Name: .... **Maths Class:** ....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

Assessment 1
March 2013

TIME ALLOWED: 70 minutes

Instructions:

- ***Start each question on a new page.***
- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- It is suggested that you spend no more than 5 minutes on Part A.
- Approved calculators may be used.

PART A: (5 Marks)

Answers to these multiple choice should be completed on the multiple choice answer sheet supplied with your answer booklet.

All questions are worth 1 mark

(a)	The value of i^{2014} is A. 1 B -1 C. i D. $-i$
(b)	As the eccentricity of a standard ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ approaches zero, i.e., as $e\to 0$, what happens to the ellipse? A. It becomes a point B. It becomes a hyperbola C. It becomes more elliptical D. It becomes a circle.
(c)	The Cartesian form of the conic given by $x = 4\sec\theta$ and $y = 3\tan\theta$ is A. $\frac{x^2}{16}-\frac{y^2}{9}=1$ B. $\frac{x^2}{9}-\frac{y^2}{16}=1$ C. $\frac{x^2}{16}+\frac{y^2}{9}=1$ D. $\frac{x^2}{9}+\frac{y^2}{16}=1$
(d)	The length of the major axis of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is: A. 2 B. 3 C. 4 D. 6
(e)	What is the solution to the equation $z^2 = i\,\overline{z}$? (A) $(0,0)$ and $(0,1)$ (B) $(0,0)$ and $(0,-1)$ (C) $(0,0)$, $(0,-1)$, $(\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2},\frac{1}{2})$ (D) $(0,0)$, $(0,1)$, $(\frac{\sqrt{3}}{2},\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2},\frac{1}{2})$

PART B

(START EACH QUESTION ON A NEW PAGE)

QUESTION 1: (15 Marks)

Marks

3 (a) Let $x = 5 - i$ and $y = 3 + 4i$.

Find (i) $|y|$ (ii) $\bar{x}$ (iii) $\frac{y}{x}$ (give your answer in the form $a + ib$)

(b) On separate Argand Diagrams, sketch the solutions to:

1 (i) $|z - 1| < 2$

1 (ii) $\frac{\pi}{4} < \arg(z - 1) < \frac{\pi}{3}$

1 (c) (i) If a point P on the hyperbola $xy = c^2$ has its x-value as $x = ct$, give its y-value

1 (ii) Find the equation of the tangent at P

2 (iii) If this tangent cuts the co-ordinate axes at A and B, show that PA=PB.

2 (d) If $z = 1 + \sqrt{3}i$, find

(i) arg z (ii) z^6, in simplest form

4 (e) If $|z| = 1$ and $\arg z = \theta$, show that $\arg\left[\frac{(z+1)^2}{z}\right] = 0$

QUESTION 2: (15 Marks) ***(Start on a new page)***

Marks

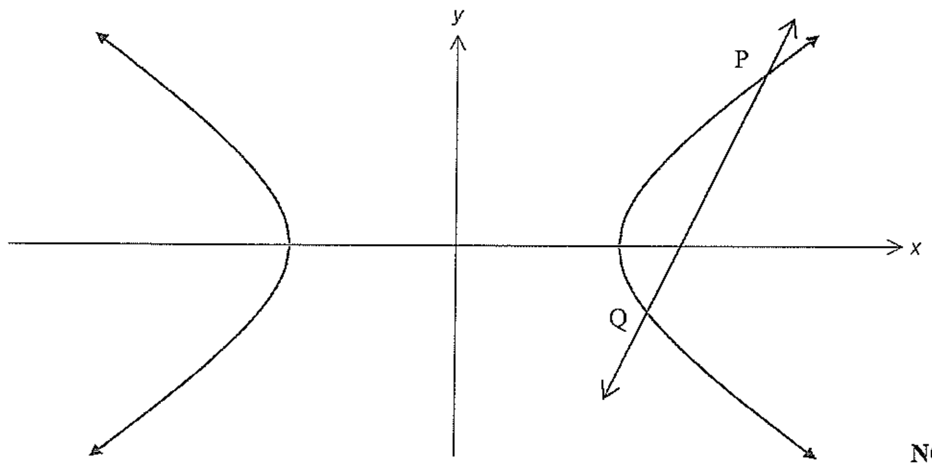
3 (a) Find the gradient of the tangent to the curve $x^4 + y^4 - 5xy^2 = 0$ at the point where x=2 and y = $\sqrt{2}$

1 (b) (i) Find the argument and modulus of $1 - i$

2 (ii) Hence, by using De Moivre's Theorem, or otherwise, simplify the expression

$$(1-i)^8 + (1+i)^8$$

(c) P $(4\sec\theta, 3\tan\theta)$ and Q $(4\sec\alpha, 3\tan\alpha)$ are points on the Hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ with parameters θ and α, where $\theta + \alpha = \frac{\pi}{2}$ and $\alpha \neq \frac{\pi}{4}$.



