SYDNEY TECHNICAL HIGH SCHOOL

(Est 1911)

MATHEMATICS EXTENSION II

HSC ASSESSMENT TASK 1

MARCH 2002

Time allowed : 70 minutes

Instructions :

- **Show all necessary working in every question.**
- **Start each question on a new page.**
- **Attempt all questions.**
- **All questions are of equal value.**
- **Full marks may not be awarded for careless or badly arranged work.**
- **Approved calculators may be used.**
- **This test forms part of your HSC assessment.**
- **These questions are to be handed in with your answers.**

Name : ____________________

Question 1	Question 2	Question 3	Total

Question 1 (16 marks) - *Use a separate sheet of paper* Marks

a) Evaluate i^{2002}. **1**

b) If $z = 2 + 2\sqrt{3}\,i$ find

i) $\dfrac{z}{1+i}$ in the form $a + ib$. **1**

ii) $\bar{z}$. **1**

iii) $|z|$ **1**

iv) $Arg(z)$ **1**

v) $\dfrac{1}{z^4}$ in modulus argument form. (use principle argument) **2**

c) On an Argand diagram sketch the locus specified by

$$Arg(z - 1 + i) = \frac{3\pi}{4}$$ **2**

d) i) For what values of λ is the equation **2**

$$\frac{x^2}{14-\lambda} + \frac{y^2}{\lambda - 6} = 1$$ the locus of an ellipse ?

ii) For what values of λ does the equation **1**

$\dfrac{x^2}{14-\lambda} + \dfrac{y^2}{\lambda - 6} = 1$ represent an ellipse with its foci on the y axis ?

e) Find the volume generated when the area bounded by $x = y^2 - 1$ **4**

and the y axis is rotated about the line $x = 1$.

Question 2 (16 marks) - *Use a separate piece of paper.* Marks

a) OABC is a rectangle with $OA = 3 \times OC$. **3**

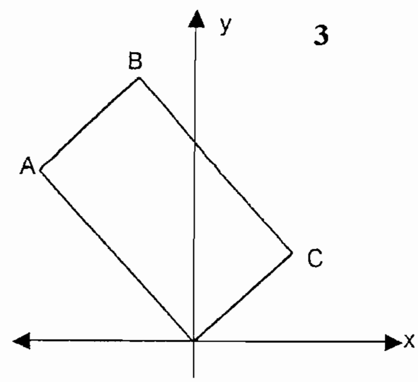
O is the origin.

If C represents the complex number w, find in terms of w the complex number represented by;

i) the point A

ii) the point B

iii) the vector from A to C



b) For the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ **3**

Find i) the eccentricity.

ii) the coordinates of the foci.

iii) the equation of the directrices.

c) i) Show that the equation of the tangent to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ **4**

at the point $P(x_1, y_1)$ is given by $\dfrac{xx_1}{4^2} + \dfrac{yy_1}{3^2} = 1$.

ii) Given that the tangent found in part i) cuts both directrices **4**

of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ above the x axis, find the area

enclosed by this tangent, the two directrices and the x axis.

d) z is a complex number where $|z| = 1$ and $Arg(z) = A$. **2**

Find, in term of A the value of $Arg(z+1)$. Justify your answer.

Question 3 (16 marks) - *Use a separate piece of paper.* Marks

a) On an Argand diagram sketch the locus specified by **2**

$$|z-1-i|>|z|$$

b) Find the modulus of $\dfrac{(2-i)^8}{(2+i)^6}$. **3**

c) i) Solve $z^5=-1$ over the complex field. **2**

ii) If ω is the complex root of $z^5=-1$ with smallest **3**

positive argument, show that the other complex roots equal

$-\omega^2$, ω^3 and $-\omega^4$.

iii) Using ω as in part ii) simplify $(1-\omega+\omega^2-\omega^3)^8$ **3**

d) Describe and sketch on an Argand diagram the locus **3**

of z given that $\mathrm{Re}(z-\frac{1}{z})=0$ where $z\neq 0$.

