Name:__________________________ Maths Class Teacher:______________________

SYDNEY TECHNICAL HIGH SCHOOL

Extension 2 Mathematics

HSC Assessment Task 1

March 2011

General Instructions

- Working time – 70 minutes
- Write using **black or blue pen**
- Board-approved calculators may be used
- **All necessary working** should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Start each question on a **new page**.
- Place your papers **in order** with the question paper on top and staple or pin them.

Total Marks - 50

- Attempt Questions 1 – 3
- Mark values are shown with the questions.

(For markers use only)

Q1	Q2	Q3	Total
17	16	17	50

Question 1 **17 Marks**

(a) In each case below. $z = 3 - 2i$. **4**

Express the following in the form $x + iy$ where x and y are real numbers:

(i) $\overline{(iz)}$

(ii) $(z - 1)^2$

Evaluate:

(iii) arg (z) (in radians to one decimal place)

(iv) $|z|$ (leave in exact form)

(b) (i) Express $1 + i\sqrt{3}$ in modulus argument form. **2**

(ii) Hence evaluate $\left(1 + i\sqrt{3}\right)^5$ **2**

(Express your answer in the domain defined for the argument.)

(c) For $z = x + iy$,

(i) Express $\frac{1}{z}$ as a complex number. **1**

(ii) Hence find the solutions for z if $\text{Re}\left(z - \frac{1}{z}\right) = 0$, **3**

and show on the Argand diagram.

(d) Sketch the following loci.

(i) $|\arg z| \le \frac{\pi}{4}$ **1**

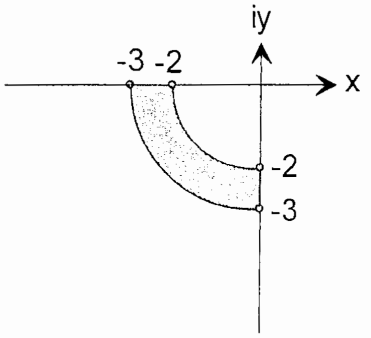
(ii) $|z + 2| + |z - 2| = 6$ **1**

(e) Sketch the region where the inequalities $|z - 2 + i| \le 5$ and $|z - 1| \ge |z + 1|$ both hold. **3**

Question 2 **16 Marks**

(a) Give the inequalities which describe this region on the Argand diagram. **3**

(Give your answer in terms of z.)



(b) If $1 - 2i$ is a root of the equation $2z^3 - 5z^2 + 12z - 5 = 0$

(i) Explain why $1 + 2i$ is also a root. **1**

(ii) Find all roots of the equation. **2**

(c) (i) Show on an Argand diagram the positions of the roots of $z^3 = -1$. **1**

(ii) Explain algebraically why these are among the roots of $z^6 = 1$. **2**

(iii) By referring to the roots of $z^6 = 1$, find the roots of $z^4 + z^2 + 1 = 0$ in mod-arg form. **3**

(d) (i) Solve $z^4 = 1$ for all z. **1**

(ii) Hence, or otherwise, solve $z^4 = (z - 1)^4$. **3**

Question 3 **17 Marks**

(a) Sketch the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. State the following: **5**

(i) the eccentricity

(ii) the coordinates of the foci

(iii) the equations of the directrices.

(b) Find the equation of the tangent to the curve $x^2 - xy^2 - 8y + 32 = 0$ at the point (1, 3). **3**

(c) Prove that the equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ can be expressed as $\frac{x_1 y}{a^2}-\frac{x y_1}{b^2}=\frac{x_1 y_1}{a^2}-\frac{x_1 y_1}{b^2}$. **4**

(d) The tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ cuts the y axis at A while the normal at P cuts the y axis at B. If S is a focus of the ellipse, show that $\angle ASB = 90°$. (Equation of tangent $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$) **5**

End of Exam

SOLUTIONS EXTENSION 2 ASSESSMENT 1 2011

1 a) $z = 3 - 2i$

(i) $iz = 3i + 2$

$= 2 + 3i$

$\therefore \overline{iz} = 2 - 3i$

(ii) $(z-1)^2 = (3 - 2i - 1)^2$

$= (2 - 2i)^2$

$= 4 - 4 - 8i$

$= -8i$

(iii) $\arg z = \tan^{-1}\left(\frac{-2}{3}\right)$

$\doteq -33^\circ 41'$

or -0.6^c

(iv) $|z| = \sqrt{3^2 + (-2)^2}$

$= \sqrt{9+4}$

$= \sqrt{13}$

b)(i) $z = 1 + i\sqrt{3}$

$\therefore |z| = \sqrt{1+3}$

$= 2$

$\arg z = \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$

$\therefore z = 2 \text{ cis } \frac{\pi}{3}$

b(ii)

