Name:________________________________ Class:______________________

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 ASSESSMENT TASK

EXTENSION 2

MATHEMATICS

MARCH 2006

Instructions

- Attempt all questions
- Answers to be written on the paper provided
- Start each question on a new page
- Marks may not be awarded for careless or badly arranged working
- Indicated marks are a guide and may be changed slightly if necessary
- These questions must be handed in attached to the top of your solutions.

Q1	Q2	Q3	Total
/16	/17	/17	

Question 1 (16 marks)

a) If $z = -1 + i$ find, (4)

(i) $\bar{z}$

(ii) $|z|$

(iii) $\arg(iz)$

b) i. Express $z = -\sqrt{3} - i$ in modulus – argument form (3)

ii. Hence write z^{12} in the form $x + iy$, where x and y are real

c) Find all complex numbers z, such that $z^3 = 64\,i$ (3)

d) Given that a and b are real numbers, find a and b if, $\dfrac{5+2i}{a+bi} = 1+i$ (3)

e) On an argand diagram shade the region containing all points representing complex numbers z such that $2 \le \text{Re}(z) \le 5 \quad \textit{and} \quad -2 \le \text{Im}(z) \le 4$

Question 2 (17 marks) **(Start a new page)**

a) Evaluate $\int_0^1 x\sqrt{1-x}\, dx$ using a suitable substitution. (3)

b) Consider the ellipse $3x^2 + 4y^2 = 12$ (6)

i. Determine the eccentricity of the ellipse

ii. Find the coordinates of the foci S and S^1 and also the equation of of the directrices.

iii. Sketch the ellipse showing all important information.

c)

i. Show that the equation of the tangent to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ at the point P $(3\cos\theta, 2\sin\theta)$ is given by $\dfrac{x\cos\theta}{3} + \dfrac{y\sin\theta}{2} = 1$ (3)

The ellipse above cuts the y axis at the points A and B (5)
The tangents to the ellipse at A and B meet the tangent to the ellipse at $P(3\cos\theta, 2\sin\theta)$ at the points C and D respectively.

ii. Draw a neat diagram showing the positions of P, A, B, C and D.

iii. Show that AC $\times$ BD = 9

Start a new page

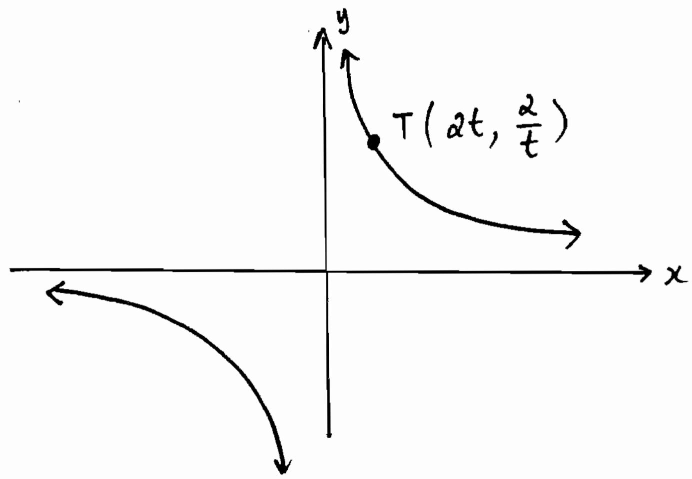
Question 3 (17 marks)

a) Show that the locus specified by $3|z-(4+4i)| = |z-(12+12i)|$ is a circle, (4)

Write down its radius and the coordinates of its centre

b) Find the equation of the tangent to the curve $x^2 + xy^2 - 6y = 0$ at the point $(2,1)$ (4)

c) Consider the diagram below



i. Show that the tangent to the hyperbola $xy = 4$ at the point T $(2t, \frac{2}{t})$ has equation $x + t^2 y = 4t$. (2)

ii. This tangent cuts the x – axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2 x - y = 4t^3$. (2)

iii. This line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates M $(2t, -2t^3)$. (2)

iv. Find the equation of the locus of M as T moves on the rectangular hyperbola, stating any restrictions that may apply. (3)