SOLUTIONS

QUESTION 1

a. $i^{2002} = (i^2)^{1001}$

$= (-1)^{1001}$

$= -1$

b. i) $\dfrac{2+2\sqrt{3}i}{1+i}$

$= \dfrac{2+2\sqrt{3}i}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{2-2i+2\sqrt{3}i+2\sqrt{3}}{2}$

$= (1+\sqrt{3}) + i(\sqrt{3}-1)$

ii) $\bar{z} = 2 - 2\sqrt{3}i$

iii) $|z| = \sqrt{2^2 + (2\sqrt{3})^2}$

$= 4$

iv) $\text{Arg}(z) = \dfrac{\pi}{3}$

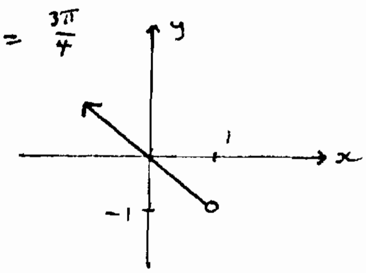
v) $\dfrac{1}{z^4} = z^{-4}$

$= \left[4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{-4}$

$= 4^{-4}\left(\cos\frac{-4\pi}{3} + i\sin\frac{-4\pi}{3}\right)$

$= \frac{1}{256}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

c. $\text{Arg}(z-(1-i)) = \dfrac{3\pi}{4}$



d. i) $14-\lambda > 0$ and $\lambda - 6 > 0$

$\lambda < 14$ and $\lambda > 6$

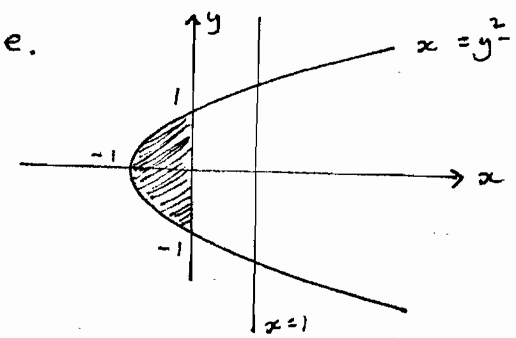
$\therefore\ 6 < \lambda < 14,\ \lambda \neq 10$

ii) $\lambda - 6 > 14 - \lambda$ (plus part i)

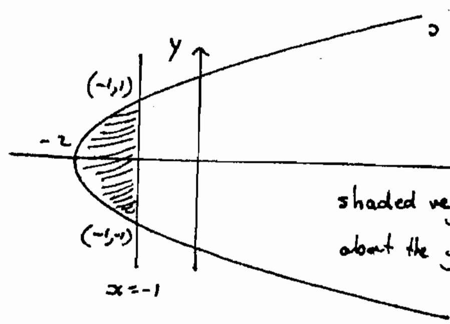
$\lambda > 10$

$\therefore\ 10 < \lambda < 14$

e.



translate 1 unit left



shaded region is rotated about the y axis

$\therefore\ V = 2\pi\int_0^1 (y^2-2)^2 - (-1)^2\ dy$

$= 2\pi\int_0^1 y^4 - 4y^2 + 3\ dy$

$= 2\pi\left[\frac{1}{5}y^5 - \frac{4}{3}y^3 + 3y\right]_0^1$

$= 2\pi\left[\frac{1}{5} - \frac{4}{3} + 3\right]$

$= \dfrac{56\pi}{15}$ cu units

QUESTION 2

a i) $3iw$

ii) $3iw + w$

or $w(1+3i)$

iii) $w - 3iw$

or $w(1-3i)$

b. i) $b^2 = a^2(1-e^2)$

$9 = 16(1-e^2)$

$e = \frac{\sqrt{7}}{4}$

ii) foci $(\pm ae, 0)$

$(\pm\sqrt{7}, 0)$

iii) directrices $x = \pm \frac{a}{e}$

$x = \pm \frac{16}{\sqrt{7}}$

c. differentiating

$$\frac{2x}{16} + \frac{2y\frac{dy}{dx}}{9} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-9x}{16y}$$

