CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Half-Yearly Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 25th February 2016

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 60 Marks

- All questions may be attempted.

Section I – 8 Marks

- Questions 1–8 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 52 Marks

- Questions 9–12 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Nine.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 74 boys

Examiner

RCF

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The expression $i^3 + i^6 + i^9 + i^{12} + i^{15}$ is equal to: **1**

(A) $-i$

(B) i

(C) 1

(D) -1

QUESTION TWO

Suppose $z = r(\cos\theta + i\sin\theta)$ is a complex number. The complex number z^2 will be: **1**

(A) $r^2(\cos\theta^2 + i\sin\theta^2)$

(B) $2r(\cos 2\theta + i\sin 2\theta)$

(C) $r^2(\cos 2\theta + i\sin 2\theta)$

(D) $2r(\cos\theta^2 + i\sin\theta^2)$

QUESTION THREE

What are the solutions to the quadratic equation $(z-2-i)(z+3+2i) = 0$? **1**

(A) $z = 2-i$ or $-3+2i$

(B) $z = -2-i$ or $3+2i$

(C) $z = 2+i$ or $-3-2i$

(D) $z = -2+i$ or $3-2i$

QUESTION FOUR

Which of the following is a primitive of $\dfrac{e^{\sqrt{x}}}{\sqrt{x}}$? **1**

(A) $2e^{\sqrt{x}}$

(B) $\left(e^{\sqrt{x}}\right)^2$

(C) $\ln\left(e^{-\sqrt{x}}\right)$

(D) $\sqrt{x}e^{\sqrt{x}}$

Exam continues next page ...

QUESTION FIVE

Consider the function $f(x) = 2\cos^{-1}\left(\dfrac{1-x}{2}\right)$. What is the gradient of the graph $y = f(x)$ at its y-intercept? **1**

(A) $-\dfrac{4}{\sqrt{3}}$

(B) $\dfrac{4}{\sqrt{3}}$

(C) $-\dfrac{2}{\sqrt{3}}$

(D) $\dfrac{2}{\sqrt{3}}$

QUESTION SIX

A primitive of $2\cot x$ is: **1**

(A) $\dfrac{1}{2}\ln(\cos^2 x)$

(B) $2\ln(\cos x)$

(C) $\dfrac{-2}{\sin^2 x}$

(D) $2\ln(\sin x)$

QUESTION SEVEN

The Kappa curve has the Cartesian equation **1**

$$x^2y^2 + y^4 = a^2x^2$$

where a is a known constant. Which of the following is $\dfrac{dy}{dx}$?

(A) $\dfrac{-xy}{x^2 + 2y^2}$

(B) $\dfrac{x(a^2 - y^2)}{y(x^2 + 2y^2)}$

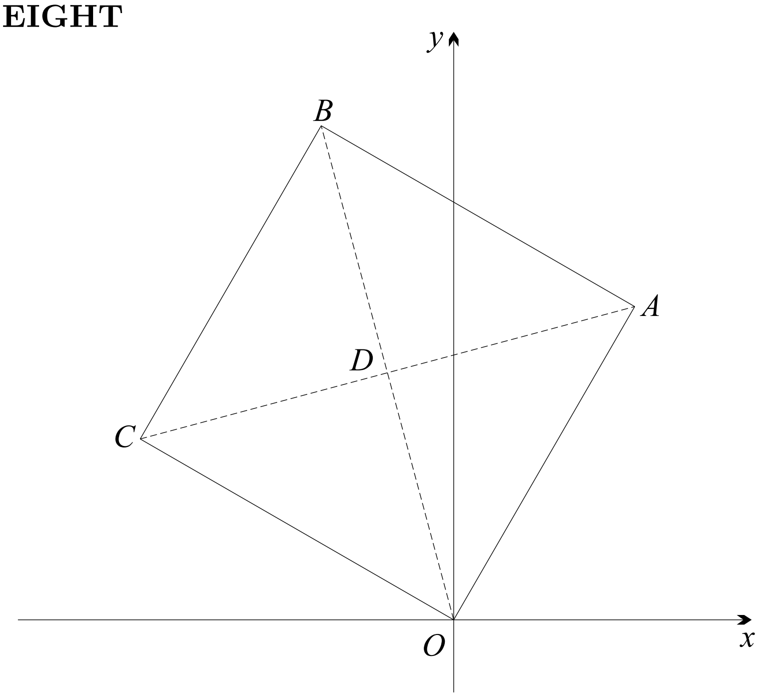
(C) $\dfrac{2(a^2 - y^2)}{xy(x^2 + 2y^2)}$

