Your Name: ………………………………………. **Teacher:** ……………………..

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 EXTENSION 2 MATHEMATICS ASSESSMENT TASK

MARCH 2007

Time allowed: 70 mins

Instructions

- Attempt all questions.
- Answers to be written on the paper provided.
- Start each question on a new page.
- Marks may not be awarded for careless or badly arranged working.
- Indicated marks are a guide and may be changed slightly if necessary during the marking process.
- **These questions must be handed in attached to the top of your solutions.**

Q1	Q2	Q3	TOTAL
/17	/16	/17	/50

QUESTION 1

(a) Suppose that $z = 2 - i$. **2**

Express the following in the form $x + iy$ where x and y are real numbers:

i) $\overline{(iz)}$

ii) $\dfrac{1}{z}$

(b) i) Express $-1 - i\sqrt{3}$ in modulus argument form. **4**

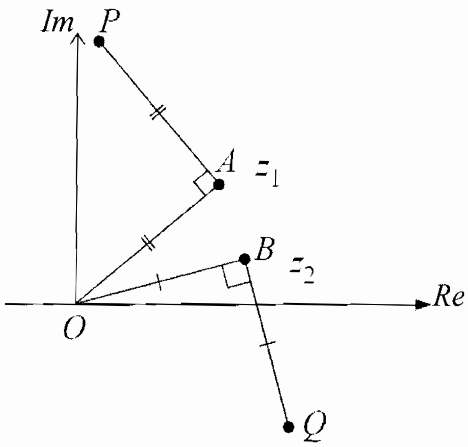
ii) Hence evaluate $\left(-1 - i\sqrt{3}\right)^3$

(c) i) Simplify $(-2i)^3$. **4**

ii) Hence or otherwise find all complex numbers, z, such that $z^3 = 8i$. Express your answers in the form $x + iy$.

(d) Sketch the region where the inequalities $|z-3+i| \leq 5$ and $|z+1| \leq |z-1|$ both hold. **3**

(e) **4**



The points A and B in the complex plane correspond to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1+i)z_1$.

(ii) Find a similar expression for the complex number corresponding to Q.

(iii) Let M be the midpoint of PQ. Give an expression in terms of z_1 and z_2 for the complex number that corresponds to M.

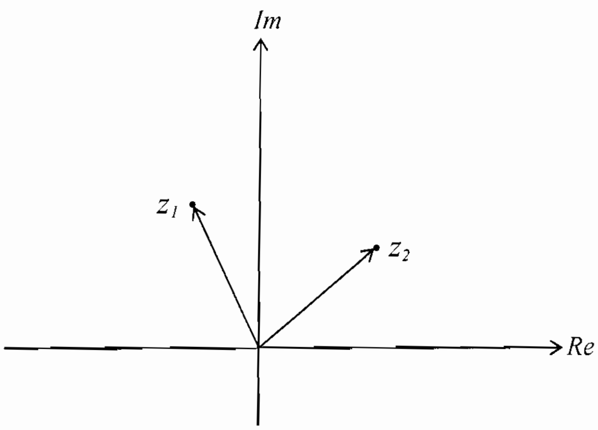
QUESTION 2 (Begin on a new page)

(a) If $1 - 3i$ is a root of the equation $2z^3 - 3z^2 + 18z + 10 = 0$ **2**

(i) Explain why $1 + 3i$ is also a root.

(ii) Find all roots of the equation.

(b) z_1 and z_2 are complex numbers such that $\dfrac{z_1 + z_2}{z_1 - z_2} = 2i$. **6**



(i) Copy the diagram onto your answer page.

On the diagram show the vectors $z_1 + z_2$ and $z_1 - z_2$.

(ii) Show that $|z_1| = |z_2|$.

(iii) If the angle between the vectors z_1 and z_2 is α, show that $\tan\dfrac{\alpha}{2} = \dfrac{1}{2}$.

(c) (i) Solve $z^5 = -1$ over the complex field showing the roots on an Argand diagram. **2**

(ii) If ω is the complex root of $z^5 = -1$ with the smallest positive argument, **3**

show that the other complex roots are $-\omega^2, \omega^3$ and $-\omega^4$.

(iii) Using ω as in part (ii), simplify $\left(1 - \omega + \omega^2 - \omega^3\right)^8$. **3**

Paper continues next page . . .