$\therefore$ gradient of tangent at $P(x_1, y_1)$ is

$$m_T = \frac{-9x_1}{16y_1}$$

$\therefore$ equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{-9x_1}{16y_1}(x - x_1)$$

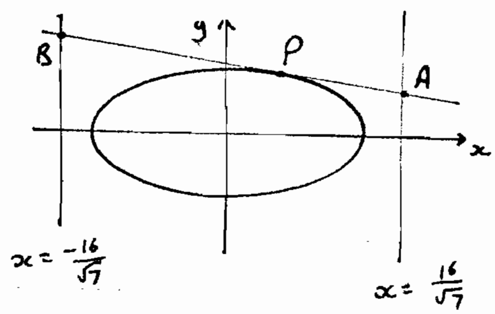
$$16yy_1 - 16y_1^2 = -9xx_1 + 9x_1^2$$

$$9xx_1 + 16yy_1 = 9x_1^2 + 16y_1^2 \quad (\div 144)$$

$$\frac{xx_1}{16} + \frac{yy_1}{9} = \frac{x_1^2}{16} + \frac{y_1^2}{9}$$

but (x_1, y_1) lies on $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\therefore \frac{xx_1}{4^2} + \frac{yy_1}{3^2} = 1$$

ii) 

to find A: sub $x = \frac{16}{\sqrt{7}}$ into tangent

$$\therefore \frac{\frac{16}{\sqrt{7}}x_1}{16} + \frac{yy_1}{9} = 1$$

$$y = \frac{9}{y_1}\left(1 - \frac{x_1}{\sqrt{7}}\right)$$

$$\therefore A\left(\frac{16}{\sqrt{7}}, \frac{9}{y_1}\left(1 - \frac{x_1}{\sqrt{7}}\right)\right)$$

to find B: sub $x = \frac{-16}{\sqrt{7}}$ into tangent

$$\frac{\frac{-16}{\sqrt{7}}x_1}{16} + \frac{yy_1}{9} = 1$$

$$y = \frac{9}{y_1}\left(1 + \frac{x_1}{\sqrt{7}}\right)$$

$$\therefore B\left(\frac{-16}{\sqrt{7}}, \frac{9}{y_1}\left(1 + \frac{x_1}{\sqrt{7}}\right)\right)$$

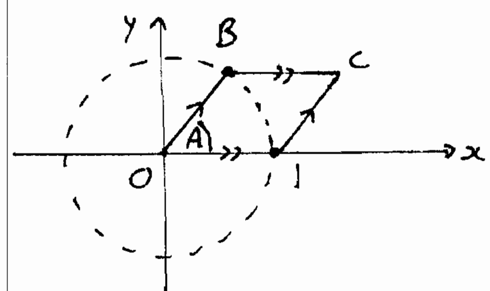
$$\therefore \text{Area} = \frac{h}{2}(a+b)$$

$$= \frac{1}{2}\cdot\frac{32}{\sqrt{7}}\left(\frac{9}{y_1}\left(1-\frac{x_1}{\sqrt{7}}\right)+\frac{9}{y_1}\left(1+\frac{x_1}{\sqrt{7}}\right)\right)$$

$$= \frac{144}{\sqrt{7}\,y_1}(1+1)$$

$$= \frac{288}{\sqrt{7}\,y_1} \text{ sq units}$$

d.

