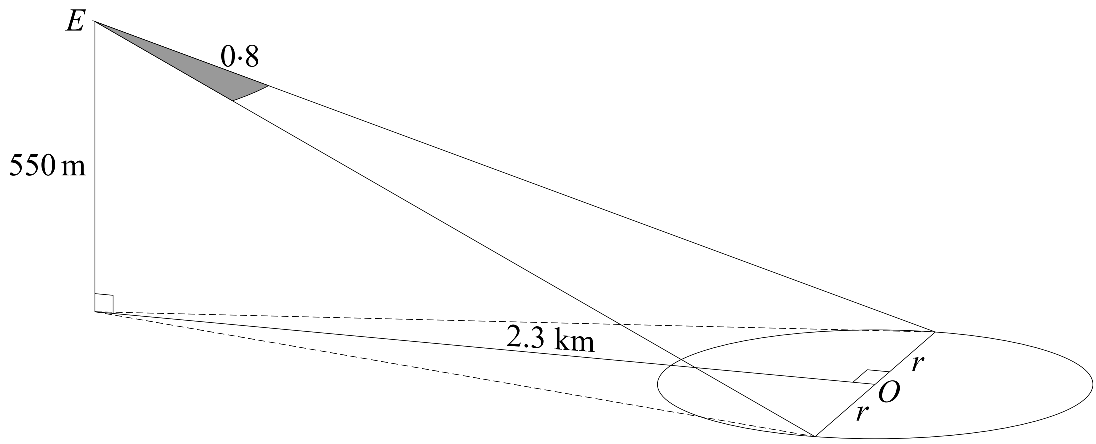
(e) Let A, B and C be the angles of triangle ABC.

(i) Prove that $\tan A = -\dfrac{\tan B + \tan C}{1 - \tan B \tan C}$. **2**

(ii) Hence prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) **2**



On the Serengeti plain in 1855 an explorer E on a rocky outcrop 550 metres above the plain observed a herd of buffalo in the shape of a circle. He estimated the distance of the centre of the herd, O, to be 2·3 kilometres horizontally from the foot of the outcrop. He also estimated the angle subtended by the diameter of the herd to be 0·8 radians. Find the radius of the circle formed by the buffalo. Give your answer correct to the nearest metre.

(b) Use mathematical induction to prove that if $t_n = 8^n - 2^n$, for all integers $n \geq 1$, then t_n is divisible by 6. **3**

(c) (i) Given that $z = r \operatorname{cis} \theta$ express $\dfrac{1}{z}$ in terms of r and θ. **1**

(ii) The origin O and the points P, Q and R, representing the complex numbers z, $z + \dfrac{1}{z}$ and $\dfrac{1}{z}$ respectively, form a quadrilateral. Assume that $0 < \arg(z) < \frac{\pi}{2}$ and $|z| \geq 1$.

(α) Sketch the quadrilateral $OPQR$ on the complex plane. **2**

(β) Find the complex number z for $OPQR$ to be a square. **1**

(d) Let $f(x) = \dfrac{x^2 + 9}{(x-2)(x^2 + 2x + 5)}$.

(i) Find the values of the constants A, B and C so that **3**

$$f(x) = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 2x + 5}.$$

(ii) Hence evaluate $\displaystyle\int_3^5 f(x)\,dx$. **3**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) By using the sum to product formula $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$ and a suitable double angle formula, or otherwise, solve the equation **4**

$$\sin 3\theta + \sin\theta + \cos 4\theta = 1\,,\ \text{for } 0^\circ \le \theta \le 180^\circ\,.$$

(b) (i) Use the substitution $x = \theta + \frac{\pi}{4}$ to show that **2**

$$\int \frac{\cos x}{\cos x + \sin x}\,dx = \frac{1}{2}\int \frac{\cos\theta - \sin\theta}{\cos\theta}\,d\theta\,.$$

(ii) Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{1+\tan x}\,dx = \frac{1}{8}(\pi + \log_e 4)\,.$ **3**

(c) Three points A_1, A_2, A_3 representing the complex numbers z_1, z_2, z_3 in the Argand diagram are equally spaced around the circumference of the circle $|z| = 1$.