QUESTION 3 (Begin on a new page)

(a) Sketch the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. State the eccentricity and show clearly the coordinates of the foci and the equations of the directrices. **4**

(b) The point C (x_0, y_0) lies outside the ellipse in (a). Give a full geometrical description of the locus of the point C if the chord of contact from C always passes through the point (2, 2). **3**

(c) Prove that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\mathrm{P}(x_1, y_1)$ is $\frac{x_1 y}{a^2} - \frac{x y_1}{b^2} = \frac{x_1 y_1}{a^2} - \frac{x_1 y_1}{b^2}$. **4**

(d) The tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\mathrm{P}(x_1, y_1)$ cuts the y axis at A while the normal at P cuts the y axis at B. If S is a focus of the ellipse, show that $\angle \mathrm{ASB} = 90°$. (You do *not* have to derive the equation of the tangent.) **5**

(e) Deduce the geometrical relationship between the points A, P, S and B in part (d). **1**

End of Paper

QUESTION 1

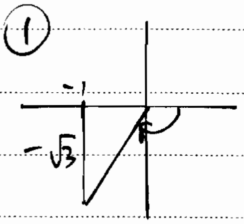
a) i) $\overline{iz} = \overline{2i - i^2}$

$= 1 - 2i$ (1)

ii) $\frac{1}{z} = \frac{1}{2-i} \times \frac{2+i}{2+i}$

$= \frac{2+i}{5}$ (1)

b) i) $-1 - i\sqrt{3}$



Modulus $= 2$

$\arg(-1-i\sqrt{3}) = -\frac{2\pi}{3}$

$\therefore -1 - i\sqrt{3} = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ (2)

(ii) $(-1-i\sqrt{3})^3 = \left(2\,\text{cis}\,\frac{-2\pi}{3}\right)^3$

$= 8\,\text{cis}\,-2\pi$

$= 8\,\text{cis}\,0$

$= 8$ (2)

c) i) $(-2i)^3 = -8i^3$

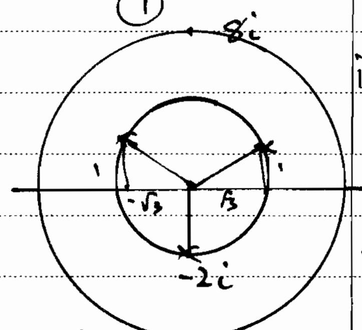
$= 8i$ (1)

ii) $z^3 = 8i$

Roots are

$-2i,\ \sqrt{3}+i,\ -\sqrt{3}+i$

(3)

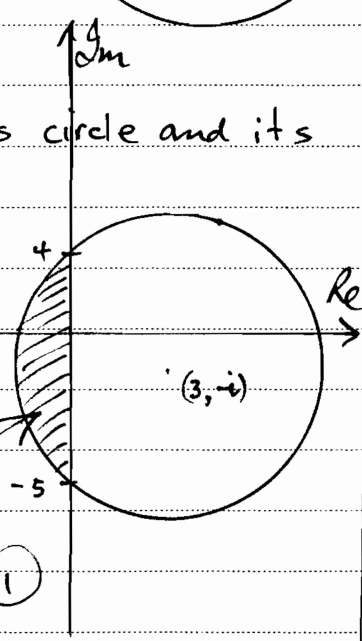


d)

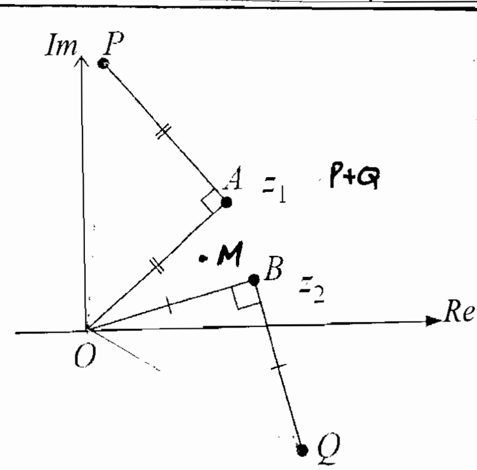
$|z-(3-i)| \leq 5$ is circle and its interior. (1)

$|z+1| \leq |z-1|$ is everything on and to the left of the Im axis (1)

Required region. (1)



(e)



i) $\overrightarrow{AP}$ represents the complex number iz_1 and P represents the complex number obtained by adding $\overrightarrow{OA}$ and $\overrightarrow{AP}$ (1)

ie $z_1 + iz_1 = (1+i)z_1$

ii) Q represents the complex number $z_2 - iz_2$

ie $(1-i)z_2$ (1)

iii) M is midpoint of vector to P+Q ie $\frac{P+Q}{2}$

$\therefore M: \left(\frac{z_1+z_2}{2} + i\,\frac{z_1-z_2}{2}\right)$ (2)

QUESTION 2

a) i) Since coefficients are real, roots occur in conjugate pairs.

