SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1

MARCH 2012

Mathematics Extension 2

General Instuctions

- Working time - 70 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Start each question on a new page

Total marks - 52

- Attempt Questions 1 – 4
- All questions are of equal value

Name : ______________________

Teacher : ______________________

Question 1	Question 2	Question 3	Question 4	**Total**

Question 1 (13 marks)

a) For the ellipse with equation $x^2 + 4y^2 = 16$ find

i) the eccentricity **2**

ii) the coordinates of the foci **1**

iii) the equation of the directrices **1**

iv) the length of the chord of the ellipse which passes through the focus and is perpendicular to the major axis of the ellipse **2**

b) Factorise $x^2 + 6x + 25$ over the complex field **2**

c) i) Express $1 + i\sqrt{3}$ in modulus argument form. **2**

ii) Find the smallest positive integer value of n such that **3**

$$Im\left(\frac{-1+i}{1+i\sqrt{3}}\right)^n = 0$$

Question 2 (13 marks) - Start a new page

a) If $z = 2 + i$ and $w = 3 - 2i$ find simplified expressions for

i) $z + \bar{w}$ 1

ii) $\frac{z}{w}$ 2

b) Sketch the locus of z described by the following -

i) $0 \leq Arg(z - 2i) \leq \frac{\pi}{6}$ 2

ii) $Im(z^2) = |z - \bar{z}|$ 3

c) Solve $z^2 = 7 + i\sqrt{72}$ over the complex field, 2

giving your answer in the form $x + iy$ where x and y are real.

d) Given $\cos(x - y) = y\cos x$ 3

show that $\frac{dy}{dx} = \frac{\sin(x-y) - y\sin x}{\sin(x-y) - \cos x}$

Question 3 (13 marks) - Start a new page

a) $P(a\cos\theta\,, b\sin\theta)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$.

i) Show that the equation of the normal to the above ellipse at the point P is given by the equation $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ 3

ii) The normal found in part i) meets the major axis of the ellipse at the point G.

If S is a focus of the ellipse and e its eccentricity, 4

show that $SG = eSP$

b) Find all the solutions of $z^6=-1$ 3

c) Simplify $\frac{\left(\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^2}{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}$ 3

Question 4 (13 marks) - Start a new page

a) i) Draw a neat sketch of the locus represented by 2

$$|z + \sqrt{2} - i\sqrt{2}| = 1$$

ii) For z on the locus in part i) find

α) the minimum value of $|z|$ 1

β) the minimum value of $Arg(z)$ 1

b) z is a complex number such that $Arg(z) = \theta$ where $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Find expressions for the following in terms of θ,

i) $Arg(iz + z)$ 2

ii) $Arg(iz - z)$ 2

c) For the general ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ describe the effect on the ellipse as $e \to 0$ (e is the eccentricity) 1

d) Show by Mathematical Induction that 4

$$(1 - a_1)(1 - a_2) \ldots\ldots\ldots\ldots (1 - a_n) > 1 - (a_1 + a_2 + \ldots\ldots\ldots\ldots + a_n)$$

for all positive integers n where $n > 1$, if a_k satisfies $0 < a_k < 1$ for $1 \le k \le n$.

SOLUTIONS

$\boxed{1}$

a) i) $x^2 + 4y^2 = 16$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$b^2 = a^2(1 - e^2)$$

$$4 = 16(1 - e^2)$$

$$\frac{1}{4} = 1 - e^2$$

$$e^2 = \frac{3}{4}$$

$$e = \frac{\sqrt{3}}{2}$$

ii) $S(\pm ae, 0)$

$S(\pm 2\sqrt{3}, 0)$

iii) $x = \pm \frac{a}{e}$

$x = \pm \frac{8}{\sqrt{3}}$

iv) when $x = 2\sqrt{3}$

$$12 + 4y^2 = 16$$

$$4y^2 = 4$$

$$y^2 = 1$$

$$y = \pm 1$$

$\therefore$ length $= 2$ units

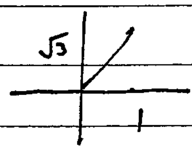
b) $x^2 + 6x + 25$

$x^2 + 6x + 9 + 16$

$(x+3)^2 + 4^2$

$(x+3+4i)(x+3-4i)$

c) i) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$



ii) $-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$$\frac{-1+i}{1+i\sqrt{3}} = \frac{\sqrt{2}\ \text{Cis}\ \frac{3\pi}{4}}{2\ \text{Cis}\ \frac{\pi}{3}}$$