(D) $\dfrac{x(a^2 - y^2)}{y^3}$

Exam continues overleaf ...

QUESTION EIGHT

1



The diagram above shows a square $OABC$ in the complex plane. The vertex A represents the complex number $1 + i\sqrt{3}$. Which of the following complex numbers represents point D, the intersection of the diagonals?

(A) $\left(\dfrac{1-\sqrt{3}}{2}\right) + \left(\dfrac{\sqrt{3}-1}{2}\right)i$

(B) $\left(\dfrac{1-\sqrt{3}}{2}\right) + \left(\dfrac{1+\sqrt{3}}{2}\right)i$

(C) $\left(\dfrac{1+\sqrt{3}}{2}\right) + \left(\dfrac{\sqrt{3}-1}{2}\right)i$

(D) $\left(\dfrac{1+\sqrt{3}}{2}\right) + \left(\dfrac{1-\sqrt{3}}{2}\right)i$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION NINE (13 marks) Use a separate writing booklet. **Marks**

(a) Let $w = 1 + 3i$ and $z = 2 - 2i$.

(i) Evaluate $i^3 z + z$. **1**

(ii) Find $\overline{w - z}$. **1**

(iii) Evaluate $|w + z|^2$. **1**

(iv) Find $\text{Re}\,(wz)$. **1**

(v) Express $\dfrac{z}{w}$ in the form $a + ib$, where a and b are real numbers in simplest form. **2**

(b) (i) Express $u = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$ and $v = 1 - i\sqrt{3}$ in modulus–argument form. **1**

(ii) Hence, or otherwise, find $\arg\,(u^2 v)$. **1**

(c) Sketch the locus of those points z in the complex plane such that $|z - i| = |z + 1|$. **1**

(d) (i) Find the two square roots of $-7 + 24i$. **2**

(ii) Hence, or otherwise, solve the quadratic equation $z^2 + (1 + 2i)z + (1 - 5i) = 0$. **2**

Exam continues overleaf ...

QUESTION TEN (13 marks) Use a separate writing booklet. **Marks**

(a) (i) Find $\displaystyle\int \frac{x}{25-4x^2}\,dx$. **1**

(ii) Find $\displaystyle\int \frac{x}{(25-4x^2)^3}\,dx$. **1**

(iii) Find $\displaystyle\int \frac{1}{\sqrt{25-4x^2}}\,dx$. **2**

(b) Use partial fractions to find $\displaystyle\int \frac{x+1}{x^2+4x-21}\,dx$. **3**

(c) (i) Use the compound angle formulae to show that **1**

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta).$$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos 6x \cos 5x\,dx$. **2**

(d) Use integration by parts to evaluate $\displaystyle\int_0^1 4xe^{2x}\,dx$. **3**

QUESTION ELEVEN (13 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that $\dfrac{d}{du}\log_e\left(u+\sqrt{a^2+u^2}\right)=\dfrac{1}{\sqrt{a^2+u^2}}$, where a is a constant. **1**

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4+\sin^2 x}}\,dx$. **1**

(b) Use the substitution $t=\tan\frac{x}{2}$ to find $\displaystyle\int \frac{1}{1+\cos x+\sin x}\,dx$. **3**

(c) A complex number satisfies $|z-4|\le 2$ and $\operatorname{Im}(z)\le 0$.