(i) Show that $z_1 + z_2 + z_3 = 0\,.$ **2**

(ii) The point P represents the complex number w and $|w| = 3$. Find, by using the fact that $z\overline{z} = |z|^2$, or otherwise, the value of ${PA_1}^2 + {PA_2}^2 + {PA_3}^2$. **4**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

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2013
Half-Yearly Examination
FORM VI
MATHEMATICS EXTENSION 2
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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

SECTION I : 1 MARK EACH

1. $|-1-\sqrt{3}i| = 2$

$$\arg(-1-\sqrt{3}i) = -\frac{\pi}{2} - \tan^{-1}\tfrac{1}{\sqrt{3}}$$

So $\arg(-1-\sqrt{3}i) = -\frac{2\pi}{3}$

So $\quad -1-\sqrt{3}i = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

Hence **D**

2. $\quad ir\operatorname{cis}\theta = \operatorname{cis}\frac{\pi}{2} \times r\operatorname{cis}\theta$

So $\quad iz = r\operatorname{cis}(\theta + \frac{\pi}{2})$

Hence **C**

3. $\displaystyle\int \frac{x}{1+x^2} = \log_e(1+x^2) \times \tfrac{1}{2} + c$

$$= \log_e \sqrt{(1+x^2)} + c$$

(for some real constant c.)

Hence **A**

4. $\quad z^2 - 2iz + 3 = 0$

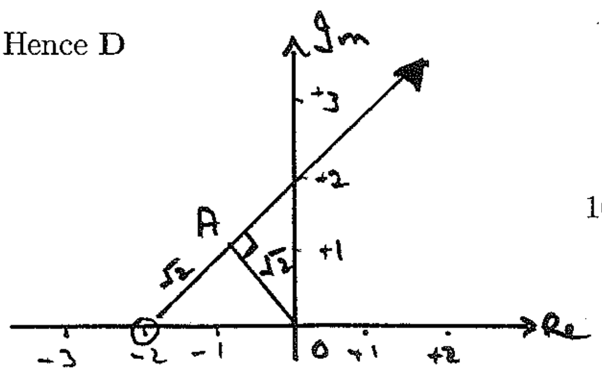
So $\quad z = \dfrac{2i \pm \sqrt{-4-12}}{2}$

So $\quad z = i \pm 2i$

So $\quad z = -i$ or $3i$

Hence **D**

5.



Minimum $|z| = OA$

$= \sqrt{2}$ ($\triangle OAB$ is a $(1-1-\sqrt{2})$ $\triangle \times \sqrt{2}$)

Hence **D**

6. $\dfrac{d}{dx} x\cos e^x = \cos e^x - x \times \sin e^x \times e^x$

Hence **A**

7. $$\frac{3x+11}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

So $\quad 3x + 11 \equiv A(x+1) + B(x-3)$

Equate the coefficients.

So $\quad A = 5$ and $B = -2$

Hence **B**

8. $$\sin x + 2\sin y = 1$$

So $\quad \cos x - 2\cos y \times \dfrac{dy}{dx} = 0$

When $\quad x = \pi$ and $y = \dfrac{\pi}{6}$

$$-1 + 2 \times \tfrac{\sqrt{3}}{2}\frac{dy}{dx} = 0$$

So $\quad \dfrac{dy}{dx} = \tfrac{1}{\sqrt{3}}$

Hence **A**

9. $\arg(z_1z_2z_3) = \arg(z_1) + \arg(z_2) + \arg(z_3)$

$$= \tfrac{\pi}{4} + \tfrac{\pi}{3} - \tfrac{\pi}{6}$$

$$= \tfrac{5\pi}{12}$$

Hence **C**

10. $\dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$

$$= \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta - 2i\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \cos 2\theta - i\sin 2\theta$$

Hence **B**

SECTION II

Each Question Is Out Of 15

11. (a) (i)
$$uw = (3-i)(4+3i)$$
$$= 15 + 5i$$
So $\text{Im}(uw) = 5$ $\boxed{1}$

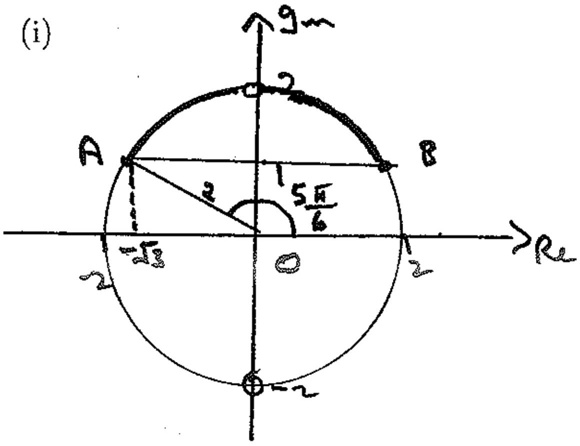
$5i$ (earns zero)

