SYDNEY TECHNICAL HIGH SCHOOL

Mathematics Extension 2

HSC ASSESSMENT TASK 1
MARCH 2008

General Instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions
- All questions are of equal value

NAME :________________________

QUESTION 1	QUESTION 2	QUESTION 3	TOTAL

Question 1 (16 marks)

a) If $z = 1 - \sqrt{3}\, i$ 5

i) find $|z|$

ii) find $\arg(z)$

iii) find z^5 in the form $a + ib$

iv) find a possible value of n ($n > 1$) such that $\arg(z) = \arg(z^n)$

b) Solve the equation : $z^2 + 4z - 1 + 12i = 0$. 4

c) Find the equation of the ellipse with eccentricity $\frac{4}{5}$ 2

and foci at $(-8, 0)$ and $(8, 0)$.

d) Find the gradient of the tangent to $4x + xy^2 = y^3$ at the point $(1, 2)$. 2

e) The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 3

at the point $P\ (x_1, y_1)$ is given by $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$.

This normal meets the minor axis of the ellipse at G .

The line parallel to the major axis of the ellipse which passes

through the point P meets the minor axis of the ellipse at N .

Show that $\frac{OG}{ON} = \frac{-a^2 e^2}{b^2}$

Question 2 (16 marks)

a) Sketch on an Argand diagram the region specified by 2

$$|z-2| < |z+2i|$$

b) i) Sketch the locus of the point z such that $|z-(3+2i)| = 2$ 2

ii) Determine the values of k for which the simultaneous equations 2

$|z-2i| = k$ and $|z-(3+2i)| = 2$ have exactly two solutions.

c) On an Argand diagram the quadrilateral OABC is a square, 2

where O is the origin.

If A represents the complex number $5+2i$ find the complex number

represented by the points B and C given that they both have positive arguments.

d) i) Solve $z^3 = 1$ over the complex field. 2

ii) Given that ω is the complex roots of $z^3 = 1$ with smallest positive argument :

α) Show that $1+\omega+\omega^2 = 0$ 1

β) Evaluate $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ 3

e) Use De Moivre's Theorem to show that 2

$$(\cot\theta + i)^n + (\cot\theta - i)^n = \frac{2\cos n\theta}{\sin^n\theta}$$

Question 3 (16 marks)

a) Given that z is a complex number, show that the locus defined by 3

$$z\bar{z} + 10 (z + \bar{z}) = 21$$

is a circle and state its centre and radius.

b) Sketch the locus of the point z such that $\arg(z+2i) = \dfrac{3\pi}{4}$ 2

c) Sketch the locus of z given that $\dfrac{z}{z+4}$ is purely imaginary. 2

d) i) Show that the equation of the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ 3

at the point $P(a\cos\theta, b\sin\theta)$ is given by $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$

ii) If this tangent at P meets the x axis at A 4

and S is a focus of the ellipse

show that $\dfrac{PS}{AS}$ is independent of the values of a and b.

