SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS EXTENSION 2

HSC ASSESSMENT TASK 1

MARCH 2009

General Instructions

- Working time allowed – 70 minutes
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown
- Start each question on a new page
- Attempt all questions
- All questions are of equal value
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NAME:____________________________________

Question 1	Question 2	Question 3	Total

QUESTION 1 | Mark

a) Let $\propto = 5 - 3i$ and $\beta = 2 + i$

(i) Find $\propto + \beta$ 1

(ii) Find $\frac{\propto}{\beta}$ in the form $x + iy$ 1

(iii) If $z = x + iy,$ sketch the region defined by $Im(z \propto) < 3$ 2

b) The complex number $z = 1 + 2i$ is a root of the equation $z^2 - aiz + b = 0$ where a and b are real numbers.

(i) Find the values of a and b 2

(ii) Find the other root of the equation 1

c) Sketch the region defined by 3

$1 < |z - (1 + i\sqrt{3}| < 2 \; and \; 0 \leq \arg z \leq \frac{\pi}{3}$

d) Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

(i) Express z in modulus – argument form 1

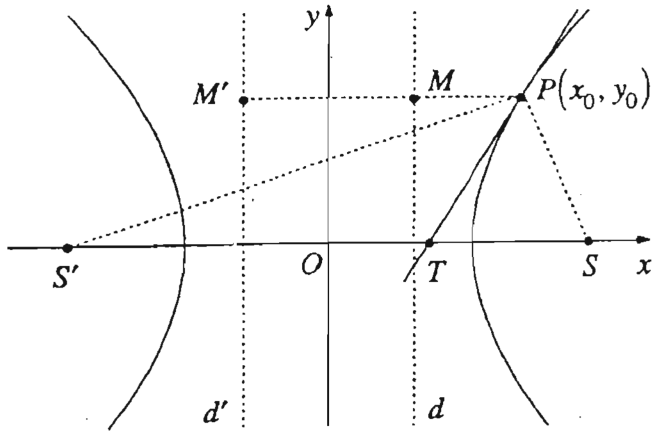
(ii) Hence or otherwise show that z is a root of the equation $z^4 = -1$ 1

(iii) Find the other roots of $z^4 = -1$ 2

(iv) Find the side length of the square formed by plotting the solutions to part (iii) on an Argand diagram and joining them together. 1

Question 2

a) Find the gradient of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point (1,2) 2

b) 

MARK

The point $P\ (x_\circ, y_\circ)$ lies on the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

T'-e tangent to the hyperbola at P cuts the x axis at T and has equation

$$\frac{x_\circ x}{16} - \frac{y_\circ y}{9} = 1$$

The two foci of the hyperbola are S and S', and the two directrices are d *and* d'. The points M and M' are the closest points to P on the directrices d *and* d'.

(i) Find the co ordinates of the foci 2

(ii) Find the equations of the directrices 1

(iii) Show that T has co ordinates $\left(\frac{16}{x_\circ}, 0\right)$ 1

(iv) Using the focus- directrix definition, or otherwise, show that 3

$$\frac{PS}{PS'} = \frac{TS}{TS'}$$

(c) Find the equation of the ellipse with eccentricity $\frac{3}{4}$ and directrices at $x = \pm 16$ 2

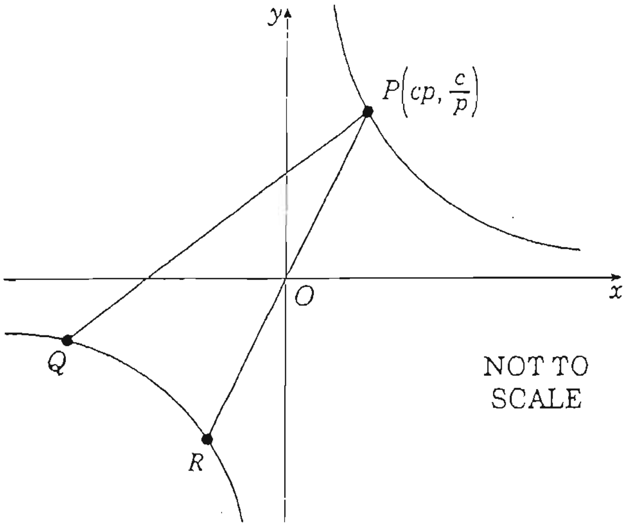
(d) (i) Express $z = \dfrac{1+\sqrt{3}i}{1+i}$ in the form rcis θ . 3

(ii) Find the smallest positive integer n such that z^n is a real number 1

QUESTION 3 MARK

(a) If the line $kx + my + n = 0$ is a tangent to the hyperbola $xy = c^2$, prove that $n^2 = 4c^2 km$. 2

(b)



The point P $\left(cp, \frac{c}{p}\right)$ where $p \neq \pm 1$, is a point on the hyperbola $xy = c^2$, and the normal to the hyperbola at P intersects the 2nd branch at Q. The line through P and the origin O intersects the second branch at R.