End of Test

March 2006 Ext II

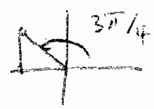
Solutions

Question 1

a) i. $\overline{z} = -1 - i$ ✓

ii. $\sqrt{2}$ ✓

iii. $\arg(z) = \frac{3\pi}{4}$

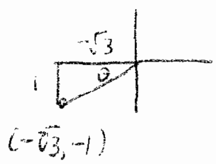


$\arg(iz) = \pi/2 + 3\pi/4$

$= 5\pi/4$ ✓

$= -\frac{3\pi}{4}$ ✓

b) i. $z = -\sqrt{3} - i$



$\angle = -\frac{5\pi}{6}$

$|z| = 2$

$\therefore\ z = 2\,\text{cis}\left(-\frac{5\pi}{6}\right)$ ✓

ii. $z^{12} = 2^{12}\,\text{cis}\left(12 \times -\frac{5\pi}{6}\right)$ ✓

$= 4096(1 + i0)$

$= 4096$ ✓ ie 2^{12}

c) $z^3 = 64i$

$z^3 - 64i = 0$

$z^3 + (4i)^3 = 0$

$(z + 4i)(z^2 - 4iz - 16) = 0$ ✓

$z = -4i$ ✓, $z = 2i \pm 2\sqrt{3}$

d) $5 + 2i = (1+i)(a+bi)$ ✓

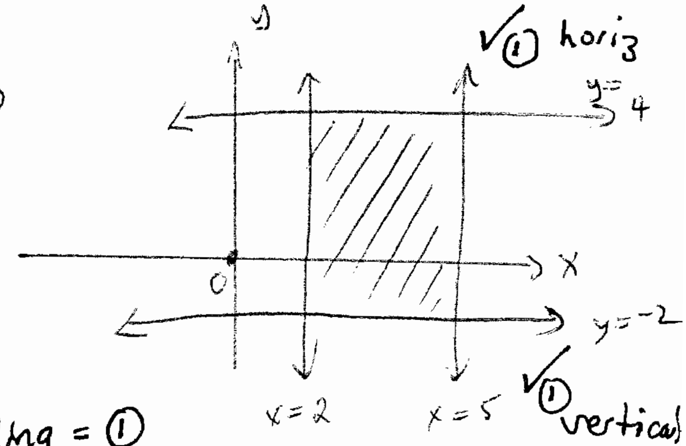
$= a + bi + ai - b$

$\therefore\ a - b = 5$ ✓

$a + b = 2$

$2a = 7$

e)



shading = ①

Question 2

a) $u = 1 - x$ $\qquad x = 1 - u$

$\frac{du}{dx} = -1$ $\qquad x = 1 \quad u = 0$

$\qquad x = 0 \quad u = 1$

$-du = dx$ ①

$\therefore\ \int_1^0 (1-u)\sqrt{u}\,.\,-du$ ①

$-\int_1^0 \sqrt{u} - u^{3/2}\,du$

$= \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$= \left(2/3 - 2/5\right) - (0)$