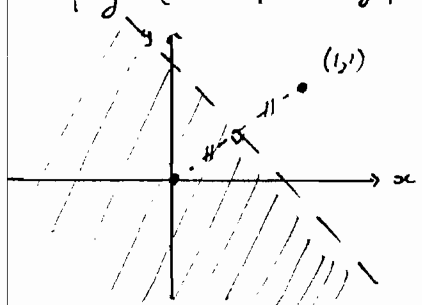


B represents z
C represents $z+1$
shape is a rhombus
diagonals bisect angles

$$\therefore \operatorname{Arg}(z+1) = \frac{A}{2}$$

QUESTION 3

. $|z-(1+i)| \geq |z|$



b. $|2-i| = \sqrt{5}$, $|2+i| = \sqrt{5}$

$$\therefore \left|\frac{(2-i)^8}{(2+i)^6}\right| = \frac{\sqrt{5}^8}{\sqrt{5}^6}$$

$$= 5$$

c. i) roots of $z^5 = -1$ are

$$z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$
$$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$$
$$z_3 = -1$$
$$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$
$$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$

ii) $\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = w$

$$\therefore \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} =$$
$$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^3$$
$$= w^3$$

$$\therefore \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$$
$$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^7$$
$$= w^7$$
$$= w^5 \cdot w^2 \qquad (w^5 = -1)$$
$$= -w^2$$

$$\therefore \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$$
$$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^9$$
$$= w^9$$
$$= w^5 \cdot w^4$$
$$= -w^4$$

iii) sum of roots of $z^5+1=0$ equals 0

$$\therefore -1 + w - w^2 + w^3 - w^4 = 0$$
or $$1 - w + w^2 - w^3 = -w^4$$

$$\therefore (1-w+w^2-w^3)^8 = (-w^4)^8$$

$$= w^{32}$$
$$= w^{30} \cdot w^{2}$$
$$= (w^{5})^{6} \cdot w^{2}$$
$$= (-1)^{6} \cdot w^{2}$$
$$= w^{2}$$
$$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

d. $\operatorname{Re}\left(z - \frac{1}{z}\right) = 0 \qquad z \neq 0$

$$\operatorname{Re}\left(x+iy - \frac{1}{x+iy}\right) = 0$$

$$\operatorname{Re}\left(x+iy - \frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right) = 0$$

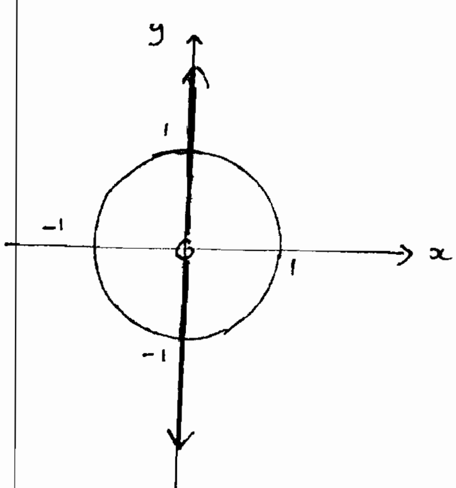
$$\operatorname{Re}\left(x+iy - \frac{x-iy}{x^2+y^2}\right) = 0$$

$$\therefore \quad x - \frac{x}{x^2+y^2} = 0$$

$$x(x^2+y^2) - x = 0$$
$$x(x^2+y^2-1) = 0$$

$\therefore \quad x = 0 \quad$ or $\quad x^2+y^2 = 1$

$(z \neq 0)$



SOLUTIONS MARCH 2002

QUESTION 1

a. $i^{2002} = (i^2)^{1001}$

$= (-1)^{1001}$

$= -1$ 1 mark

b. i) $\dfrac{2+2\sqrt{3}i}{1+i}$

$= \dfrac{2+2\sqrt{3}i}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{2-2i+2\sqrt{3}i+2\sqrt{3}}{2}$