(i) Sketch the locus of z. **2**

(ii) Show that $-\dfrac{\pi}{6}\le \arg z\le 0$. **1**

(d) By rationalising the numerator of the integrand, evaluate $\displaystyle\int_{\frac{1}{2}}^{1}\sqrt{\frac{x}{2-x}}\,dx$. **2**

(e) Use a suitable substitution to evaluate $\displaystyle\int_3^9 \frac{3}{(9+x)\sqrt{x}}\,dx$. **3**

Exam continues overleaf ...

QUESTION TWELVE (13 marks) Use a separate writing booklet. **Marks**

(a) Determine the locus specified by $\text{Im}\left(z + \frac{1}{z}\right) = 0$. **3**

(b) Let $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$.

(i) Prove that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ for $n \geq 2$. **2**

(ii) Hence evaluate $I_4 = \int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$. **2**

(c) Find $\int \sec^5 x \, dx$. **3**

You may use the fact that $\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \log(\sec x + \tan x)) + C$.

(d) In the Fibonacci sequence **3**

$$1, 1, 2, 3, 5, 8, \ldots$$

the terms of the sequence are defined recursively by the equation

$$F_{n+1} = F_n + F_{n-1}$$

where $F_1 = 1, F_2 = 1$ and $n \geq 2$.

Use Induction to prove that, for $n \geq 2$,

$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}.$$

End of Section II

END OF EXAMINATION

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ○ B ○ C ○ D ○

Question Two

A ○ B ○ C ○ D ○

Question Three

A ○ B ○ C ○ D ○

Question Four

A ○ B ○ C ○ D ○

Question Five

A ○ B ○ C ○ D ○

Question Six

A ○ B ○ C ○ D ○

Question Seven

A ○ B ○ C ○ D ○

Question Eight

A ○ B ○ C ○ D ○

EXTENSION 2 H-Y 2016

① $i^3+i^6+i^9+i^{12}+i^{15} = (-i)+(-1)+i+1+(-i)$
$= (-i)$ (A) ✓

② $z^2 = r^2 \text{cis}2\theta + i\sin 2\theta$ (C) ✓

③ $(z-(2+i))(z-(-3-2i)) = 0$
$\therefore z = 2+i$ or $-3-2i$ (C) ✓

④ $\frac{d}{dx}(e^{\sqrt{x}}) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ $\therefore F(x) = 2e^{\sqrt{x}}$
(A) ✓

⑤ $y = 2\cos^{-1}\left(\frac{1-x}{2}\right)$
$\frac{dy}{dx} = 2\left(\frac{-1}{\sqrt{1-\left(\frac{1-x}{2}\right)^2}} \times \left(-\frac{1}{2}\right)\right)$
$= \frac{1}{\sqrt{1-\left(\frac{1-x}{2}\right)^2}}$
$\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{\sqrt{1-\frac{1}{4}}}$
$= \frac{2}{\sqrt{3}}$ (D) ✓

⑥ $\int \frac{2\cos x}{\sin x}\,dx = 2\ln(\sin x)$ (D) ✓

⑦ $x^2y^2 + y^4 = a^2x^2$
$\left(\frac{d}{dx}\right)$ $2xy^2 + x^2 2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 2a^2x$
$\therefore \frac{dy}{dx}(2x^2y + 4y^3) = 2a^2x - 2xy^2$
$\frac{dy}{dx} = \frac{a^2x - xy^2}{x^2y + 2y^3}$
$= \frac{x(a^2-y^2)}{y(x^2+2y^2)}$ (B) ✓

⑧ A represents $z = 1+i\sqrt{3}$
C represents $iz = -\sqrt{3}+i$ (Rotation anticlockwise of 90°)
B represents $z+iz = (1-\sqrt{3}) + i(1+\sqrt{3})$
D represents $\frac{1}{2}(z+iz) = \left(\frac{1-\sqrt{3}}{2}\right) + i\left(\frac{1+\sqrt{3}}{2}\right)$ (B) ✓

Multi Choice Summary

① A ② C ③ C
④ A ⑤ D ⑥ D
⑦ B ⑧ B
(1 mark each)

Question ⑨

a) $w = 1+3i$ $z = 2-2i$

(i) $i^3z + z = (-i+1)z$ or $-i(2-2i) + (2-2i)$
$= (-i+1)(2-2i)$ $= -2i-2+2-2i$
$= -2i + 2i^2 - 2i + 2$ $= -4i$
$= -4i$ ✓