$= \frac{4}{15}$ ①

b) $\frac{x^2}{4}+\frac{y^2}{3}=1$ ✓

$a=2 \quad b=\sqrt{3}$

$\therefore\ b^2=a^2(1-e^2)$

$3/4=1-e^2$

$e^2=\frac{1}{4}$ ✓

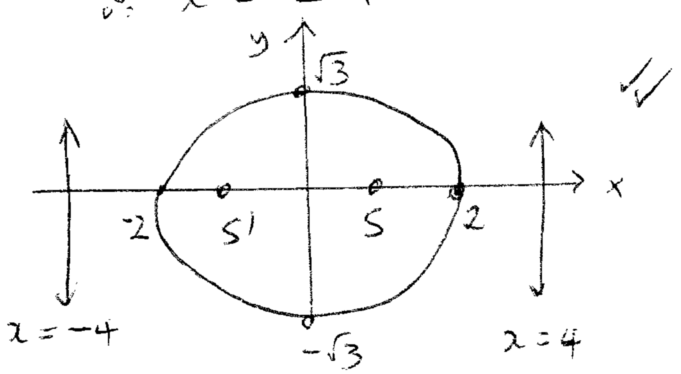
$e=\frac{1}{2}$

$S(ae,0) \quad S'(-ae,0)$

$S=(1,0) \quad S'(-1,0)$ ✓

directrices $x=\pm a/e$

$\therefore\ x=\pm 4$ ✓



✓✓

c) $\frac{2x}{9}+\frac{2y}{4}\cdot\frac{dy}{dx}=0$ ✓

$\frac{dy}{dx}=-\frac{2x}{9}\times\frac{4}{2y}$

$=-\frac{4}{9}\frac{x}{y}$

$M_T=-\frac{4\cdot 3\cos\theta}{9\cdot 2\sin\theta}$

$=\frac{-2\cos\theta}{3\sin\theta}$ ✓

✓

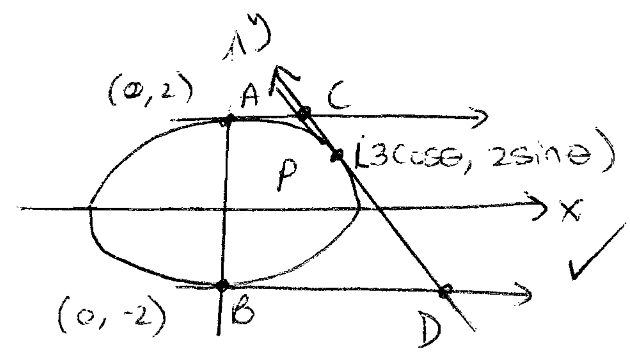
$y-2\sin\theta=\frac{-2\cos\theta}{3\sin\theta}(x-3\cos\theta)$

$3\sin\theta\, y-6\sin^2\theta=-2\cos\theta\, x+6\cos^2\theta$

$2\cos\theta\, x+3\sin\theta\, y=6$

$\therefore\ \frac{\cos\theta\, x}{3}+\frac{\sin\theta\, y}{2}=1$

d)



✓

tangent at A $y=2$

at B $y=-2$

tangent at P

$\frac{x\cos\theta}{3}+\frac{y\sin\theta}{2}=1$ (1)

Finding C sub $y=2$ into (1)

$\frac{x\cos\theta}{3}+\sin\theta=1$

$\frac{x\cos\theta}{3}=1-\sin\theta$

✓ $x=\frac{3(1-\sin\theta)}{\cos\theta}$

$C=\left[\frac{3(1-\sin\theta)}{\cos\theta}, 2\right]$

Finding D sub $y=-2$ into (1)

✓ $\frac{x\cos\theta}{3}-\sin\theta=1$

$D=\left[\frac{3(1+\sin\theta)}{\cos\theta}, -2\right]$

$$KC \times BD = \frac{3(1-\sin\theta)}{\cos\theta} \times \frac{3(1+\sin\theta)}{\cos\theta}$$

$$= 9\frac{(1-\sin^2\theta)}{\cos^2\theta}$$

$$= 9$$

Question 3

i) $3|x+iy-4-4i| = |x+iy-12-12i|$

$3|(x-4)+(y-4)i| = |(x-12)+(y-12)i|$

$9[(x-4)^2+(y-4)^2] = (x-12)^2+(y-12)^2$

$9[x^2-8x+16+y^2-8y+16] = x^2-24x+144$
$+y^2-24y+144$

$8x^2-48x+8y^2-48y=0$

$x^2-6x+y^2-6y=0$

$(x-3)^2+(y-3)^2=18$

$\therefore$ centre $(3,3)$ radius $= 3\sqrt{2}$.

b) Implicit diff

$\cdots + 1.y^2 + x.2y.\frac{dy}{dx} - 6\frac{dy}{dx} = 0$ (1)