$= (1+\sqrt{3}) + i(\sqrt{3}-1)$ 1 mark

ii) $\bar{z} = 2-2\sqrt{3}i$ 1 mark

iii) $|z| = \sqrt{2^2 + (2\sqrt{3})^2}$

$= 4$ 1 mark

iv) $\text{Arg}(z) = \dfrac{\pi}{3}$ 1 mark

v) $\dfrac{1}{z^4} = z^{-4}$

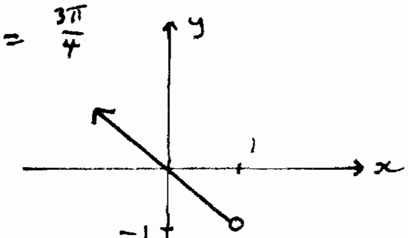
$= \left[4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{-4}$ must have argument $\frac{2\pi}{3}$

$= 4^{-4}\left(\cos\frac{-4\pi}{3} + i\sin\frac{-4\pi}{3}\right)$

$= \dfrac{1}{256}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ 2 marks

c. $\text{Arg}(z-(1-i)) = \dfrac{3\pi}{4}$

must go through origin
must have open circle



2 marks

d. i) $14 - \lambda > 0$ and $\lambda - 6 > 0$

$\lambda < 14$ and $\lambda > 6$

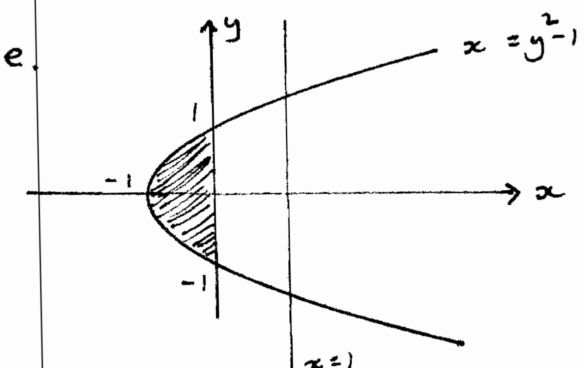
$\therefore \quad 6 < \lambda < 14, \quad \lambda \neq 10$ — 2 marks

ii) $\lambda - 6 > 14 - \lambda$ (plus part i)

$\lambda > 10$

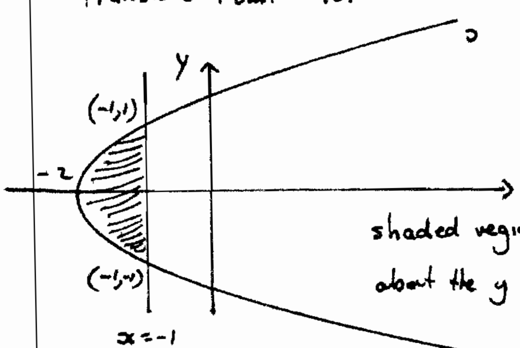
$\therefore \quad 10 < \lambda < 14$ — 1 mark

e.



Can use any other metho[d]

translate 1 unit left



shaded region is rotated about the y axis

$$\therefore \quad V = 2\pi \int_0^1 (y^2 - 2)^2 - (-1)^2 \, dy$$

$$= 2\pi \int_0^1 y^4 - 4y^2 + 3 \, dy$$

$$= 2\pi \left[\tfrac{1}{5} y^5 - \tfrac{4}{3} y^3 + 3y \right]_0^1$$

$$= 2\pi \left[\tfrac{1}{5} - \tfrac{4}{3} + 3 \right]$$

$$= \frac{56\pi}{15} \text{ cu units}$$

4 marks

QUESTION 2

a i) $3iw$ 1 mark

ii) $3iw + w$
or $w(1+3i)$ 1 mark

iii) $w - 3iw$
or $w(1-3i)$ 1 mark

b. i) $b^2 = a^2(1-e^2)$
$9 = 16(1-e^2)$

$e = \frac{\sqrt{7}}{4}$ 1 mark

ii) foci $(\pm ae, 0)$

$(\pm\sqrt{7}, 0)$ 1 mark

iii) directrices $x = \pm \frac{a}{e}$

$x = \pm \frac{16}{\sqrt{7}}$ 1 mark

c. differentiating
$$\frac{2x}{16} + \frac{2y\frac{dy}{dx}}{9} = 0$$
$$\therefore \frac{dy}{dx} = \frac{-9x}{16y}$$