OR

$1+3i$ could be shown to be a root by substitution. (1)

ii) Known factors of $2z^3 - 3z^2 + 18z + 10 = 0$ are $(z-1-3i)(z-1+3i)$

ie $(z^2 - 2z + 10)$ is a factor

$\therefore$ Third factor is $(2z+1)$

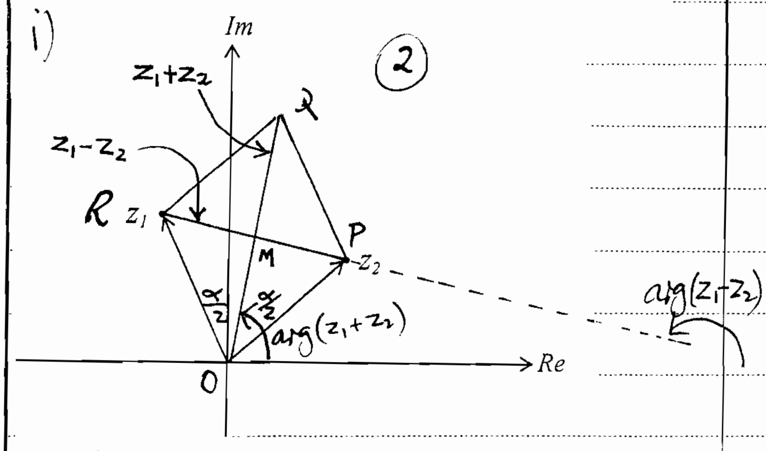
$\therefore$ Final root is $-\frac{1}{2}$ (1)

QUESTION 2 (cont.)

(b)

z_1 and z_2 are complex numbers such that $\frac{z_1 + z_2}{z_1 - z_2} = 2i$

i) (2)



ii) Since $\frac{z_1+z_2}{z_1-z_2} = 2i$

$\arg(z_1+z_2) - \arg(z_1-z_2) = \arg 2i = \frac{\pi}{2}$

ie $\angle OMP = 90°$

$\therefore$ OPQR is a rhombus (2)

$\therefore$ $|z_1| = |z_2|$ (all sides equal)

iii) Since $\frac{Z_1+Z_2}{Z_1-Z_2} = 2i$,

$\left|\frac{z_1+z_2}{z_1-z_2}\right| = |2i| = 2$

ie $|z_1+z_2| = 2|z_1-z_2|$

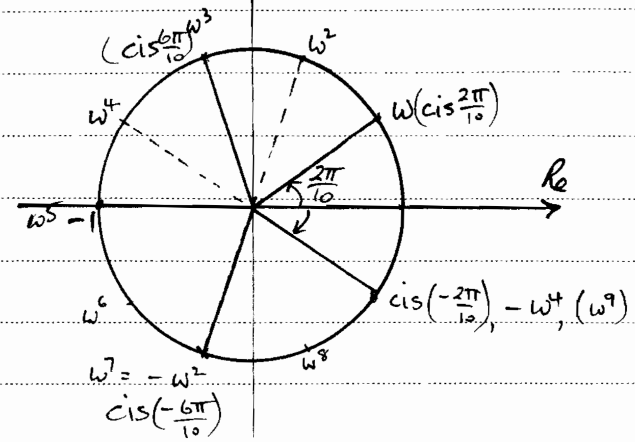
ie. $OQ = 2\,RP$

$\therefore$ $MP = \frac{1}{2}OM$ (2)

$\therefore$ $\tan\frac{\alpha}{2} = \frac{MP}{OM} = \frac{1}{2}$.

(c)(i) $z^5 = -1$

$\therefore z = \text{cis}\frac{2\pi}{10}, \text{cis}\frac{6\pi}{10}, -1, \text{cis}\frac{-2\pi}{10}, \text{cis}(-\frac{6\pi}{10})$. (2)



(ii) If $\text{cis}\frac{2\pi}{10}$ is ω then $\text{cis}\frac{6\pi}{10}$ becomes ω^3 and the roots $\text{cis}\frac{-2\pi}{10}$ and $\text{cis}\frac{-6\pi}{10}$ can be seen to be the opposites of ω^4 and ω^2 resp.