$$= \frac{1}{\sqrt{2}}\ \text{Cis}\ \frac{5\pi}{12}$$

for $\text{Im}\left(\frac{-1+i}{1+i\sqrt{3}}\right)^n = 0$

$n \times \frac{5\pi}{12} = m\pi$ where m is an integer

$\therefore\ n = 12$

c) $z^2 = 7 + i\sqrt{72}$

$(x+iy)^2 = 7 + i\sqrt{72}$

$x^2 - y^2 + 2ixy = 7 + i\sqrt{72}$

$x^2 - y^2 = 7 \qquad 2xy = \sqrt{72}$

$xy = 3\sqrt{2}$

$\therefore \ x = 3, \ y = \sqrt{2}$

$\therefore \ z = 3 + i\sqrt{2}, \ -3 - i\sqrt{2}$

d) $\cos(x-y) = y\cos x$

$-\sin(x-y)\left(1 - \frac{dy}{dx}\right) = \frac{dy}{dx}\cos x - y\sin x$

$-\sin(x-y) + \sin(x-y)\frac{dy}{dx} = \frac{dy}{dx}\cos x - y\sin x$

$y\sin x - \sin(x-y) = \frac{dy}{dx}\left(\cos x - \sin(x-y)\right)$

$$\frac{dy}{dx} = \frac{y\sin x - \sin(x-y)}{\cos x - \sin(x-y)}$$

$\boxed{2}$ a) i) $z + \overline{w}$

$= 2 + i + 3 + 2i$

$= 5 + 3i$

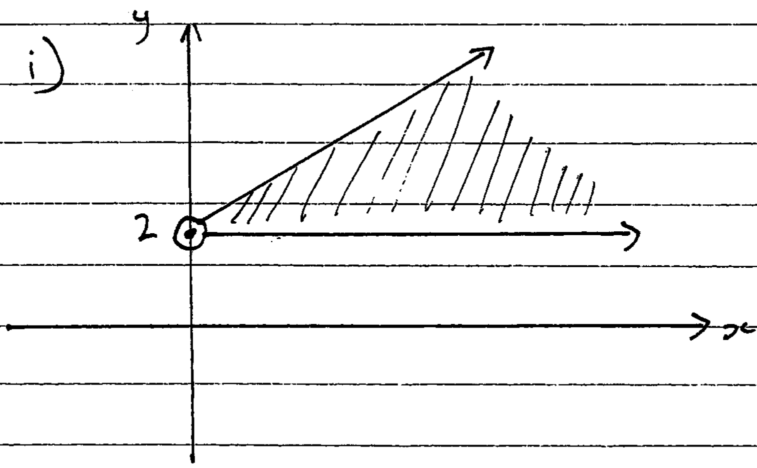
ii) $\dfrac{z}{w} = \dfrac{2+i}{3-2i}$

$= \dfrac{2+i}{3-2i} \times \dfrac{3+2i}{3+2i}$

$= \dfrac{6 + 4i + 3i - 2}{9+4}$

$= \dfrac{4+7i}{13}$

b) i)



ii) $\mathrm{Im}(z^2) = |z - \overline{z}|$

$\mathrm{Im}(x^2 - y^2 + 2ixy) = |x + iy - (x - iy)|$

$2xy = |2y|$

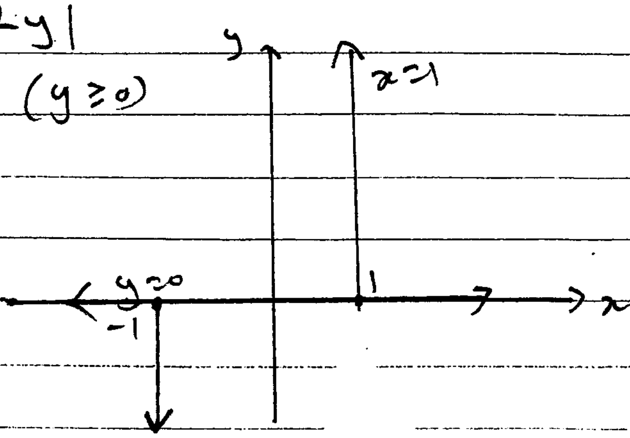
$xy + y = 0 \quad (y \le 0)$

$y = 0, \; x = -1$

$xy - y = 0 \quad (y \ge 0)$

$y(x-1) = 0$

$y = 0, \; x = 1$



$\boxed{3}$ a) i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

$$\therefore \quad m_T = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$$\therefore m_N = \frac{a\sin\theta}{b\cos\theta}$$