(i) Show that the equation of the normal is
$py - c = p^3\ (x - cp)$ 2

(ii) Show that the y coordinates of P and Q satisfy the equation. 3

$$py^2 - c(1 - p^4)y - p^3c^2 = 0$$

(iii) Find the coordinates of Q. 1

(iv) Show that Q, R and P are concyclic 2

(e) (i) If w is a complex cube root of unity (ie: a root of $z^3 = 1$), prove that w^2 is also a root. 1

(ii) Prove that $1 + w + w^2 = 0$ 1

(iii) Hence or otherwise form a quadratic equation whose roots are given by $\alpha = 2 + w$ and $\beta = 2 + w^2$ 3

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Question 1

a) $\alpha = 5-3i \qquad \beta = 2+i$

(i) $\alpha + \beta$

$5-3i+2+i$

$= 7-2i$

(ii) $\dfrac{\alpha}{\beta}$

$\dfrac{5-3i}{2+i} \times \dfrac{2-i}{2-i}$

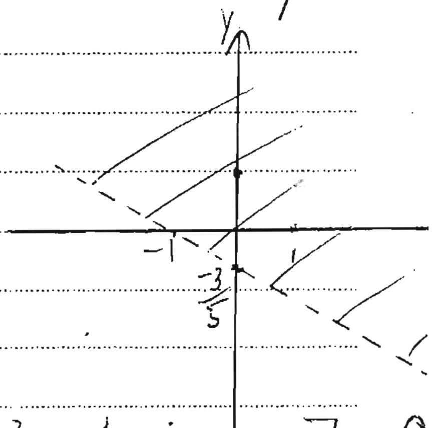
$= \dfrac{10-11i-3}{5}$

$= \dfrac{7}{5} - \dfrac{11}{5}i$

(iii) $\text{Im}\left[(x+iy)(5-3i)\right] < 3$

$\text{Im}\left[5x-3xi+5yi+3y\right] < 3$

$-3x+5y<3$



b) (i) $z = 1+2i$ is a root of

$z^2 - aiz + b = 0$

$\therefore (1+2i)^2 - ai(1+2i) + b = 0$

$-1+4i - ai + 2a + b = 0$

$(2a+b-1) + i(4-a) = 0$

Equating real and imaginary parts to 0

$\Rightarrow a = 4, \quad b = -7$

(ii) $z^2 - 4aiz - 7 = 0$

Let the other root be $\alpha = x+iy$

$\therefore$ Sum of roots

$\alpha + 1 + 2i = 4ai$

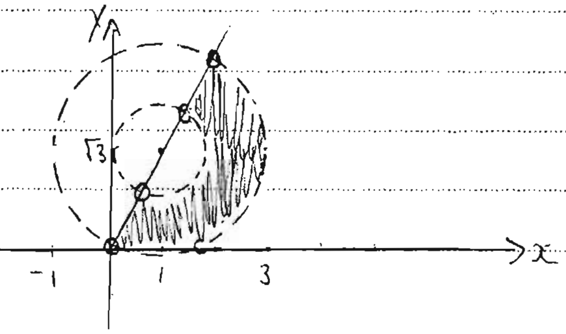
$(x+iy) + 1 + 2i - 16i = 0$

$(x+1) + i(y-14) = 0$

$\therefore x = -1, \ y = 14$

$\therefore$ Other root is $-1+14i$

c)



d) (i) $|z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{\frac{1}{2} + \frac{1}{2}}$