$(1 + i\sqrt{3})^5 = \left(2 \text{ cis } \frac{\pi}{3}\right)^5$

$= 32 \text{ cis } \frac{5\pi}{3}$

$= 32 \text{ cis } \left(\frac{-\pi}{3}\right)$

c) (i) $\frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

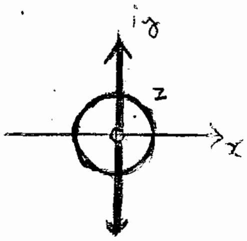
$= \frac{x - iy}{x^2+y^2}$

(ii) $Re\left(z - \frac{1}{z}\right) = Re\left[(x+iy) - \frac{x-iy}{x^2+y^2}\right], z \neq 0$

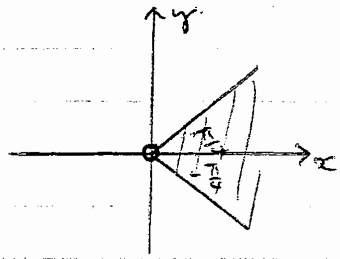
$= x - \frac{x}{x^2+y^2}$

$\therefore x\left(1 - \frac{1}{x^2+y^2}\right) = 0$

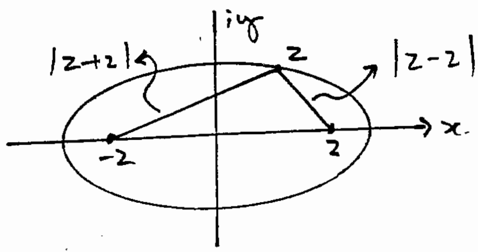
$\therefore x = 0$ or $x^2 + y^2 = 1$



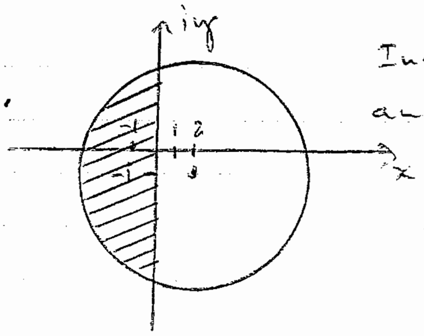
d) (i) $|\arg z| \leq \frac{\pi}{4}$



(ii) $|z+2| + |z-2| = 6$



e) $|z - 2 + i| \leq 5$ and $|z-1| \leq |z+1|$



Inside circle centre $2 - i$ with radius 5 (including circle) and left of y-axis.

Q2

a) $2 \le |z| \le 3$
and $-\pi \le \arg z \le -\frac{\pi}{2}$

d) (i) $z^4 = 1$
$\therefore z = \pm 1, \pm i$

(ii) $z^4 = (z-1)^4$
$\therefore \frac{z^4}{(z-1)^4} = 1$
$\therefore \left(\frac{z}{z-1}\right)^4 = 1$
$\therefore \frac{z}{z-1} = \pm 1, \pm i$

For $\frac{z}{z-1} = 1$
$z = z-1$ no solution

For $\frac{z}{z-1} = -1$
$z = 1-z$
$\therefore z = \frac{1}{2}$

For $\frac{z}{z-1} = i$
$z = iz - i$
$z - iz = -i$
$z(1-i) = -i$
$\therefore z = \frac{-i}{1-i}$

For $\frac{z}{z-1} = -i$
$z = -iz + i$
$\therefore z + iz = i$
$\therefore z(1+i) = i$
$\therefore z = \frac{i}{1+i}$

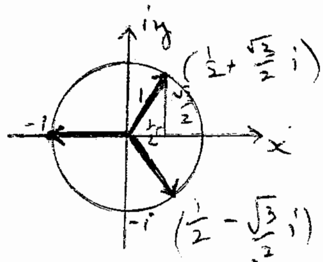
$\therefore z = \frac{1}{2}, \frac{-i}{1-i}, \frac{i}{1+i}$

b) $2z^3 - 5z^2 + 12z - 5 = 0$

(i) The coefficients are real
$\therefore$ complex roots consist of conjugates.
$\therefore$ If $1-2i$ is a root then $1+2i$ is also a root.

(ii) $(z - (1-2i))(z - (1+2i))$
$= z^2 - (1-2i)z - (1+2i)z + (1-2i)(1+2i)$
$= z^2 - 2z + 5$
$2z^3 - 5z^2 + 12z - 5 = (z^2 - 2z + 5)(2z - 1)$
$\therefore$ roots are $z = \frac{1}{2}, 1-2i, 1+2i$

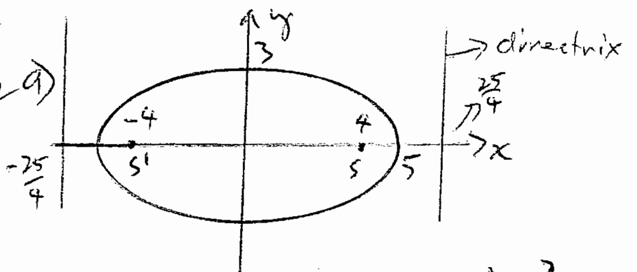
c) (i) $z^3 = -1$



(ii) $z^6 = 1$
$\therefore z^6 - 1 = 0$
$\therefore (z^3+1)(z^3-1) = 0$
$\therefore z^3 = 1$ or -1
$\therefore$ the 6 roots of $z^6 = 1$ include the 3 roots of $z^3 = -1$