(ii) $\overline{w-z} = \overline{(1+3i)-(2-2i)}$
$= \overline{-1+5i}$
$= -1-5i$ ✓

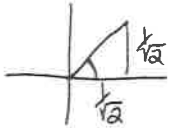
(iii) $|w+z|^2 = |3+i|^2$
$= (3+i)(3-i)$
$= 9+3i-3i-i^2$
$= 10$ ✓

(iv) $Re(wz) = Re\{(1+3i)(2-2i)\}$
$= Re\{2+6i-2i+6\}$
$= Re\{8+4i\}$
$= 8$ ✓

(v) $\frac{z}{w} = \frac{2-2i}{1+3i} \times \frac{(1-3i)}{(1-3i)}$
$= \frac{2-2i-6i-6}{1+9}$ ✓
$= \frac{-4-8i}{10}$
$= \frac{-(2+4i)}{5}$ ✓
$= \left(-\frac{2}{5}\right) + \left(-\frac{4}{5}\right)i$

ie $a = \left(-\frac{2}{5}\right)$, $b = \left(-\frac{4}{5}\right)$

b) (i) $u = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

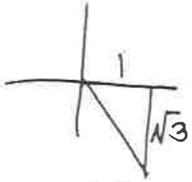


$\Rightarrow r = 1$

$\theta = \frac{\pi}{4}$

$u = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$v = 1 - i\sqrt{3}$



$\Rightarrow r = 2$

$\theta = \left(-\frac{\pi}{3}\right)$

$v = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$ ✓ (Both Correct)

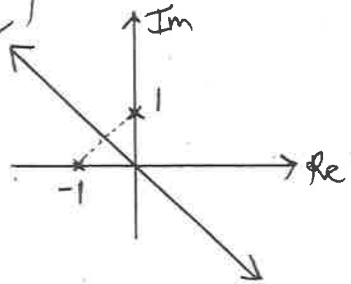
(ii) $\arg(u^2 v) = 2 \times \frac{\pi}{4} + \left(-\frac{\pi}{3}\right)$

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$ ✓

c) Locus of points equidistant from $(-1,0)$ and $(0,1)$

ie perpendicular bisector



✓ Sketch

d) (i) $(a+ib)^2 = -7+24i$

$a^2 - b^2 + 2abi = -7 + 24i$

$a^2 - b^2 = (-7)$

$ab = 12$ ✓

$\therefore b = 4, a = 3$ or $b = (-4), a = (-3)$

square roots are $3+4i$ and $-(3+4i)$ ✓

(ii) $z^2 + (1+2i)z + (1-5i) = 0$

$\Delta = (1+2i)^2 - 4 \times 1 \times (1-5i)$

$= 1 - 4 + 4i - 4 + 20i$ ✓

$= -7 + 24i$

$z = \frac{-(1+2i) + 3 + 4i}{2}$ or $\frac{-(1+2i) - (3+4i)}{2}$

$= \frac{2+2i}{2} = 1+i$ or $\frac{-4-6i}{2} = -2-3i$ ✓

10 a) (i) $\int \frac{x}{25-4x^2}\,dx = -\frac{1}{8}\ln|25-4x^2| + C$ ✓

(ii) $\int \frac{x}{(25-4x^2)^3}\,dx = \frac{1}{16}(25-4x^2)^{-2} + C$ ✓

$= \frac{1}{16(25-4x^2)^2} + C$

(iii) $\int \frac{1}{\sqrt{25-4x^2}} = \frac{1}{2}\int \frac{1}{\sqrt{\frac{25}{4} - x^2}}\,dx$

Do not penalise omission of constant

$= \frac{1}{2}\sin^{-1}\left(\frac{x}{5/2}\right) + C$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{5}\right) + C$ ✓✓ 1 → $\frac{2x}{5}$, 1 → Coeff correct

b) $\int \frac{x+1}{x^2+4x-21}\,dx = \int \frac{x+1}{(x+7)(x-3)}\,dx$ Let $\frac{x+1}{(x+7)(x-3)} = \frac{A}{(x+7)} + \frac{B}{(x-3)}$ ✓