$\therefore$ gradient of tangent at $P(x_1, y_1)$ is
$$m_T = \frac{-9x_1}{16y_1}$$

$\therefore$ equation of tangent
$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{-9x_1}{16y_1}(x - x_1)$$

$$16yy_1 - 16y_1^2 = -9xx_1 + 9x_1^2$$

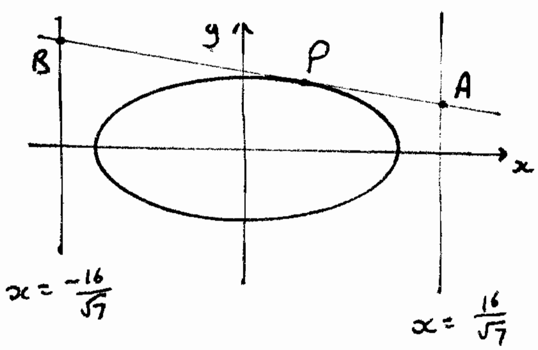
$$9xx_1 + 16yy_1 = 9x_1^2 + 16y_1^2 \quad (\div 144)$$

$$\frac{xx_1}{16} + \frac{yy_1}{9} = \frac{x_1^2}{16} + \frac{y_1^2}{9}$$

but (x_1, y_1) lies on $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$$\therefore \frac{xx_1}{4^2} + \frac{yy_1}{3^2} = 1$$

4 marks

ii) 

to find A: sub $x = \frac{16}{\sqrt{7}}$ into tangent

$$\therefore \frac{\frac{16}{\sqrt{7}}x_1}{16} + \frac{yy_1}{9} = 1$$

$$y = \frac{9}{y_1}\left(1 - \frac{x_1}{\sqrt{7}}\right)$$

$$\therefore A\left(\frac{16}{\sqrt{7}}, \frac{9}{y_1}\left(1 - \frac{x_1}{\sqrt{7}}\right)\right)$$

to find B: sub $x = \frac{-16}{\sqrt{7}}$ into tangent

$$\frac{\frac{-16}{\sqrt{7}}x_1}{16} + \frac{yy_1}{9} = 1$$

$$y = \frac{9}{y_1}\left(1 + \frac{x_1}{\sqrt{7}}\right)$$

$$\therefore B\left(\frac{-16}{\sqrt{7}}, \frac{9}{y_1}\left(1 + \frac{x_1}{\sqrt{7}}\right)\right)$$

$$\therefore \text{Area} = \frac{h}{2}(a+b)$$

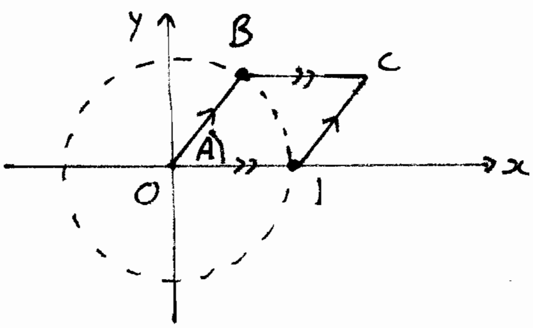
$$= \frac{1}{2} \cdot \frac{32}{\sqrt{7}} \left(\frac{9}{y_1}\left(1 - \frac{x_1}{\sqrt{7}}\right) + \frac{9}{y_1}\left(1 + \frac{x_1}{\sqrt{7}}\right)\right)$$

$$= \frac{144}{\sqrt{7}\, y_1} (1 + 1)$$

$$= \frac{288}{\sqrt{7}\, y_1} \quad \text{sq units}$$

4 marks

d.



can be done other ways need some attempt at justifying answer.