$\therefore$ Roots may be designated as $\omega, \omega^3, -1, -\omega^2, -\omega^4$. (3)

(iii) Now $z^5+1 = 0$ and since coefficient of z^4 is zero the sum of the roots is zero! ✓

ie $\omega + \omega^3 - 1 - \omega^2 - \omega^4 = 0$

$\therefore$ $-\omega^4 = 1 - \omega - \omega^3 + \omega^2$ ✓

$\therefore$ $(1-\omega-\omega^3+\omega^2)^8 = (-\omega^4)^8$

$= \omega^{32}$

$= (\omega^5)^6 \times \omega^2$

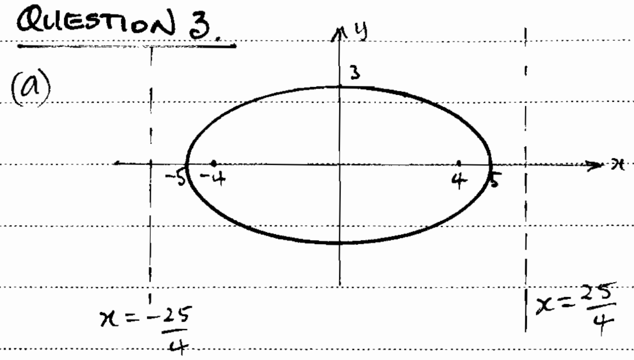
$= 1 \times \omega^2$ (3)

$= \text{cis}\frac{4\pi}{10}$ ✓

or $\text{cis}\frac{2\pi}{5}$

QUESTION 3.

(a)



$a = 5, b = 3$

$a^2e^2 = 25 - 9 = 16 \quad (\to ae = 4)$

$\therefore \quad e = \frac{4}{5}$ (4)

(b) Chord of contact: $\frac{xx_0}{25} + \frac{yy_0}{9} = 1$

This always contains the point $(2,2)$

$\therefore \quad \frac{2x_0}{25} + \frac{2y_0}{9} = 1$

ie (x_0, y_0) lies on the st. line $\frac{2x}{25} + \frac{2y}{9} = 1$ (3)

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{d}{dx}\frac{x^2}{a^2} + \frac{d}{dx}\frac{y^2}{b^2} = \frac{d}{dx}1$

ie $\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{xb^2}{ya^2}$ ✓

$\therefore$ at (x_1, y_1) the slope of the normal is $\frac{y_1a^2}{x_1b^2}$ ✓

$\therefore$ Eqn of Normal is $y - y_1 = \frac{y_1a^2}{x_1b^2}(x - x_1)$ ✓

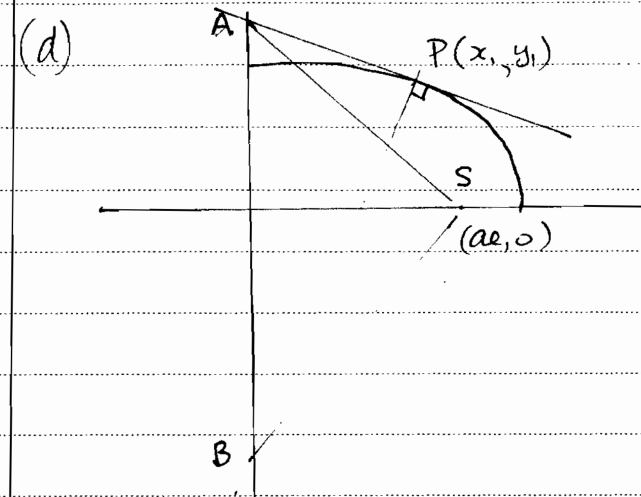
ie $x_1yb^2 - x_1y_1b^2 = xy_1a^2 - x_1y_1a^2$

ie $\frac{x_1y}{a^2} - \frac{x_1y_1}{a^2} = \frac{xy_1}{b^2} - \frac{x_1y_1}{b^2}$ ✓ on

dividing through by a^2b^2.

Hence required form follows on rearranging.

(d)



Tangent at P $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

cuts at A when $x = 0$

ie $y = \frac{b^2}{y_1}$ (1)

Normal at P, $\frac{x_1y}{a^2} - \frac{xy_1}{b^2} = \frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}$,

cuts at B when $x = 0$

ie $y = \frac{a^2}{x_1}\left(\frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}\right)$

$= y_1\left(1 - \frac{a^2}{b^2}\right)$ (1)

Slope AS $= \frac{b^2/y_1}{-ae}$ (1)

Slope BS $= \frac{y_1\left(1 - \frac{a^2}{b^2}\right)}{-ae}$ (1)

The product of these is $\frac{b^2 - a^2}{a^2e^2}$.

But $a^2e^2 = a^2 - b^2$

$\therefore$ Product of Slopes is $\frac{b^2 - a^2}{a^2 - b^2} = -1$ (1)

$\therefore$ AS $\perp$ BS.

(e) APSB is a cyclic quadrilateral with AB as diameter of the