(ii) $-iw = 3 - 4i$ $\boxed{1}$

(iii) $|u+w|^2 = |7+2i|^2$
$$= 49 + 4$$
$$= 53 \quad \boxed{1}$$

(iv)
$$\frac{u}{w} = \frac{3-i}{4+3i}$$
$$= \frac{(3-i)(4-3i)}{16+9}$$
$$= \frac{12-3-4i-9i}{25}$$
So $\dfrac{u}{w} = \dfrac{9}{25} - i\dfrac{13}{25}$ $\boxed{2}$

(b) (i)



The required locus is the minor arc AB with a "hole" where $z = 2i$.

$\boxed{2}$

(ii) The point on the locus that has maximum argument is labelled A in the diagram above.
A represents the complex number z where

$|z| = 2$, $\text{Im}(z) = 1$ and $\frac{\pi}{2} < \arg(z) < \pi$

So the z that has maximum argument is $-\sqrt{3} + i$ $\boxed{2}$

(c) (i)
$$\text{LHS} = \overline{(z_1 z_2)}$$
$$= \overline{(a+ib)(x+iy)}$$
$$= \overline{ax - by + i(bx+ay)}$$
$$= ax - by - i(bx+ay)$$
$$= (a-ib)(x-iy)$$
$$= \overline{z_1}\,\overline{z_2}$$
$$= \text{RHS} \quad \boxed{2}$$

(ii) (α)
$$(8+7i)(5+4i)$$
$$= 40 - 28 + (35+32)i$$
$$= 12 + 67i \quad \boxed{1}$$

(β) From(i)
$$\overline{(8+7i)(5+4i)}$$
$$= \overline{12+67i}$$
$$= 12 - 67i \quad \boxed{1}$$

(iii)
$$12^2 + 67^2$$
$$= (12+67i)(12-67i)$$
$$= (8+7i)(5+4i)(8-7i)(5-4i)$$
$$= (8+7i)(8-7i)(5+4i)(5-4i)$$
$$= (64+49)(25+16)$$
$$= 113 \times 41 \quad \boxed{2}$$

12\. (a) (i) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1-\sin x}{\cos x + x}\, dx$

$= \Big[\log_e |\cos x + x|\Big]_0^{\frac{\pi}{2}}$

$= \log_e |0 + \frac{\pi}{2}| - \log_e |1+0|$

$= \log_e \frac{\pi}{2} (\approx 0{\cdot}452)$ $\boxed{1}$

(ii) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^3 2x\, dx$

$\displaystyle= \int_0^{\frac{\pi}{4}} (1-\sin^2 2x)\cos 2x\, dx$

$\displaystyle= \int_0^{\frac{\pi}{4}} \cos 2x - \sin^2 2x \cos 2x\, dx$

$= \Big[\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\Big]_0^{\frac{\pi}{4}} = \frac{1}{2} - \frac{1}{6} - 0 + 0$

$= \frac{1}{3}$ $\boxed{2}$

(b) (i) $\displaystyle\int \frac{1}{4x^2+25}\, dx$

$= \frac{1}{10}\tan^{-1}\frac{2}{5}x + c$, (for some real constant c.) $\boxed{2}$

(ii) $\displaystyle\int \frac{x^2}{(1+x^2)^2}\, dx$

$\displaystyle= \int \frac{\tan^2 u}{(1+\tan^2 u)}\sec^2 u\, du$

$\displaystyle= \int \frac{\tan^2 u \sec^2 u}{\sec^4 u}\, du$

$\displaystyle= \int \frac{\tan^2 u}{\sec^2 u}\, du$

$\displaystyle= \int \sin^2 u\, du$

$\displaystyle= \frac{1}{2}\int 1 - \cos 2u\, du$

$= \dfrac{u}{2} - \frac{1}{4}\sin 2u + c$, (for some real constant c.)

$= \dfrac{u}{2} - \frac{1}{2}\sin u \cos u + c$, (for some real constant c.)