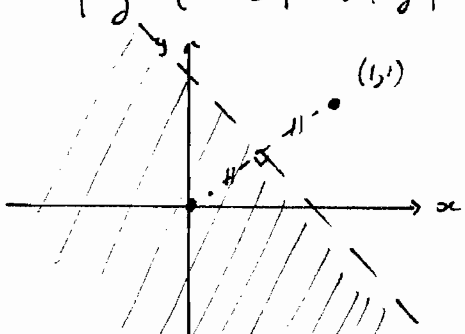
B represents z
C represents $z+1$
shape is a rhombus
diagonals bisect angles

$$\therefore \operatorname{Arg}(z+1) = \frac{A}{2}$$

2 marks

QUESTION 3

a. $|z - (1+i)| > |z|$



2 marks

b. $|2 - i| = \sqrt{5}, \quad |2+i| = \sqrt{5}$

$$\therefore \left| \frac{(2-i)^8}{(2+i)^6} \right| = \frac{\sqrt{5}^{\,8}}{\sqrt{5}^{\,6}}$$

$$= 5$$

3 marks

c) i) roots of $z^5 = -1$ are

roots can be in any form

$z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$z_3 = -1$

$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

2 marks

ii) $\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = w$

$\therefore \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} =$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^3$

$= w^3$

$\therefore \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^7$

$= w^7$

$= w^5 . w^2 \qquad (w^5 = -1)$

$= -w^2$

$\therefore \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^9$

$= w^9$

$= w^5 . w^4$

$= -w^4$

3 marks

iii) sum of roots of $z^5 + 1 = 0$ equals 0

$\therefore -1 + w - w^2 + w^3 - w^4 = 0$

or $1 - w + w^2 - w^3 = -w^4$

$\therefore (1 - w + w^2 - w^3)^8 = (-w^4)^8$

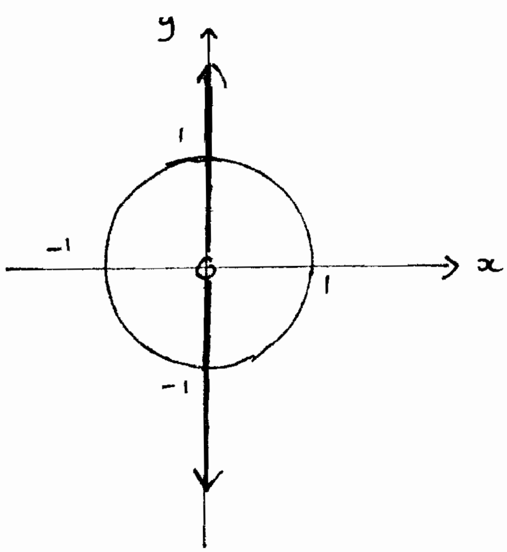
$$= w^{32}$$
$$= w^{30} . w^{2}$$
$$= (w^{5})^{6} . w^{2}$$
$$= (-1)^{6} . w^{2}$$
$$= w^{2}$$
$$= \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$$

give full marks for w^2

3 marks.

d. $Re\left(z - \frac{1}{z}\right) = 0 \qquad z \neq 0$

$$Re\left(x + iy - \frac{1}{x+iy}\right) = 0$$
$$Re\left(x + iy - \frac{1}{x+iy} + \frac{x-iy}{x-iy}\right) = 0$$
$$Re\left(x + iy - \frac{x-iy}{x^2+y^2}\right) = 0$$
$$\therefore \quad x - \frac{x}{x^2+y^2} = 0$$
$$x(x^2+y^2) - x = 0$$
$$x(x^2+y^2-1) = 0$$
$$\therefore \quad x = 0 \quad \text{or} \quad x^2+y^2 = 1$$
$$(z \neq 0)$$



origin needs to be excluded.

3 marks