(iii) $z^6 = 1$ has 1 and -1 as roots.
Now $z^6 - 1 = 0$
$\therefore z^6 - 1 = (z+1)(z-1)(z^4+z^2+1)$ —— A
Let $z = \text{cis}\,\theta$
$\therefore (\text{cis}\,\theta)^6 = 1$
$\therefore \text{cis}\,6\theta = 1$
Now $\text{cis}(2\pi k) = 1$ where $k = 0, 1 \ldots 5$
$\therefore z = (\text{cis}\,2\pi k)^{\frac{1}{6}}$ for $k = 0 \ldots 5$
$= \text{cis}\left(\frac{\pi k}{3}\right)$ " " "
$= 1, \text{cis}\frac{\pi}{3}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{3\pi}{3} = -1, \text{cis}\frac{4\pi}{3}$ and $\text{cis}\frac{5\pi}{3}$
In A the factors $z+1$ and $z-1$ use the zeroes ± 1
$\therefore$ the roots of z^4+z^2+1 are $\text{cis}\frac{\pi}{3}, \text{cis}\frac{2\pi}{3}, \text{cis}\frac{4\pi}{3}, \text{cis}\frac{5\pi}{3}$

Q3 a)



$a = 5, \ b = 3$

$a^2e^2 = a^2 - b^2$

$= 25 - 9$

$= 16$

$1 - e^2 = \frac{b^2}{a^2}$

$= \frac{9}{25}$

ii) $\therefore ae = 4 \rightarrow$ focus $(\pm 4, 0)$

$\therefore e^2 = 1 - \frac{9}{25}$

$= \frac{16}{25}$

(i) $\therefore e = \frac{4}{5} \rightarrow$ eccentricity

(iii) directrices $\Rightarrow x = \pm \frac{a}{e}$

$x = \pm \frac{5}{4}$

b) $x^2 - xy^2 - 8y = 0 \quad (1, 3)$

$\therefore 2x - y^2 - 2xy \cdot \frac{dy}{dx} - 8\frac{dy}{dx} = 0$

$\therefore 2x - y^2 - \frac{dy}{dx}(2xy + 8) = 0$

$\therefore \frac{dy}{dx} = \frac{2x - y^2}{2xy + 8}$

$= \frac{2 - 9}{6 + 8}$ at $(1, 3)$

$= \frac{-7}{14}$

$= -\frac{1}{2}$

eqtn of tangent $\Rightarrow y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{2}(x - 1)$

$2y - 6 = -x + 1$

$\therefore x + 2y - 7 = 0$

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

$\therefore$ slope of normal at (x_1, y_1)

$= \frac{a^2y_1}{b^2x_1}$

$\therefore$ eqtn of normal $\Rightarrow$

$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$

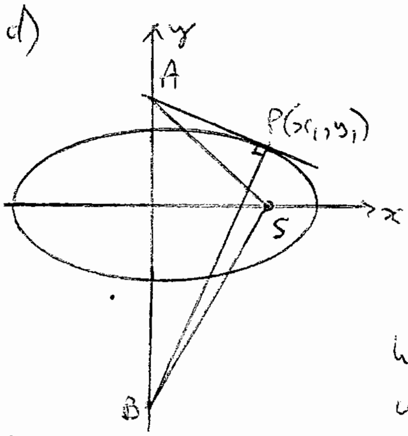
$\therefore b^2x_1y - b^2x_1y_1 = a^2xy_1 - a^2x_1y_1$

$\therefore \frac{x_1y}{a^2} - \frac{x_1y_1}{a^2} = \frac{xy_1}{b^2} - \frac{x_1y_1}{b^2}$

$\therefore \frac{x_1y}{a^2} - \frac{xy_1}{b^2} = \frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}$

QED

d)



tangent at P,

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

When $x = 0$, $y = \frac{b^2}{y_1}$

Normal at P,

$\frac{x_1y}{a^2} - \frac{xy_1}{b^2} = \frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}$

When $x = 0$,

$y = \frac{a^2}{x_1}\left(\frac{x_1y_1}{a^2} - \frac{x_1y_1}{b^2}\right)$

$= y_1\left(1 - \frac{a^2}{b^2}\right)$

$\therefore$ slope AS $= \frac{\frac{b^2}{y_1}}{-ae}$ slope BS $= \frac{y_1\left(1 - \frac{a^2}{b^2}\right)}{-ae}$

Now slope AS $\times$ slope BS

$= \frac{b_1}{y_1 \times -ae} \times \frac{y_1\left(1 - \frac{a^2}{b^2}\right)}{-ae}$

$= \frac{b^2 - a^2}{a^2e^2}$

$= -1$ as $a^2e^2 = a^2 - b^2$

$\therefore AS \perp BS$