NOT TO SCALE

2 (i) Find the co-ordinates of Q in terms of θ, in simplest trigonometric form.

2 (ii) Prove that the gradient of the chord PQ is $\frac{3}{4}(\cos\theta + \sin\theta)$

3 (iii) Find the equation of the chord PQ, in gradient/intercept form, and hence find the coordinates of a point on PQ that is independent of the value of θ.

2 (iv) As $\theta \rightarrow \frac{\pi}{2}$, show that the chord PQ approaches a line parallel to an asymptote of the hyperbola.

QUESTION 3: (15 Marks) *(Start on a new page)*

Marks

(a) For the ellipse $\frac{x^2}{4} + y^2 = 1$,

1 (i) Find the eccentricity, e.

2 (ii) Find an expression for $\frac{dy}{dx}$, and hence find the slope of the tangent at P(x_0, y_0)

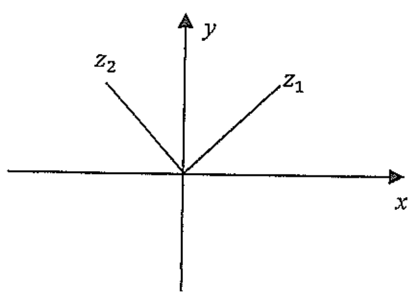
1 (iii) Prove that the equation of the tangent at P is $\frac{xx_0}{4} + y\,y_0 = 1$

1 (iv) The tangent at P meets the Directrix cutting the positive x-axis at Q.
Prove that the y-value of Q is $y_Q = \frac{\sqrt{3} - x_0}{\sqrt{3}y_0}$

1 (v) If $x_0 > 0$, and $y_0 > 0$, find the range of values of x_0, so that Q lies below the x-axis.

(b) z_1 and z_2, shown on the Argand Diagram below, are complex numbers such that

$$\frac{z_1 + z_2}{z_1 - z_2} = 2i,$$



(i) Copy the diagram onto your answer sheet (NO MARKS)

2 (ii) On the diagram, plot the points $z_1 + z_2$ and $z_1 - z_2$

2 (iii) Show that $|z_1| = |z_2|$

5 (c) The sequence $1, \sqrt{3}, \sqrt{1 + 2\sqrt{3}}, \ldots\ldots\ldots.$

has its nth position given by $x_n = \sqrt{1 + 2x_{n-1}}$

By the process of Mathematical Induction, prove that $x_n < 4$ for all n ≥1

Multiple Choice Answer Sheet

Name ______________________________

Completely fill the response oval representing the most correct answer.

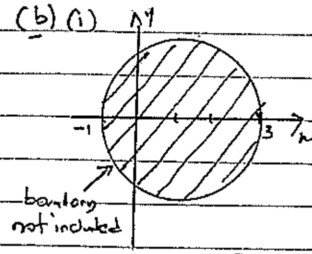
1. A ○ B ● C ○ D ○
2. A ○ B ○ C ○ D ●
3. A ● B ○ C ○ D ○
4. A ○ B ○ C ○ D ●
5. A ○ B ○ C ● D

PART B

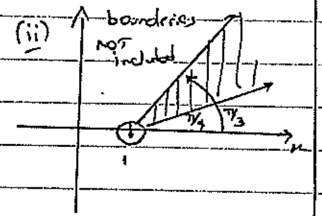
QUESTION 1:

(a)(i) 5 (ii) $5+i$ (iii) $\frac{11+23i}{26}$ — 1 MARK each

(b) (i)



(ii)



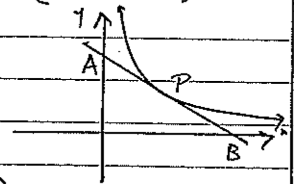
1 MARK EACH

(c) (i) $y = c^2/x$
$= c/t$ — 1 MARK

(ii) $t^2 y + x = 2ct$ (or similar) — 1 MARK ONLY (working NOT required).