e) Given that $|z-2| = 2$ and $0 < \arg(z) < \dfrac{\pi}{2}$ 2

find the value of k if $\arg(z-2) = k \times \arg(z^2 - 2z)$

The End

Solutions - March 2008 Ext 2

Question

ii) $-\frac{\pi}{3}$

iii) $z^5 = \left[2 \text{ cis}\left(-\frac{\pi}{3}\right)\right]^5$

$= 32 \text{ Cis}\left(-\frac{5\pi}{3}\right)$

$= 16 + 16\sqrt{3}\, i$

iv) $n = 7$

b) $z = \dfrac{-4 \pm \sqrt{16 - 4(1)(-1+12i)}}{2}$

$= -2 \pm \sqrt{5 - 12i}$

$\sqrt{5-12i} = a + ib$

$5 - 12i = a^2 - b^2 + i(2ab)$

$\therefore\ a^2 - b^2 = 5$

$ab = -6$

$\therefore\ a = 3,\ b = -2$

$\therefore\ z = -2 \pm (3 - 2i)$

$= -5 + 2i,\ 1 - 2i$

c) $ae = 8$ $\quad b^2 = a^2(1 - e^2)$

$a \cdot \frac{4}{5} = 8$ $\quad = 10^2\left(1 - \frac{16}{25}\right)$

$a = 10$ $\quad = 36$

$b = 6$

$\therefore\ \dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$

$(3y^2 - 2xy)\dfrac{dy}{dx} = 4 + y^2$

$\dfrac{dy}{dx} = \dfrac{4 + y^2}{3y^2 - 2xy}$

sub $(1, 2)$

$m_T = \dfrac{4 + 4}{12 - 4}$

$= 1$

e) $N(0, y_1)$

when $x = 0$ normal becomes

$-\dfrac{b^2 y}{y_1} = a^2 - b^2$

$y_1 = \dfrac{y_1}{-b^2}(a^2 - b^2)$

$\therefore\ G\left(0, \dfrac{y_1}{-b^2}(a^2 - b^2)\right)$

$\therefore\ \dfrac{OG}{ON} = \dfrac{\frac{y_1}{-b^2}(a^2 - b^2)}{y_1}$

$= \dfrac{a^2 - b^2}{-b^2}$

$= \dfrac{-a^2 e^2}{b^2}$

but $b^2 = a^2(1 - e^2)$

$e^2 = 1 - \dfrac{b^2}{a^2}$

$= \dfrac{a^2 - b^2}{a^2}$

$\therefore\ a^2 e^2 = a^2 - b^2$

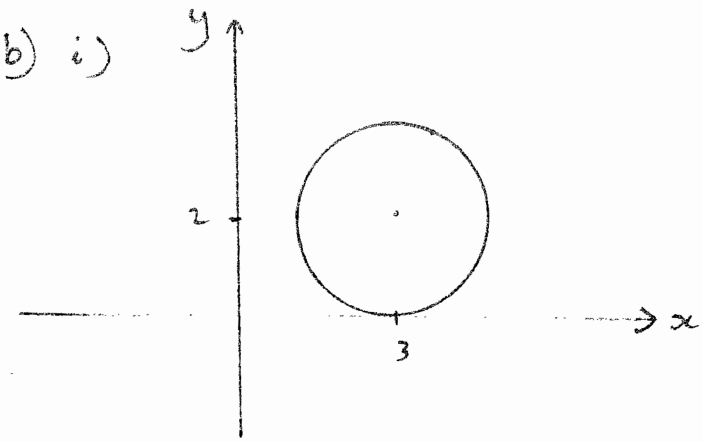
sub $z = w$

$(w-1)(w^2 + w + 1) = 0$

but $w - 1 \neq 0$ as w is complex root

$\therefore \; w^2 + w + 1 = 0$

b) i)



ii) $1 < k < 5$

c) $C \to i(5+2i)$

$= -2 + 5i$

$B \to (-2+5i) + (5+2i)$

$= 3 + 7i$

d) i) $z = 1, \; \text{Cis}\,\frac{2\pi}{3}, \; \text{Cis}\,\frac{4\pi}{3}$

or $1, \; \text{Cis}\,\frac{2\pi}{3}, \; \text{Cis}\left(-\frac{2\pi}{3}\right)$

or $1, \; \frac{1}{2}\left(-1+\sqrt{3}i\right), \; \frac{1}{2}\left(-1-\sqrt{3}i\right)$

iii) $(1-w)(1-w^2)(1-w^4)(1-w^5)$

$= (1-w)^2(1-w^2)^2$

$= (1-2w+w^2)(1-2w^2+w^4)$

$= (-3w)(-3w^2)$

$= 9w^3$

$= 9$

e) $\text{LHS} = (\cot\theta + i)^n + (\cot\theta - i)^n$

$= \left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta}\right)^n + \left(\dfrac{\cos\theta - i\sin\theta}{\sin\theta}\right)^n$

$= \dfrac{\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta}{\sin^n\theta}$

$= \dfrac{2\cos n\theta}{\sin^n\theta}$

$= \text{RHS}$

Question 3

[illegible]

$x^2 + y^2 + 20x = 21$

$x^2 + 20x + 100 + y^2 = 121$

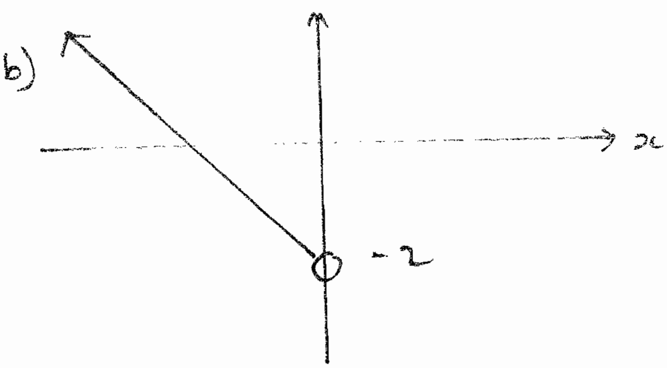
$(x+10)^2 + y^2 = 11^2$

which is a circle

radius 11 units

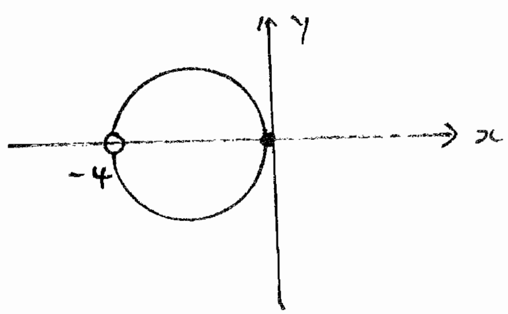
centre $(-10, 0)$

b)



c) purely imaginary

$\therefore \arg\left(\frac{z}{z+4}\right) = \pm\frac{\pi}{2}$



or $\frac{z}{z+4} = \frac{x+iy}{x+4+iy} \times \frac{x+4-iy}{x+4-iy}$

$= \frac{x(x+4)+y^2 + i\left(y(x+4) - xy\right)}{(x+4)^2 + y^2}$

purely imaginary $\Rightarrow$ real part is zero

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$m_T = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$

$= \frac{-b\cos\theta}{a\sin\theta}$

$\therefore$ equation of tangent

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab$

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

ii) $PS = e\,PN$

$= e\left(\frac{a}{e} - a\cos\theta\right)$

$= a - ae\cos\theta$

when $y = 0$ $\quad \frac{x\cos\theta}{a} = 1$

$x = \frac{a}{\cos\theta}$

$\therefore A\left(\frac{a}{\cos\theta}, 0\right)$

$\therefore AS = \frac{a}{\cos\theta} - ae$

$= \frac{a - ae\cos\theta}{\cos\theta}$

$\therefore \frac{PS}{AS} = \frac{a - ae\cos\theta}{\frac{a - ae\cos\theta}{\cos\theta}}$

$= \cos\theta$

let $\arg(z) = \alpha$

$\therefore \arg(z-2) = 2\alpha$

$\arg(z-2) = k \times \arg(z^2 - 2z)$

$= k \times [\arg(z) + \arg(z-2)]$

$\therefore \quad 2\alpha = k[\alpha + 2\alpha]$

$k = \frac{2}{3}$