$|z| = 1$

$\arg z = \tan^{-1} 1 = \frac{\pi}{4}$

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(ii) $Z = cis\frac{\pi}{4}$

$Z^4 = (cis\frac{\pi}{4})^4$

$Z^4 = cis\,\pi$ by De Moivre's Theorem

$= -1$ as req'd $\therefore$ a root

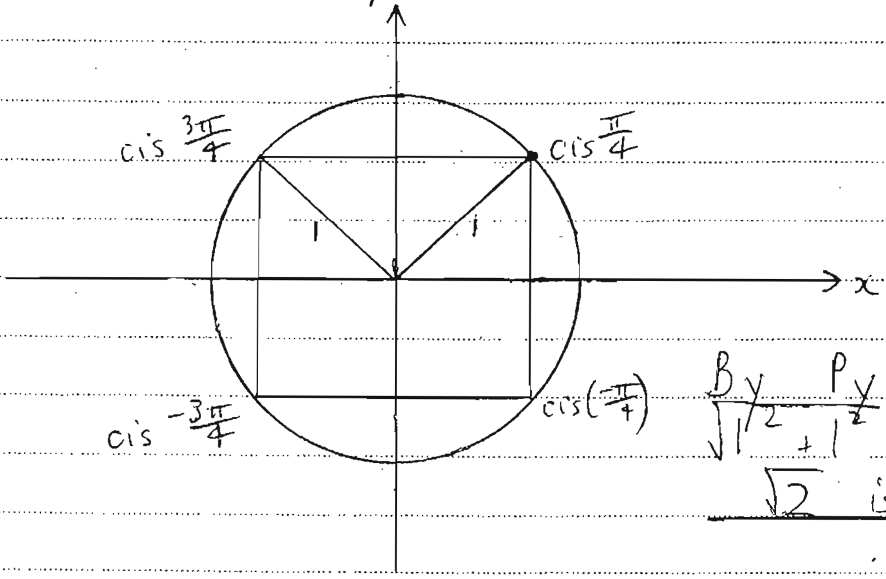
(iii) $Z^4 = -1$

$Z^4 = cis(\pi + 2k\pi)$

$Z = cis(\frac{\pi + 2k\pi}{4})$ by De Moivre's

$\therefore Z = cis\frac{\pi}{4}, \; cis\frac{-\pi}{4}, \; cis\frac{3\pi}{4}, \; cis(\frac{-3\pi}{4})$

(iv)



By Pythagoras

$\sqrt{1^2 + 1^2}$

$\sqrt{2}$ is side length

Question 2

a) $x^3 + y^3 - 3xy = 3$

Differentiating implicitly

$3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0$

$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$

$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$

$\therefore$ At $(1, 2)$,

$M_{tangent} = \frac{1}{3}$

b)(i) Foci at $(\pm ae, 0)$

$a = 4$

$b^2 = a^2(e^2 - 1)$

$9 = 16(e^2 - 1)$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

$\therefore (\pm ae, 0) \Rightarrow (\pm 5, 0)$

(ii) Directrices $x = \pm\frac{a}{e}$

$x = \pm \frac{4}{\frac{5}{4}}$

$x = \pm \frac{16}{5}$

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c(iii) T on tangent is where $y = 0$

ie: $\frac{x_0 x}{16} - 0 = 1$

$x = \frac{16}{x_0}$

$\therefore$ T is $\left(\frac{16}{x_0}, 0\right)$

c(iv) Need to show

$\frac{PS}{PS'} = \frac{TS}{TS'}$

Since $PS = ePM$ and $PS' = ePM'$

$\therefore \frac{PM}{PM'} = \frac{TS}{TS'}$

$\frac{x_0 - \frac{16}{5}}{x_0 + \frac{16}{5}} = \frac{5 - \frac{16}{x_0}}{\frac{16}{x_0} + 5}$

$\frac{5x_0 - 16}{5x_0 + 16} = \frac{5x_0 - 16}{5x_0 + 16}$ ✓

$\therefore$ Result shown

c) $16 = \frac{a}{e}$

$e = \frac{3}{4}$

$\therefore a = 12$

$b^2 = a^2(1 - e^2)$

$b^2 = 144\left(1 - \left(\frac{3}{4}\right)^2\right)$

$b^2 = 144 \times \frac{7}{16}$

$b^2 = 63$

$\therefore$ Ellipse has eq'n

$\frac{x^2}{144} + \frac{y^2}{63} = 1$

d(i) $z = \frac{1 + \sqrt{3}i}{1 + i}$

$1 + \sqrt{3}i = 2\operatorname{cis}\frac{\pi}{3}$

$1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$

$\therefore z = \sqrt{2}\operatorname{cis}\frac{\pi}{12}$

(ii) $z^n = (\sqrt{2})^n \operatorname{cis}\left(\frac{\pi}{12}\right)^n$

$= (\sqrt{2})^n \operatorname{cis}\left(\frac{\pi n}{12}\right)$

is real if

$\sin\left(\frac{\pi n}{12}\right) = 0$

ie: when $n = 12$, $n > 0$

Question 3

a) Solve simultaneously

$kx + my + n = 0$

$xy = c^2$

$y = \frac{c^2}{x}$

$\Rightarrow kx + m \times \frac{c^2}{x} + n = 0$

$kx^2 + nx + mc^2 = 0$

$\rightarrow$ If line is a tangent, only one solution $\Rightarrow \Delta = 0$

$n^2 - 4 \times k \times mc^2 = 0$

$n^2 = 4kmc^2$ as required

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b) (i) $x = cp \quad \frac{dx}{dp} = c$