$= \int \frac{3/5}{(x+7)} + \frac{2/5}{(x-3)}\,dx$ By Cover Up Rule

$4 = 10B \Rightarrow B = \frac{2}{5}$

$-6 = -10A \Rightarrow A = \frac{3}{5}$ ✓

$= \frac{3}{5}\ln(x+7) + \frac{2}{5}\ln(x-3) + C$ ✓

$= \frac{1}{5}\left[3\ln(x+7) + 2\ln(x-3)\right] + C$

c) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ①

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ ②

① + ② $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$

$\frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right] = \cos\alpha\cos\beta$ ✓ "Show"

ii) $\int_0^{\frac{\pi}{2}} \cos 6x \cos 5x \, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 11x + \cos x)\, dx$

$= \frac{1}{2}\left[\frac{1}{11}\sin 11x + \sin x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{2}\left[\left(\frac{1}{11}\sin\frac{11\pi}{2} + \sin\frac{\pi}{2}\right) - 0\right]$

$= \frac{1}{2}\left(1 - \frac{1}{11}\right)$

$= \frac{5}{11}$ ✓

d) $\int_0^1 4xe^{2x}\, dx$ $\quad u = 4x \quad \frac{dv}{dx} = e^{2x}$

$= \left[4x\left(\frac{1}{2}e^{2x}\right)\right]_0^1 - \int_0^1 4\cdot\frac{1}{2}e^{2x}\, dx$ ✓ $\quad \frac{du}{dx} = 4 \quad v = \frac{1}{2}e^{2x}$

$= \left[2xe^{2x} - e^{2x}\right]_0^1$ ✓

$= \left[(2x-1)e^{2x}\right]_0^1 = e^2 - (-1e^0) = e^2 + 1$ ✓

11) a) (i) $\frac{d}{du}\ln\left(u + \sqrt{a^2+u^2}\right)$

$= \frac{1}{u+\sqrt{a^2+u^2}} \times \left(1 + \frac{1}{2}(a^2+u^2)^{-\frac{1}{2}} \times 2u\right)$

$= \frac{1 + \frac{u}{\sqrt{a^2+u^2}}}{u + \sqrt{a^2+u^2}} \times \frac{\sqrt{a^2+u^2}}{\sqrt{a^2+u^2}}$ ✓ "Show"

$= \frac{\left(\sqrt{a^2+u^2} + u\right)}{\left(u+\sqrt{a^2+u^2}\right)\sqrt{a^2+u^2}}$

$= \frac{1}{\sqrt{a^2+u^2}}$

(ii) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4+\sin^2 x}}\, dx$ $\quad$ ie $a = 2 \quad u = \sin x \quad \frac{du}{dx} = \cos x$

$= \left[\ln\left(\sin x + \sqrt{4+\sin^2 x}\right)\right]_0^{\frac{\pi}{2}}$

$= \ln\left(1+\sqrt{5}\right) - \ln(0+2)$

$= \ln\left(\frac{1+\sqrt{5}}{2}\right)$ ✓

b) $I = \int \frac{1}{1+\cos x + \sin x}\, dx$ $\quad$ Let $t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$= \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$

$\therefore \frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{2dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2} \quad \sin x = \frac{2t}{1+t^2}$

$= \int \frac{1}{1+\left(\frac{1-t^2}{1+t^2}\right)+\left(\frac{2t}{1+t^2}\right)} \frac{2dt}{1+t^2}$ ✓