$= \frac{1}{2}\tan^{-1} x - \dfrac{x}{2(1+x^2)} + c$, (for some real constant c.) $\boxed{3}$

Let $x = \tan u$

So $dx = \sec^2 u\, du$

Note: $\tan u = x$

$\Longrightarrow \quad \sin u = \dfrac{x}{\sqrt{1+x^2}}$

and $\quad \cos u = \dfrac{1}{\sqrt{1+x^2}}$

(c) $\displaystyle\int \sin^{-1} dx$

$\displaystyle= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$

$= x\sin^{-1} x + \sqrt{1-x^2} + c$, (for some real constant c.) $\boxed{2}$

(d) $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3} = \sin^{-1} x$

Let $\alpha = \tan^{-1}\frac{1}{2}$ and $\beta = \tan^{-1}\frac{1}{3}$

So $\alpha - \beta = \sin^{-1} x$

So $x = \sin(\alpha - \beta)$

So $x = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

So $x = \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} - \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$

So $x = \frac{\sqrt{2}}{10}$ $\boxed{2}$

(e) (i) $A + B + C = \pi$ ($\angle$ sum $\triangle$)

So $\tan A = \tan\left(\pi - (B + C)\right)$

So $\tan A = -\tan(B + C)$

So $-\tan A = \dfrac{\tan B + \tan C}{1 - \tan B \tan C}$

So $\tan A = -\dfrac{\tan B + \tan C}{1 - \tan B \tan C}$, as required. $\boxed{2}$

(ii) So $\tan A - \tan A \tan B \tan C = -\tan B - \tan C$

So $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, as required. $\boxed{1}$

13. (a) $EO^2 = 550^2 + 2300^2$ (Pythagoras)

So $EP = \sqrt{5\,592\,500}$

But $r = EO \tan 0{\cdot}4$

So $r = \sqrt{5\,592\,500}\tan 0{\cdot}4$

So $r \approx 999{\cdot}8$

So $r = 1000\,\text{m}$ to the nearest metre. $\boxed{2}$

(b) Let $S(n)$ be the statement "that if $t_n = 8^n - 2^n$, for all integers $n \geq 1$, then t_n is divisible by 6."

Let k be an integer, ≥ 1 such that $S(k)$ is true.

That is $t_k = 8^k - 2^k$ is divisible by 6.

$$\begin{aligned} \text{Now } t_{k+1} &= 8^{k+1} - 2^{k+1} \\ &= 8(8^k - 2^k) + 8 \times 2^k - 2^{k+1} \\ &= 8t_k + (8-2)2^k \\ &= 8t_k + 6 \times 2^k \end{aligned}$$

Now this expression is divisible by 6 as t_k is by assumption and 6 clearly divides 6×2^k.

So $S(k)$ being true implies $S(k+1)$ is true.

Furthermore, when $n = 1$, $8^n - 2^n = 6 \Longrightarrow S(1)$ is true.

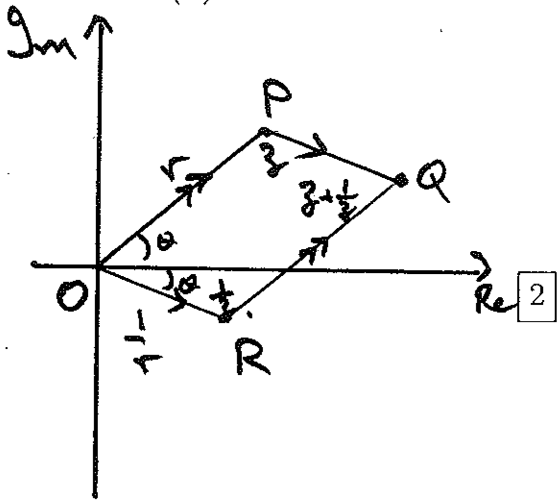
Hence by the principle of mathematical induction $8^n - 2^n$ is divisible by 6 for all positive integers n. $\boxed{3}$

(c) (i)

$$\frac{1}{z} = \frac{1}{r\,\text{cis}\,\theta}$$
$$= \frac{1}{r\,\text{cis}\,\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$
$$= \frac{1}{r}\frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$
$$= \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= \frac{1}{r}\Big(\cos(-\theta) + i\sin(-\theta)\Big)$$

or $= \dfrac{1}{r}\,\text{cis}(-\theta)$ $\boxed{1}$

(α)



$\boxed{2}$

(β) For $OPQR$ to be a square OP must equal OR

That is $|z| = \left|\frac{1}{z}\right|$ which is only possible if $|z| = 1$.

