SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 ASSESSMENT TASK

EXTENSION 2 MATHEMATICS

MARCH 2005

Instructions

* Attempt all questions.
* Answers to be written on the paper provided.
* Start each question on a new page.
* Marks may not be awarded for careless or badly arranged working.
* Indicated marks are a guide and may be changed slightly if necessary.
* These questions must be handed in attached to the top of your solutions.

Q1	Q2	Q3	TOTAL
/16	/17	/18	

QUESTION 1

a) Find $\left|(3-4i)^n\right|$ **(2)**

b) (i) On an Argand diagram shade in the region determined by the inequalities

$2 \le \text{Im}(z) \le 4$ and $\dfrac{\pi}{6} \le \arg(z) \le \dfrac{\pi}{4}$. **(3)**

(ii) Let z_0 be the complex number of maximum modulus satisfying the inequalities in (i). Express z_0 in the form $x+iy$. **(1)**

c) Find pairs of integers x and y which satisfy the condition

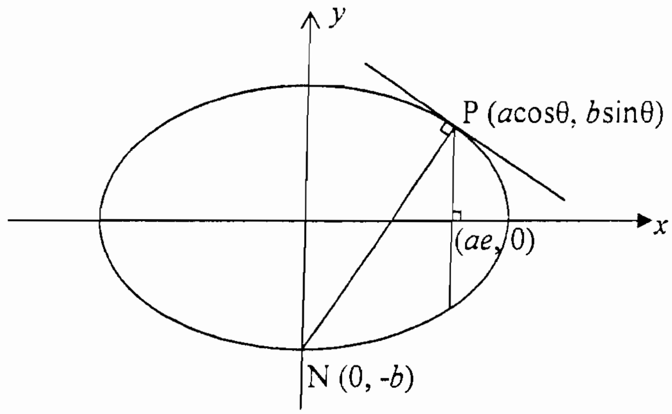
$(x+iy)^2 = -3-4i$. **(3)**

d) If $z = \cos\theta + i\sin\theta$ use De Moivres' Theorem or otherwise to simplify

$z^4 + \dfrac{1}{z^4}$. **(2)**

Question 1 (Cont)

e)



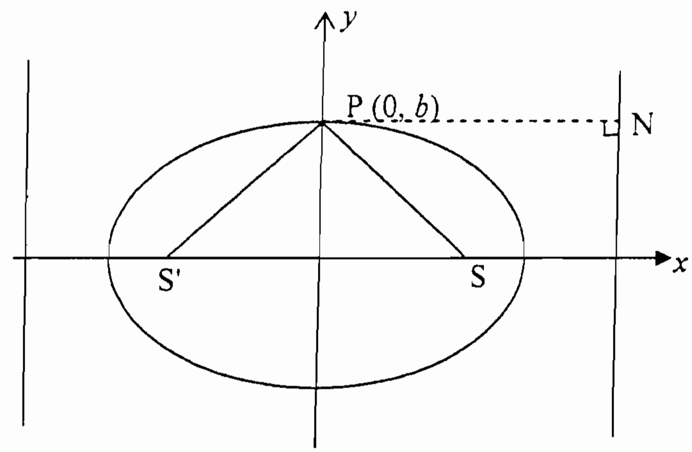
The chord through the focus $(ae,0)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at right angles to the $x-axis$ meets the ellipse at P $(a\cos\theta,\, b\sin\theta)$. The normal at P passes through the point $(0,-b)$.

(i) Show that $\cos\theta = e$ and $\sin\theta = \sqrt{1-e^2}$. **(2)**

(ii) Given the equation of the normal at P is $ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$, show that the condition for it to pass through (0, -b) is $e^4+e^2-1=0$.
(You may show instead that $e^6-2e^2+1=0$, which is another version of the above condition) **(3)**

QUESTION 2

a)



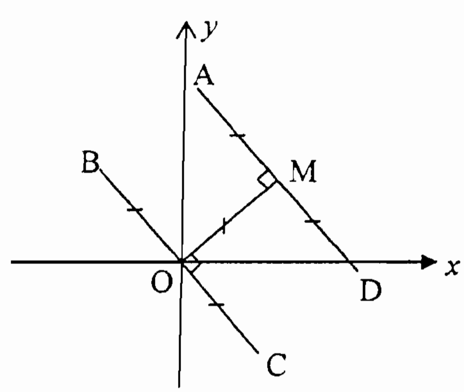
If $P(0,b)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where S and S' are the focii and N is a point on the directrix.