$y = \frac{c}{p} \quad \frac{dy}{dp} = -\frac{c}{p^2}$

$\therefore \frac{dy}{dx} = \cancel{\frac{\frac{dy}{dp}}{\frac{dx}{dp}}}$

$= \cancel{\frac{\frac{-c}{p^2}}{c}}$

$\frac{dy}{dx} = \frac{-1}{p^2}$

$y - y_1 = m(x - x_1)$

$y - \frac{c}{p} = \frac{p^2}{1}(x - cp)$ is the eq'n of the normal

$py - c = p^3(x - cp)$ as required

(ii) Solving the hyperbola simultaneously with the normal:

$\begin{cases} xy = c^2 \\ py - c = p^3(x - cp) \end{cases}$

$py - c = p^3\left(\frac{c^2}{y} - cp\right)$

$py^2 - cy = p^3(c^2 - cpy)$

$py^2 - cy = p^3c^2 - cp^4y$

$\underline{py^2 - c(1 - p^4)y - p^3c^2 = 0}$

(iii) One root of (ii) is $\frac{c}{p}$. Let y value of be α. Product of roots is

$\alpha \times \frac{c}{\cancel{p}} = -\frac{p^3c^2}{\cancel{p}}$

$\therefore \alpha = -cp^3$

$\therefore x$ value of Q is $\frac{c^2}{\alpha}$

$= \frac{c^{\cancel{2}}}{-\cancel{c}p^3} = \frac{-c}{p^3}$

$\therefore Q\left(-\frac{c}{p^3}, -c\right.$

(iii) cont'd.

QRP are concyclic if

$\angle QRP = 90°$ (Angle in a semi-circle is 90°)

ie: $m_{QR} \times m_{RP} = -1$

Since $xy = c^2$ is odd R is $\left(-cp, \frac{-c}{p}\right)$

$m_{QR} = \frac{-cp^3 + \frac{c}{p}}{-\frac{c}{p^3} + cp}$

$= \frac{\frac{1}{p} - p^3}{p - \frac{1}{p^3}}$

$m_{RP} = \frac{\frac{c}{p} + \frac{c}{p}}{cp + cp}$

$= \frac{\frac{1}{p}}{p} = \frac{1}{p^2}$

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(iii) Cont'd

$m_{QR} \times m_{RP}$

$= \dfrac{\frac{1}{p} - p^3}{p - \frac{1}{p^3}} \times \dfrac{1}{p^2}$

$= \dfrac{\frac{1}{p} - p^3}{p^3 - \frac{1}{p}}$

$= -1$ $\therefore$ QRP are concyclic

e. (i) If ω is a root of $z^3 = 1$, then $\omega^3 = 1$. If ω^2 is also a root then 1

$(\omega^2)^3 = 1$

$\Rightarrow (\omega^3)^2 = 1$

$1^2 = 1$ ✓ $\therefore$ ω^2 is a root.

(ii) The 3 roots of $z^3 = 1$ are 1, ω and ω^2. 1

Sum of roots $= 1 + \omega + \omega^2$

$= \frac{-b}{a}$ from polynomial theory

$z^3 - 1 = 0$

$\frac{-b}{a} = 0$

$\therefore 1 + \omega + \omega^2 = 0$ as required.

(iii) Roots of quadratic are α and β 3

$z^2 - (\alpha + \beta) z + \alpha\beta = 0$

$z^2 - (2 + \omega + 2 + \omega^2) z + (2 + \omega)(2 + \omega^2) = 0$

$z^2 - (1 + \omega + \omega^2 + 3) z + (4 + 2\omega^2 + 2\omega + \omega^3) = 0$

$z^2 - 3z + [5 + 2(\omega + \omega^2)] = 0$ since $\omega^3 = 1$ and using c(ii)

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$$z^2 - 3z + (5 + 2(-1)) = 0$$

$$\underline{z^2 - 3z + 3 = 0}$$