Furthermore, it is necessary that $\angle POR = \frac{\pi}{2}$

That is $2\theta = \frac{\pi}{2}$

So $\arg(z) = \frac{\pi}{4}$

Hence $z = \text{cis}\,\frac{\pi}{4}$

That is $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ $\boxed{1}$

(d) (i) The "cover-up rule" gives $A = 1$.

So $x^2 + 9 \equiv x^2 + 2x + 5 + (x-2)(Bx + C)$

So $B + 1 = 1$ (equating the coefficient of x^2)

So $B = 0$

Also $5 - 2C = 9$ (equating the constant term)

So $C = -2$

Hence $f(x) = \dfrac{1}{x-2} - \dfrac{2}{x^2+2x+5}$ $\boxed{3}$

(ii)
$$\int_3^5 f(x)\,dx = \int_3^5 \frac{1}{x-2} - \frac{2}{x^2+2x+5}\,dx$$
$$= \int_3^5 \frac{1}{x-2}\,dx - \int_3^5 \frac{2}{4+(x+1)^2}\,dx$$
$$= \left[\log_e|x-2| - \tan^{-1}\left(\frac{x+1}{2}\right)\right]_3^5$$
$$= \log_e 3 - \tan^{-1}3 - \log_e 1 + \tan^{-1}2$$
$$= \log_e 3 - \tan^{-1}3 + \tan^{-1}2\,(\approx 0{\cdot}957)$$ $\boxed{3}$

14. (a)

$$\sin 3\theta + \sin\theta + \cos 4\theta = 1$$

So $2\sin 2\theta\cos\theta + \cos 4\theta = 1$

So $2\sin 2\theta\cos\theta + 1 - 2\sin^2 2\theta = 1$

So $\sin 2\theta(\cos\theta - \sin 2\theta) = 0$

So $\sin 2\theta(\cos\theta - 2\sin\theta\cos\theta) = 0$

So $\sin 2\theta\cos\theta(1 - 2\sin\theta) = 0$

So $\sin 2\theta = 0$ or $\cos\theta = 0$ or $\sin\theta = \frac{1}{2}$

Now $0^\circ \le \theta \le 180^\circ$

So $\sin 2\theta = 0 \Longrightarrow \theta = 0^\circ,\ 90^\circ$ or 180°

$\cos\theta = 0 \Longrightarrow \theta = 90^\circ$

$\sin\theta = \frac{1}{2} \Longrightarrow \theta = 30^\circ$ or 150°

So in summary, $\theta = 0^\circ,\ 30^\circ,\ 90^\circ,\ 150^\circ$ or 180° $\boxed{4}$

(b) (i) Let $x = \theta + \frac{\pi}{4}$ $\Longrightarrow dx = d\theta$

$$\int \frac{\cos x}{\cos x + \sin x}\,dx$$

$$= \int \frac{\cos(\theta + \frac{\pi}{4})}{\cos(\theta + \frac{\pi}{4}) + \sin(\theta + \frac{\pi}{4})}\,d\theta$$

$$= \int \frac{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta}{\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{\cos\theta - \sin\theta}{\cos\theta}\,d\theta, \text{ as required.} \quad \boxed{2}$$

(ii) So

$$\int_0^{\frac{\pi}{4}} \frac{1}{1 + \tan x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos x}{\cos x + \sin x}\,dx$$

Let $x = \theta + \frac{\pi}{4}$

x	0	$\frac{\pi}{4}$
θ	$-\frac{\pi}{4}$	0

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{0} \frac{\cos\theta - \sin\theta}{\cos\theta}\,d\theta \quad \text{(From (i))}$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{0} 1 - \tan\theta\,d\theta$$