(i) Write down the value of the ratio $\frac{PS}{PN}$. **(1)**

(ii) Hence or otherwise show that $PS+PS'=2a$. **(2)**

(iii) Explain why the perimeter of the triangle PSS' is always less than $4a$ units. **(2)**

b)



In the diagram $AM = MD = OM = OB = OC$ and $AD \perp OM \perp BC$. O is the origin.

If M represents the complex number z

(i) Which point represents the complex number iz? **(1)**

(ii) Find, in terms of z, the complex number represented by the point D. **(2)**

Question 2 (Cont)

c) (i) Sketch the curve $y = (x-1)^2$ and shade the region bounded by the curve, the x axis and the line $x = 2$. **(1)**

(ii) The region in (i) is rotated about the line $y = -1$. Find the volume of the solid formed by this rotation. **(3)**

d) (i) Sketch the locus of the complex number z if $|z-1| = 1$. **(1)**

(ii) Let z be a complex number which satisfies the locus in (i) and let $\arg(z) = \theta$. Explain with the aid of your graph or otherwise why $\arg(z-1) = 2\theta$. **(2)**

(iii) Find $\arg(z^2 - 3z + 2)$ in terms of θ. **(2)**

QUESTION 3

a) (i) Express $z = \sqrt{3} + i$ in modulus/argument form. **(2)**

(ii) Show that z is a complex solution of the equation $x^7 + 64x = 0$. **(2)**

b) If $z = x + iy$

(i) Write $\frac{1}{z}$ as a complex number. **(1)**

(ii) Hence find the equations of the locus of z if $\text{Re}(z - \frac{1}{z}) = 0$. **(2)**

c) If the roots of the equation $z^8 = 1$ are $1, w, w^2, w^3, w^4, w^5, w^6, w^7$ where w is the complex root with the smallest positive argument

(i) Find w^3 in mod-arg form. **(1)**

(ii) Evaluate $w^2 + w^4 + w^6$ giving a reason. **(2)**

Question 3 (Cont).

d) (i) Differentiate $\frac{x^2}{25}+\frac{y^2}{9}=1$ implicitly. **(2)**

(ii) Derive the equation of the tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ at (x_1, y_1). **(2)**

(iii) Write down the equations of the directrices. **(1)**

(iv) If $x_1 > 0$ and $y_1 > 0$ find the values of x_1 so that the tangent at (x_1, y_1) intersects the nearest directrix below the x axis. **(3)**

End of Exam

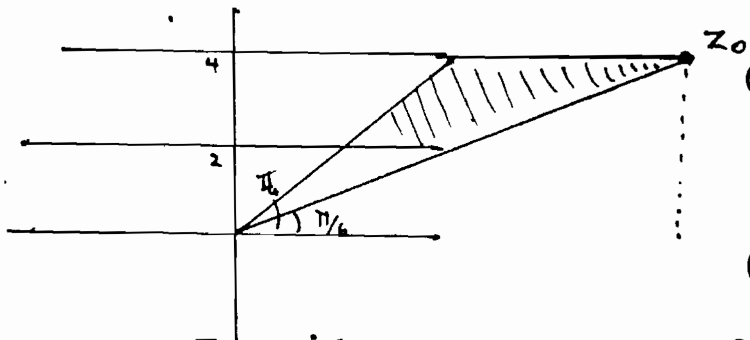
(1)

a) $|3-4i| = \sqrt{3^2+4^2}$

$= 5$ — (1)

$|(3-4i)^n| = 5^n$ — (1)

b)



(1) - understanding Im

(1) - understanding arg

(1) - region.

$z_0 = 4\sqrt{3}+4i$ (1)

(ii) $(x+iy)^2 = x^2-y^2+2ixy$

Now $x^2-y^2 = -3$

$2xy = -4$ — (1) equating

$xy = -2$

$x=-1, y=2$ or $x=1, y=-2$ (1) each answer.