(iii)

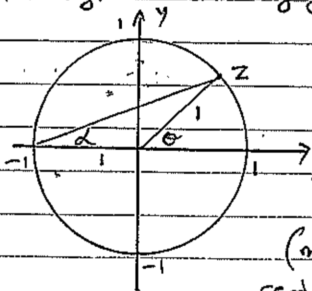
At A, $x=0$
$\therefore$ A is $(0, 2c/t)$
At B, $y=0$
$\therefore$ B is $(2ct, 0)$ — 1 MARK



MIDPOINT of AB is $(ct, c/t)$ which is P — 1 MARK
$\therefore$ PA = PB

(d) (i) $\arg z = \pi/3$ (ii) $z = 2\,\text{cis}\,\pi/3$
$z^6 = 64\,\text{cis}\,2\pi$
$= 64$ — 1 MARK each

(e) $|z| = 1$ and $\arg z = 0$



From the diagram,
$\theta = 2\alpha$. ← 2 for arriving here.
(METHOD I: external angle of a triangle)
(METHOD II: angle at the centre is twice that at the circumf

$\therefore \arg\left[\frac{(z+1)^2}{z}\right] = \arg(z+1)^2 - \arg z$
$= 2\arg(z+1) - \arg z$
$= 2\alpha - \theta$
$= 0$ — 2 for simplificati

QUESTION 2

(a) $4x^3 + 4y^3\frac{dy}{dx} - 5\left(y^2 + x\,2y\frac{dy}{dx}\right) = 0$ — 1 MARK

$\frac{dy}{dx}\left(4y^3 - 10xy\right) = 5y^2 - 4x^3$

$dy/dx = \frac{5y^2 - 4x^3}{4y^3 - 10xy}$ — 1 MARK

At $(2, \sqrt{2})$ $m_T = \frac{10 - 32}{8\sqrt{2} - 20\sqrt{2}}$

$= 11/6\sqrt{2}$ OR $\frac{11\sqrt{2}}{12}$ — 1 MARK

(b) (i) $\arg(1-i) = -\pi/4$ $|1-i| = \sqrt{2}$ — 1 MARK for EACH (= 2)

(ii) $(1-i)^8 + (1+i)^8 = \left[\sqrt{2}\,\text{cis}(-\pi/4)\right]^8 + \left[\sqrt{2}\,\text{cis}\,\pi/4\right]^8$ — 1 MARK

$= 16\,\text{cis}(-2\pi) + 16\,\text{cis}\,2\pi$

$= 32$ — 1 MARK

(c) (i) Q x-coordinate $= 4\sec(\alpha)$

$= 4\sec(\pi/2 - \theta)$

$= 4\operatorname{cosec}\theta$

y-value is $3\cot\theta$ — 1 for each = 2

(ii) $m_{PQ} = \frac{3\cot\theta - 3\tan\theta}{4\operatorname{cosec}\theta - 4\sec\theta}$

$= \frac{3}{4}\left[\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} \Big/ \frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right]$

$= \frac{3}{4}\left(\frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}\right)$ — 2 MARKS

$= \frac{3}{4}(\cos\theta + \sin\theta)$

(iii) EQUATION of chord PQ is

$\therefore y - 3\tan\theta = \frac{3}{4}(\cos\theta + \sin\theta)(x - 4\sec\theta)$

$4y - 12\tan\theta = 3(\cos\theta + \sin\theta)(x - 4\sec\theta)$

$4y = 3x(\cos\theta + \sin\theta) - 12$ ← (2) to get to here (or $y =$)

This goes through $(0, -3)$ ← (1) MARK

(iv) $m_{PQ} = \frac{3}{4}(\cos\theta + \sin\theta)$

As $\theta \to \pi/2$ $m_{PQ} \to 3/4$ ← (1) MARK

as the asymptotes are $y = \pm 3x/4$

PQ is parallel to $y = 3x/4$ — (1) for reasoning the slope of $y = 3x/4$

QUESTION 3:

(a) (i) $b^2 = a^2(1 - e^2)$

$e^2 = -1/4 + 1$

$e = \sqrt{3}/2$ — 1 MARK

(ii) $2x/4 + \frac{2y\,dy}{dx} = 0$

$\therefore dy/dx = -x/2 \times \frac{1}{2y}$

$= -x/4y$

$m_T = -x_0/4y_0$ ← 1 MARK

(iii) $y - y_0 = -x_0/4y_0\,(x - x_0)$

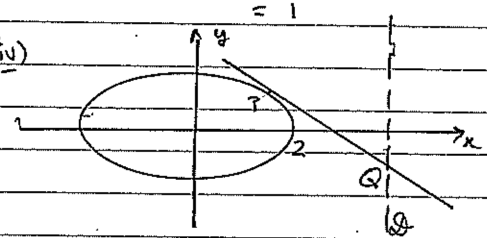
$4yy_0 - 4y_0^2 = -xx_0 + x_0^2$

$x_0^2 + 4y_0^2 = xx_0 + 4yy_0$

$\therefore xx_0/4 + yy_0 = x_0^2/4 + y_0^2$

$= 1$ — 1 MARK for an effort to prove.