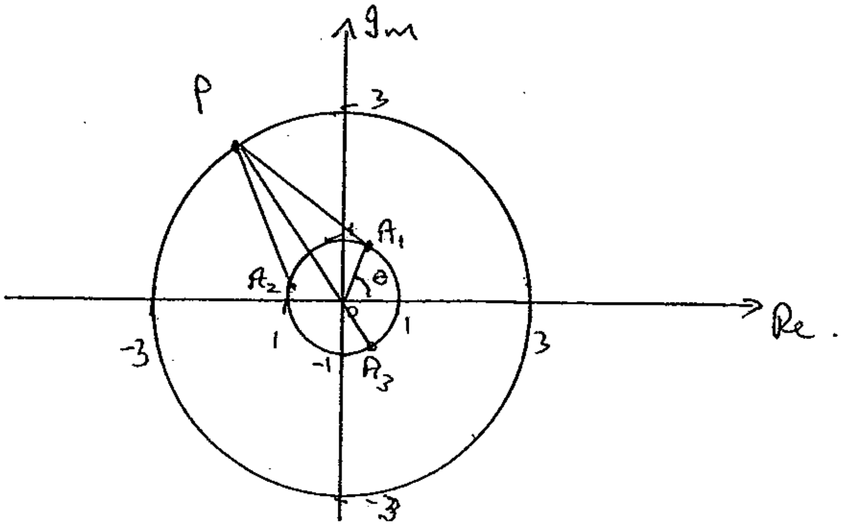
$$= \frac{1}{2}\Big[\theta + \log_e|\cos\theta|\Big]_{-\frac{\pi}{4}}^{0}$$

$$= \frac{1}{2}\left(0 + \log_e 1 - \left(-\frac{\pi}{4} + \log_e \frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{8}(\pi + 2\log_e 2)$$

$$= \frac{1}{8}(\pi + \log_e 4), \text{ as required.} \quad \boxed{3}$$

(c)



(i) Let $z_1 = \text{cis}\,\theta$, $z_2 = \text{cis}(\theta + \frac{2\pi}{3})$ and $z_3 = \text{cis}(\theta - \frac{2\pi}{3})$.

Hence $z_1 + z_2 + z_3 = \text{cis}\,\theta + \text{cis}(\theta + \frac{2\pi}{3}) + \text{cis}(\theta - \frac{2\pi}{3})$

But $\text{cis}(\alpha + \beta) = \text{cis}\,\alpha\,\text{cis}\,\beta$

So $z_1 + z_2 + z_3 = \text{cis}\,\theta\Big(1 + \cos\frac{2\pi}{3} + \cos(-\frac{2\pi}{3}) + i\left(\sin\frac{2\pi}{3} + \sin(-\frac{2\pi}{3})\right)\Big)$

So $z_1 + z_2 + z_3 = \text{cis}\,\theta\left(1 - \frac{1}{2} - \frac{1}{2} + i(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2})\right)$

Hence $z_1 + z_2 + z_3 = 0$, as required. $\boxed{2}$

OR

The triangle formed by z_1, z_2 and z_3 is equilateral.
Hence the centroid of the triangle is the origin, that is the complex number 0.

But the centroid is given by $\frac{1}{3}\Big(z_1 + z_2 + z_3\Big)$

Hence $z_1 + z_2 + z_3 = 0$, as required. $\boxed{2}$

(ii) Now $PA_1{}^2 = |z_1 - w|^2$, $PA_2{}^2 = |z_2 - w|^2$ and $PA_3{}^2 = |z_3 - w|^2$.

$|z_1 - w|^2$
$= (z_1 - w)(\overline{z_1 - w})$
$= (z_1 - w)(\overline{z}_1 - \overline{w})$
$= z_1\overline{z}_1 - w\overline{z}_1 - z_1\overline{w} + w\overline{w}$
$= |z_1|^2 + |w|^2 - (w\overline{z}_1 + z_1\overline{w})$
$= 1 + 9 - (w\overline{z}_1 + z_1\overline{w})$
$= 10 - (w\overline{z}_1 + z_1\overline{w})$

Using the similar results for $|z_2 - w|^2$ and $|z_3 - w|^2$ we have:

$|z_1 - w|^2 + |z_2 - w|^2 + |z_3 - w|^2$
$= 30 - (w\overline{z}_1 + z_1\overline{w} + w\overline{z}_2 + z_2\overline{w} + w\overline{z}_3 + z_3\overline{w})$
$= 30 - (w(\overline{z}_1 + \overline{z}_2 + \overline{z}_3) + \overline{w}(z_1 + z_2 + z_3))$
$= 30 - w(\overline{z_1 + z_2 + z_3})$ $\Big(\text{from (i)}\Big)$
$= 30 - 0$ (as $\overline{0} = 0$))

So $PA_1{}^2 + PA_2{}^2 + PA_3{}^2 = 30$ $\boxed{4}$