$2x + y^2 + \frac{dy}{dx}(2xy-6) = 0$

$\frac{dy}{dx} = \frac{-2x-y^2}{2xy-6}$ (1)

$\therefore$ at $(2,1)$ $M_T = \frac{-2(2)-1}{2(2)(1)-6}$

$= \frac{5}{2}$ (1)

$\therefore$ eq tangent

$y-1 = 5/2(x-2)$ $\leftarrow$ (1)

c) $y = 4/x$ $y' = \frac{-4}{x^2}$ at $x = 2t$

i. $M_T = \frac{-4}{4t^2}$

$= -1/t^2$ (1)

$\therefore$ eq $y - \frac{2}{t} = -\frac{1}{t^2}(x-2t)$ (1)

$t^2y - 2t = -x + 2t$

$\therefore x + t^2y = 4t$ QED

ii. x-int $y=0$ $x = 4t$ $\therefore Q(4t,0)$

$M\perp = t^2$ (1)

eq: $y - 0 = t^2(x-4t)$

$y = t^2x - 4t^3$

$\therefore t^2x - y = 4t^3$

iii. Solve $t^2x - y = 4t^3$ and $xy = 4$ simultaneously

$y = t^2x - 4t^3$ & $xy = 4$

$\therefore x(t^2x - 4t^3) = 4$

$t^2x^2 - 4t^3x - 4 = 0$ (1)

as the 2 x values for this give R & S let them be α & β

$\alpha + \beta = \frac{4t^3}{t^2} = 4t$

$\therefore$ midpt $\frac{\alpha+\beta}{2} = 2t$ (1)

and y value $= -2t^3$

iv. Locus of m

$x = 2t \longrightarrow \frac{x}{2} = t$

Sub into $y = -2t^3$

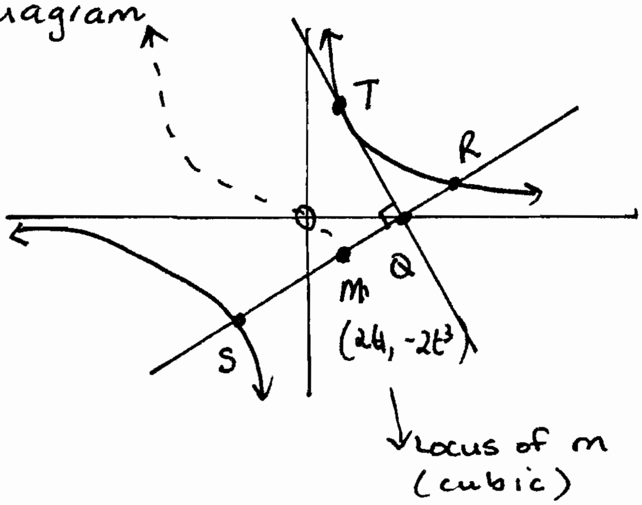
$$y = -2\left(\frac{x}{2}\right)^3$$

$$y = -2 . \frac{x^3}{8}$$

$$y = \frac{x^3}{-4}$$

$$-4y = x^3 \qquad 1$$

diagram



Locus of m moves on the cubic equation in the 2nd & 4th quadrants 1 1

restriction → excluding the origin as $t \neq 0$.

Alternative Sol^n to Q3 (c) iii

$$t^2x^2 - 4t^3x - 4 = 0$$

$$x = \frac{4t^3 \pm 4t\sqrt{t^4+1}}{2t^2}$$

$$= 2t^{\cancel{2}} \pm \frac{2}{t}\sqrt{t^4+1}$$

$$R\left[2t + \frac{2}{t}\sqrt{t^4+1} \; , \; -2t^3 + 2t\sqrt{t^4+1}\right]$$

$$S\left[2t - \frac{2}{t}\sqrt{t^4+1} \; , \; -2t^3 - 2t\sqrt{t^4+1}\right]$$

$$\text{Midpt } M = \left[\frac{4t}{2} \; , \; \frac{-4t^3}{2}\right]$$

$$= (2t, -2t^3)$$