(d) $z^4 = \cos 4\theta + i\sin 4\theta$ — (1)

$z^{-4} = \cos(-4\theta) + i\sin(-4\theta)$

$= \cos 4\theta - i\sin 4\theta$ — (1)

$\therefore z^4+z^{-4} = 2\cos 4\theta$ — (1)

e) i) comparing x-value of P and S

$ae = a\cos\theta$

$\therefore \cos\theta = e$ — (1)

Now $\sin^2\theta+\cos^2\theta = 1$

$\therefore \sin^2\theta = 1-e^2$

$\sin\theta = \sqrt{1-e^2}$ — (1)

ii) at $(0,-b)$

$b^2\cos\theta = (a^2-b^2)\cos\theta\sin\theta$

now $\cos\theta = e$ and $\sin\theta = \sqrt{1-e^2}$

$\therefore b^2 e = (a^2-b^2)e\sqrt{1-e^2}$

for the ellipse $b^2 = a^2(1-e^2)$ — (1)

$a^2(1-e^2)e = [a^2 - a^2(1-e^2)]e\sqrt{\ }$

$a^2e(1-e^2) = a^2e^3\sqrt{1-e^2}$

since $0<e<1$ for ellipse

$1-e^2 = e^2\sqrt{1-e^2}$

$\therefore (1-e^2)^2 = e^4(1-e^2)$

$1-2e^2+e^4 = e^4-e^6$

$\therefore e^6-2e^2+1=0$

(2) fo[r] wor[king]

Question 2

a) i) e — ①

ii) Now $PS = e\,PN$

and $PS' = e\,PN'$ (mark on diagram) — ①

$\therefore PS + PS' = e[PN + PN']$

$= e\left[\frac{a}{e} + \frac{a}{e}\right]$ — ①

$= 2a$

iii) Perimeter $PS'S = 2a + 2ae$ — ①

but $e < 1$ for ellipse — ①

$\therefore PS'S < 4a$

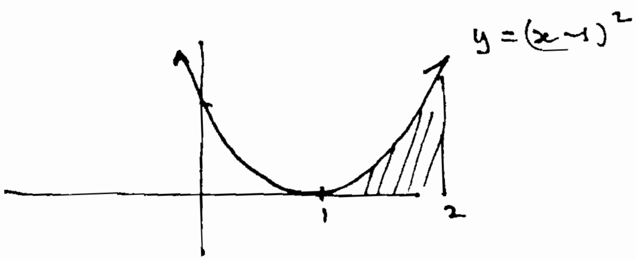
b) i) B ①

ii) $\vec{OD} = \vec{OM} + \vec{MD}$ — ① addition

$= z + (-iz)$ — ①

$= z - iz$

c i



①

ii) Volume of slice $= \pi(R^2 - r^2)\Delta x$

$= \pi((y+1)^2 - 1^2)\Delta x$

$= \pi((y+1-1)(y+1+1))\Delta x$

$= \pi\, y\cdot(y+2)\Delta x$

but $y = (x-1)^2$

$= \pi (x-1)^2((x-1)^2+2)\Delta x$

$= \pi[(x-1)^4 - 2(x-1)^2]\Delta x$

Volume $= \lim_{\Delta x \to 0} \sum_{1}^{2} \pi[(x-1)^4 - 2(x-1)^2]\Delta x$

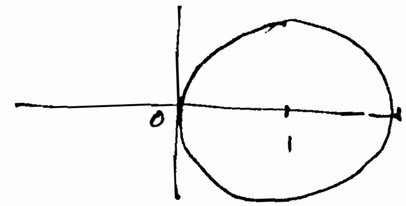
$= \pi \int_1^2 (x-1)^4 + 2(x-1)^2\,dx$ ① idea of annulus ① expression

$= \pi\left[\frac{(x-1)^5}{5} + \frac{2(x-1)^3}{3}\right]_1^2$

$= \pi\left[\frac{1}{5} + \frac{2}{3} - 0\right]$ either ①

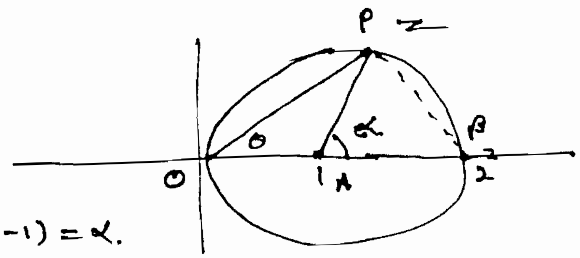
$= \frac{13\pi}{15}\,u^3$

d) i)



— ①

ii)



Let $\arg(z-1) = \alpha$.