(iv)



Directrix is $x = 4/\sqrt{3}$

Q has y-value: $\frac{4/\sqrt{3}\,.x_0}{4} + yy_0 = 1$

$x_0/\sqrt{3} + yy_0 = 1$

$y = \frac{1 - x_0/\sqrt{3}}{y_0}$

$= \frac{\sqrt{3} - x_0}{\sqrt{3}\,y_0}$ — 1 MARK

(v) For $y_0 < 0$

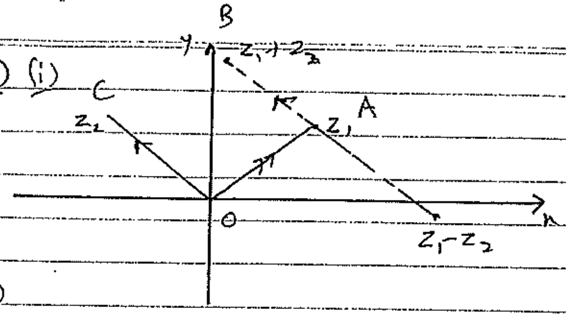
$\sqrt{3} - x_0 < 0$

$\therefore x_0 > \sqrt{3}$

But $x_0 < 2$ (ellipse)

$\sqrt{3} < x_0 < 2$ — 2 MARKS

(b) (i)



1 for each of z_1+z_2 and z_1-z_2

(ii)

2 METHODS

METHOD I Algebraic Since $\frac{z_1+z_2}{z_1-z_2} = 2i$

$$z_1+z_2 = 2iz_1 - 2iz_2$$
$$z_1(1-2i) = -z_2(1+2i)$$
$$-z_1/z_2 = \frac{1+2i}{1-2i}$$
$$|-z_1/z_2| = \left|\frac{1+2i}{1-2i}\right|$$
$$|z_1|/|z_2| = \frac{|1+2i|}{|1-2i|}$$
$$= \sqrt5/\sqrt5$$
$$= 1$$
$$\therefore |z_1| = |z_2|$$

2 MARKS
1 for getting to z_1/z_2
1 for establishing the moduli

METHOD II Geometric.

$$\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = \arg(2i)$$
$$\arg(z_1+z_2) - \arg(z_1-z_2) = \pi/2$$

this is the line from the point z_1+z_2 to the origin (ie. a diagonal)
this is the line from z_1 to z_2 (ie a diagonal)

∴ The angle between the diagonals is 90°.
∴ The shape is a RHOMBUS
$|z_1| = |z_2|$ (sides of a rhombus)

2 MARKS
1 for establishing the 90°
1 for sides of a rhombus

METHOD III Geometric.

Since $z_1+z_2 = 2i(z_1-z_2)$
then z_1+z_2 (OB) is a 90° rotation of z_1-z_2 (AC)
∴ OB ⊥ AC (diagonals intersect at 90°)
∴ the shape is a RHOMBUS
∴ $|z_1| = |z_2|$.

2 MARKS
1 for rotation
1 for the sides of a rhombus

(c) For $n=1$

$$x_n = \sqrt1 = 1 < 4$$

(1) MARK (no real need to test $n=2$)

For $n=2$

$$x_2 = \sqrt{1+2x_1}$$
$$= \sqrt3 < 4$$

∴ True for $n=1$ and $n=2$
Assume the formula is true for $n=k$.

$$\therefore x_k = \sqrt{1+2x_{k-1}} < 4$$

(1) for assumption

For $n=k+1$

$$x_{k+1} = \sqrt{1+2x_k}$$
$$< \sqrt{1+8}$$
$$= \sqrt9$$
$$< 4$$

(2) for this

∴ If the formula is true for $n=k$, it is true for $n=k+1$
But it is true for $n=1$ and $n=2$
∴ " " " " $n=3$
and so on.
ie true $\forall\, n$.

(1) for some type of conclusion which is inductive.