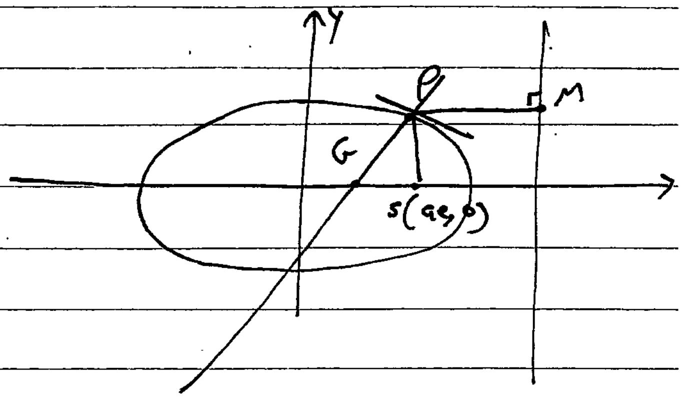
using $\quad y - y_1 = m(x - x_1)$

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta = a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$



ii) when $y=0$

$$\frac{ax}{\cos\theta} = a^2 - b^2$$

$$x = \frac{(a^2-b^2)\cos\theta}{a}$$

$$\therefore G\left(\frac{(a^2-b^2)\cos\theta}{a}, 0\right)$$

$$\therefore e.SP = e \times e \times PM$$

$$= e^2\left(\frac{a}{e} - a\cos\theta\right)$$

$$= e(a - ae\cos\theta)$$

$$SG = ae - \left(\frac{a^2-b^2}{a}\cos\theta\right)$$

$$= ae - \left(\frac{a^2e^2}{a}\right)\cos\theta$$

$$= ae - ae^2\cos\theta$$

$$= e(a - ae\cos\theta)$$

$$= e\,SP$$

$$b^2 = a^2(1-e^2)$$

$$\frac{b^2}{a^2} = 1 - e^2$$

$$e^2 = \frac{a^2-b^2}{a^2}$$

$$a^2e^2 = a^2 - b^2$$

b) $z^6 = -1$

$z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$z_2 = i$

$z_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$z_4 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$z_5 = -i$

$z_6 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

c) $$\frac{\left(\sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^2}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$$

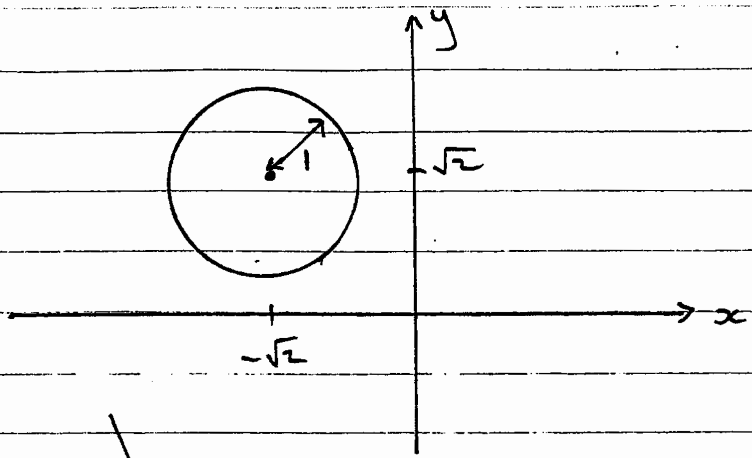
$$= \frac{\left(\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)\right)^2}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$$

$$= \frac{\left(\cos\frac{3\pi}{10} + i\sin\frac{3\pi}{10}\right)^2}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$$