ΔOPA is isosceles — ①

$\therefore \angle OPA = \theta$. — ① explanation

$\therefore \alpha = 2\theta$

$\therefore \arg(z-1) = 2\theta$

iii) $\text{Arg}(z^2 - 3z + 2) = \text{Arg}(z-1)(z-2)$

$= \text{Arg}(z-1) + \text{Arg}(z-2)$ — ①

$= 2\theta + \beta$ (on diagram)

$= 2\theta + \theta + \pi/2$ (exterior ∠ of Δ and angle in semi-circle)

$= 3\theta + \pi/2$. — ①

Question 3

a) i) $z = 2\,\text{cis}\frac{\pi}{6}$ → ① mod, ① arg.

ii) If z is a solution then

$$\left(2\,\text{cis}\frac{\pi}{6}\right)^7 + 64\left(2\,\text{cis}\frac{\pi}{6}\right) = 0$$ ①

$$\text{LHS} = 128\,\text{cis}\frac{7\pi}{6} + 128\,\text{cis}\frac{\pi}{6}$$
$$= -128\,\text{cis}\frac{\pi}{6} + 128\,\text{cis}\frac{\pi}{6}$$
$$= 0$$
$$= \text{RHS}$$

① simplification

b) i) $\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$ — ①

ii) $z - \frac{1}{z} = x+iy - \frac{x-iy}{x^2+y^2}$

$= \frac{(x^2+y^2)(x+iy) - x + iy}{x^2+y^2}$

$Re\left(z-\frac{1}{z}\right) = \frac{x(x^2+y^2) - x}{x^2+y^2}$

$\therefore \frac{x(x^2+y^2)-x}{x^2+y^2} = 0$

$\therefore x(x^2+y^2) - x = 0$

$x(x^2+y^2-1) = 0$

$\therefore x = 0$ or $x^2+y^2 = 1$

② both with restriction $x, y \neq 0$

①

c) i) $w = \text{cis}\frac{\pi}{4}$

$\therefore w^3 = \text{cis}\frac{3\pi}{4}$ — ①

ii) $1, w^2, w^4, w^6$ are the roots of the equation $z^4 = 1$

$\therefore 1 + w^2 + w^4 + w^6 = 0$ (sum of roots)

$\therefore w^2 + w^4 + w^6 = -1$ — ① answer ① reason.

d) i) $\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$ — ①

$\frac{dy}{dx} = \frac{-2x}{25} \times \frac{9}{2y}$

$= \frac{-9x}{25y}$ — ①

ii) at (x_1, y_1) $\frac{dy}{dx} = \frac{-9x_1}{25y_1}$

$\therefore$ equat $y - y_1 = \frac{-9x_1}{25y_1}(x - x_1)$

$\frac{yy_1}{9} - \frac{y_1^2}{9} = -\frac{xx_1}{25} + \frac{x_1^2}{25}$

$\therefore \frac{xx_1}{25} + \frac{yy_1}{9} = \frac{x_1^2}{25} + \frac{y_1^2}{9}$ (since x_1, y_1 lies on ellipse

$\therefore \frac{xx_1}{25} + \frac{yy_1}{9} = 1$

wit ex

iii) $x = \pm 25/4$ — ①

iv) when $x = 25/4$

$\frac{25}{4}\cdot\frac{x_1}{25} + \frac{yy_1}{9} = 1$

$\frac{x_1}{4} + \frac{yy_1}{9} = 1$

$\therefore y = \frac{9(4-x_1)}{4y_1}$ — ①

Now $y < 0$

$\therefore \frac{9(4-x_1)}{4y_1} < 0$

but $y_1 > 0$

$\therefore 4 - x_1 <$

$x_1 > 4$ — ①

$\therefore 4 < x_1 < 5.$ — ①