$\left(\times \frac{1+t^2}{1+t^2}\right)$

$$= \int \frac{2\,dt}{1+t^2+1-t^2+2t}$$

$$= \int \frac{2\,dt}{2+2t} \quad ✓$$

$$= \int \frac{dt}{1+t}$$

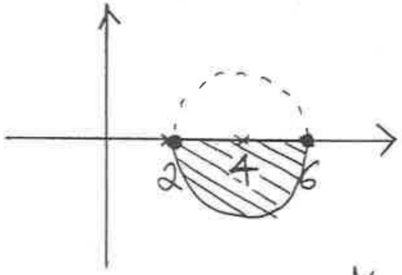
$$= \ln|1+t| + C$$

$$= \ln\left|1+\tan\tfrac{x}{2}\right| + C \quad ✓$$

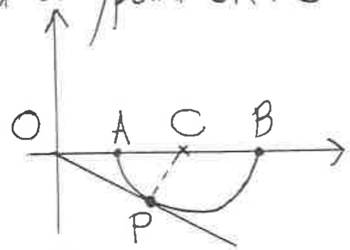
c)(i) $|z-4| \leq 2$

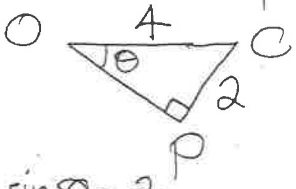
Inside circle centred at $(4,0)$ radius 2 ✓

x-axis and below ✓



(ii) Minimum and maximum arguments occur for P, at tangent from origin in 4^{th} quad, and on x axis at any point on the interval AB.





✓

$\sin\theta = \frac{2}{4}$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$.

$\therefore -\frac{\pi}{6} \leq \arg z \leq 0$

d) $$\int_{\frac{1}{2}}^{1} \sqrt{\frac{x}{2-x}}\,dx \quad \frac{(\times\sqrt{x})}{(\times\sqrt{x})} = \int_{\frac{1}{2}}^{1} \frac{x}{\sqrt{2x-x^2}}\,dx$$

$$= \int_{\frac{1}{2}}^{1} \frac{1-(1-x)}{\sqrt{2x-x^2}}\,dx$$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-(1-x)^2}}\,dx - \frac{1}{2}\int \frac{2(1-x)}{(2x-x^2)^{\frac{1}{2}}}\,dx \quad ✓$$

$$= \left[-\sin^{-1}(1-x) - \sqrt{2x-x^2}\right]_{\frac{1}{2}}^{1}$$

$$= \left(-\sin^{-1}0 - \sqrt{1}\right) - \left(-\sin^{-1}\tfrac{1}{2} - \sqrt{\tfrac{3}{4}}\right)$$

$$= -1 + \frac{\pi}{6} + \frac{\sqrt{3}}{2} \quad ✓$$

$$= \frac{\pi + 3\sqrt{3} - 6}{6}$$

e) $$I = \int_3^9 \frac{3}{(9+x)\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

x	3	9
u	$\sqrt{3}$	3

$$= \int_{\sqrt{3}}^{3} \frac{3 \times 2\,du}{9+u^2} \quad ✓$$

$$= 6\int_{\sqrt{3}}^{3} \frac{du}{9+u^2}$$

$$= 6\left[\tfrac{1}{3}\tan^{-1}\tfrac{u}{3}\right]_{\sqrt{3}}^{3} \quad ✓$$

$$= 2\tan^{-1}1 - 2\tan^{-1}\left(\tfrac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{3}$$

$$= \frac{\pi}{6}. \quad ✓$$

(12) a) $\operatorname{Im}\left(z+\frac{1}{z}\right)=0$ Domain $z \neq 0$

Let $z = x+iy$.

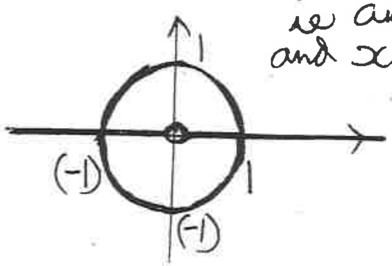
$$z+\frac{1}{z} = x+iy+\frac{1}{x+iy}$$
$$= x+iy+\frac{(x-iy)}{x^2+y^2}$$
$$= \frac{x^3+xy^2+ix^2y+iy^3+x-iy}{x^2+y^2}$$
$$= \frac{(x^3+xy^2+x)+i(x^2y+y^3-y)}{x^2+y^2} \checkmark$$

(NB $x^2+y^2 \neq 0$ since $z \neq 0$)

A neat alternative which simplifies algebra is: $\operatorname{Im}(z)+\operatorname{Im}\left(\frac{1}{z}\right)$

$$\therefore \operatorname{Im}\left(z+\frac{1}{z}\right)=0 \Rightarrow x^2y+y^3-y=0$$
$$y(x^2+y^2-1)=0$$
$$\therefore y=0 \text{ OR } x^2+y^2=1$$

ie circle radius 1 centre 0 and x axis excluding origin.