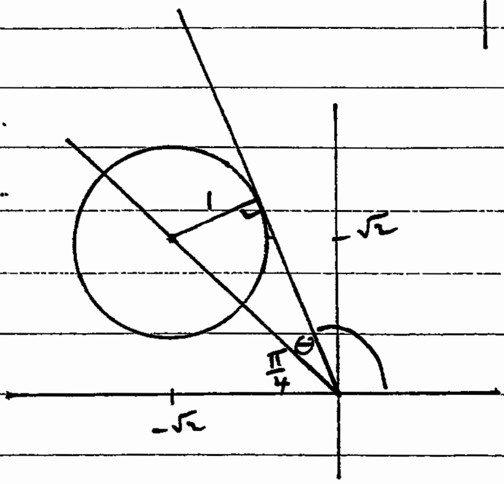
$$= \frac{\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}}{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}$$

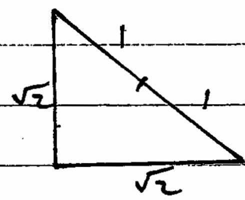
$$= \cos\frac{13\pi}{30} + i\sin\frac{13\pi}{30}$$

$\boxed{4}$ a) i)



ii)





α) $\min |z| = 1$

$\sin\theta = \frac{1}{2}$

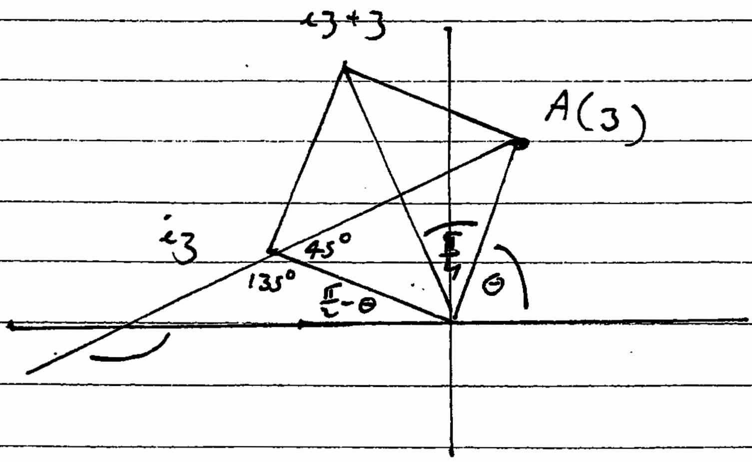
$\theta = \frac{\pi}{6}$

β) $\min \operatorname{Arg}(z) = \frac{\pi}{2} + \frac{\pi}{12}$

$= \frac{7\pi}{12}$

$\frac{\pi}{2} - \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

b)



i) $\operatorname{Arg}(iz+z) = \frac{\pi}{4} + \theta$

ii) $\operatorname{Arg}(iz-z) = -\left(\frac{3\pi}{4} + \frac{\pi}{2} - \theta\right)$

$= \theta - \frac{5\pi}{4}$

c) the ellipse tends towards a circle.

d) Step 1 test $n=2$.

$LHS = (1-a_1)(1-a_2)$ $\qquad$ $RHS = 1-(a_1+a_2)$

$= 1 - a_1 - a_2 + a_1 a_2$ $\qquad$ $= 1 - a_1 - a_2$

$> 1 - a_1 - a_2$

$= RHS$

$\therefore$ true for $n=2$.

Step 2 assume true for $n=k$

i.e. $(1-a_1)(1-a_2)\cdots(1-a_k) > 1-(a_1+a_2+\cdots+a_k)$

show true for $n=k+1$

$(1-a_1)(1-a_2)\cdots(1-a_k)(1-a_{k+1})$

$> [1-(a_1+a_2+\cdots+a_k)](1-a_{k+1})$ from assumption

$= 1-(a_1+a_2+\cdots+a_k) - a_{k+1} + (a_1+a_2+\cdots+a_k)a_{k+1}$

$> 1-(a_1+a_2+\cdots+a_k) - a_{k+1}$

$= 1-(a_1+a_2+\cdots+a_k+a_{k+1})$

which is the required result

$\therefore$ true for $n=k+1$ if true for $n=k$.

Step 3

As true for $n=2$, also true for $n=2+1$, ie $n=3$

As true for $n=3$, also true for $n=3+1$, ie $n=4$

and so on for all positive integers n, $n>1$.