✓ circle ✓ Line omitting origin

b) $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ By Parts $u = x^n$, $\frac{du}{dx} = nx^{n-1}$, $\frac{dv}{dx} = \sin x$, $v = -\cos x$

$$= \left[-\cos x \times x^n\right]_0^{\pi/2} - \int_0^{\pi/2} -\cos x \,(nx^{n-1})\, dx \checkmark$$
$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x \, dx$$

By Parts $u = x^{n-1}$, $\frac{du}{dx} = (n-1)x^{n-2}$, $\frac{dv}{dx} = \cos x$, $v = \sin x$

$$= n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}(n-1)x^{n-2}\sin x \, dx\right\}$$

✓ shown

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

as required

(ii) $I_0 = \int_0^{\pi/2} \sin x \, dx$
$$= \left[-\cos x\right]_0^{\pi/2}$$
$$= -\cos\frac{\pi}{2} - -\cos 0$$
$$= 1 \checkmark$$

$$I_2 = 2\left(\frac{\pi}{2}\right) - 2\times1\times I_0$$
$$= \pi - 2$$
$$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 4\times3\,I_2$$
$$= \frac{\pi^3}{2} - 12\pi + 24 \checkmark$$

c) $\int \sec^5 x \, dx = \int \sec^2 x \times \sec^3 x \, dx$

Let $\frac{dv}{dx} = \sec^2 x$, $v = \tan x$

$u = \sec^3 x$, $\frac{du}{dx} = 3\sec^2 x \times \sec x \tan x = 3\sec^3 x \tan x$ ✓

$$I = \sec^3 x \tan x - \int 3\sec^3 x \tan^2 x \, dx$$
$$= \sec^3 x \tan x - \int 3\sec^3 x(\sec^2 x - 1)\, dx \checkmark$$
$$= \sec^3 x \tan x + \int 3\sec^3 x \, dx - \int 3\sec^5 x \, dx \checkmark$$
$$4I = \sec^3 x \tan x + 3\int \sec^3 x \, dx$$
$$I = \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log(\sec x + \tan x) + C \checkmark$$

d) r.t.p. $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$

STEP A

Prove true for $n=2$.

LHS $F_2^2 - F_3 \times F_1 = 1^2 - 2\times 1$

$= (-1)$

RHS $(-1)^3 = (-1)$

$\therefore$ LHS = RHS $\therefore$ true for $n=2$ ✓

STEP B.

Assume result true for $n=k$ $(k \geq 2)$

ie $F_k^2 - F_{k+1}F_{k-1} = (-1)^{k+1}$

Now prove true for $n=k+1$

ie $F_{k+1}^2 - F_{k+2}F_k = (-1)^{k+2}$

$$\begin{aligned}
LHS &= F_{k+1}^2 - F_{k+2}F_k \\
&= F_{k+1}^2 - (F_{k+1}+F_k)\times F_k \\
&= F_{k+1}^2 - F_{k+1}F_k - F_k^2 \\
&= F_{k+1}(F_{k+1}-F_k) - F_k^2 \\
&= F_{k+1}F_{k-1} - F_k^2 \\
&= -(F_k^2 - F_{k+1}F_{k-1}) \\
&= -(-1)^{k+1} \\
&= (-1)^{k+2} \\
&= RHS
\end{aligned}$$

using recursive definition $F_{k+2} = F_{k+1} + F_k$

✓ One Use of Recursive Definition

using recursive definition rearranged $F_{k+1} - F_k = F_{k-1}$

using assumption ✓ Use of assumption

STEP C

By principle of mathematical induction, hypothesis proven true for all integer $n \geq 2$.