Board of Studies Student Number

Class Teacher: ....

*Do **NOT** write solutions on this question paper. Any working on the question paper will **NOT** be marked.*

DETUR GLORIA SOLI DEO

TRINITY GRAMMAR SCHOOL

Mathematics Department

2012

HALF YEARLY EXAMINATION

HSC Assessment Task 3

Year 12

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on the back of the Section I answer sheet and on page 15
- Show all necessary working in Questions 11 – 16
- Write your Board of Studies Student Number **and** Class Teacher on the writing booklet(s) **or** sheet(s) submitted
- **WEIGHTING:** 30%

Total marks – 100

Section I Pages 3 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 14

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

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Section I 10 marks

- Circle the correct response on the answer sheet provided
- Each question is worth 1 mark

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1 Using a suitable substitution, the definite integral $\int_0^{\frac{\pi}{24}} \tan 2x \sec^2 2x \; dx$ is equivalent to

A. $\int_0^{\frac{\pi}{24}} \frac{u}{2} \; du$

B. $\int_0^{2-\sqrt{3}} u \; du$

C. $\int_0^{2-\sqrt{3}} 2u \; du$

D. $\int_0^{2-\sqrt{3}} \frac{u}{2} \; du$

2 In the Argand plane, the curve $|z-(2+3i)|=1$ is intersected exactly twice by the curve with equation

A. $|z-3i|=1$

B. $|z-3i|=|z-3|$

C. $\text{Im}(z)=4$

D. $\text{Re}(z)=3$

3 The slope of the curve $2x^3 - y^2 = 7$ at the point where $y=-3$ is

A. –4

B. –2

C. 2

D. 4

4 Which one of the following relations does **NOT** have a graph that is a straight line passing through the origin?

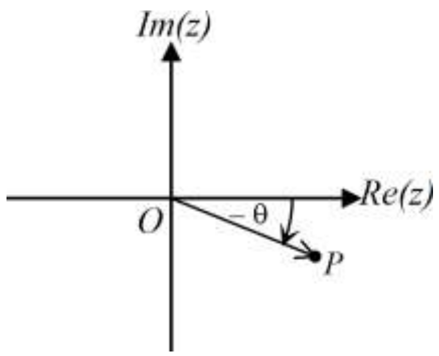
A. $z + \overline{z} = 0$

B. $3\,\mathrm{Re}(z) = \mathrm{Im}(z)$

C. $z = i\overline{z}$

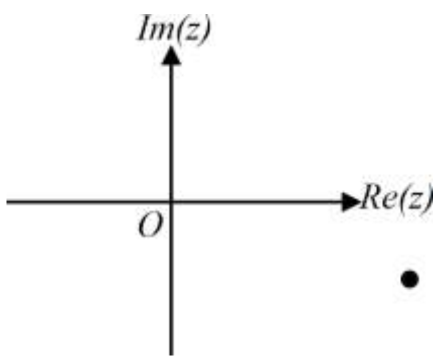
D. $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$

5 A certain complex number z, with $|\,z\,| > 1$, is represented by the point P on the following Argand diagram below.

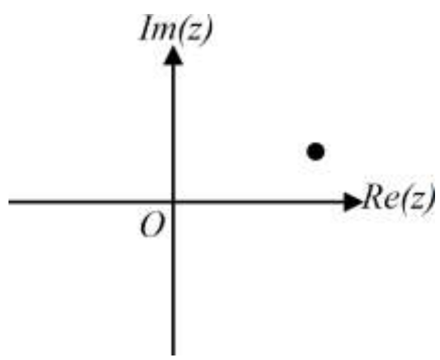


The complex number $\dfrac{1}{\overline{z}}$ is best represented by

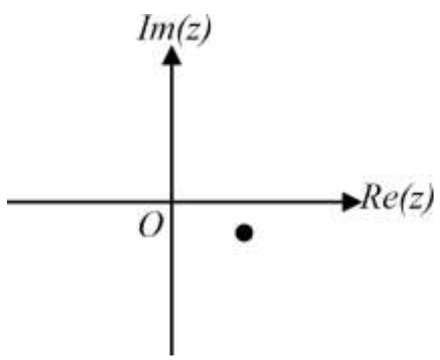
A.



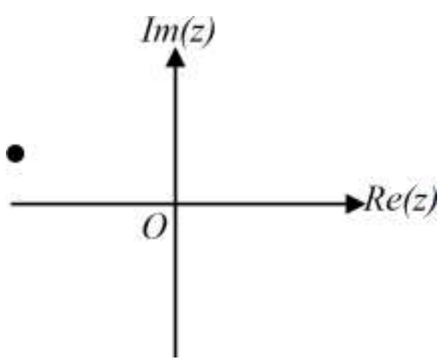
B.



C.



D.



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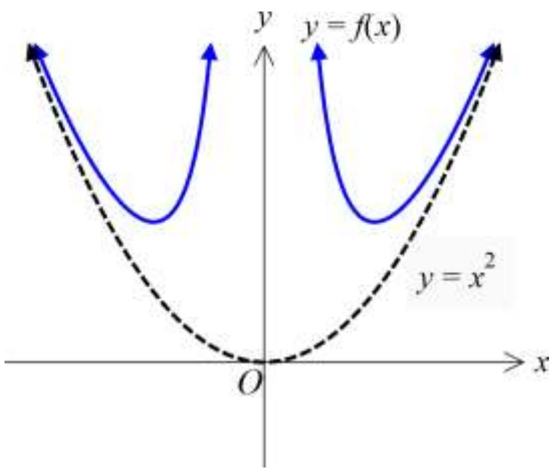


A possible equation for the graph of the curve $y = f(x)$ shown above is

A. $y = \dfrac{x^3 + a}{x}$, $a > 0$

B. $y = \dfrac{x^3 + a}{x}$, $a < 0$

C. $y = \dfrac{x^4 + a}{x^2}$, $a > 0$

D. $y = \dfrac{x^4 + a}{x^2}$, $a < 0$

7 For a certain complex number z, $\arg(z) = \dfrac{\pi}{5}$. The complex number z^7 has principal argument of

A. $-\dfrac{7\pi}{5}$

B. $-\dfrac{3\pi}{5}$

C. $\dfrac{3\pi}{5}$

D. $\dfrac{7\pi}{5}$

8 Given that $(1+i)^n = ai$, where a is a non-zero real number, then $(1+i)^{2n+2}$ simplifies to

A. a^4

B. $2a^2 i$

C. 0

D. $-2a^2 i$

9 In simplest form, $\frac{d}{dx}\cos^{-1}(\sin x)$ is equal to

A. -1, for all x

B. -1, if $\cos x < 0$

C. 1, for all x

D. 1, if $\cos x < 0$

10 Let $z = \cos\theta + i\sin\theta$. The expression $z^n + \frac{1}{z^n}$ is equivalent to

A. $-2\cos n\theta$

B. $2\cos n\theta$

C. $-2i\sin n\theta$

D. $2i\sin n\theta$

End of Section I

Section II 90 marks

- Begin each question in a new writing booklet or on a new answer sheet
- Show all necessary working
- Each question is worth 15 marks

Question 11 (15 marks)

(a) Find $\text{Im}\left(\dfrac{3+4i}{1+2i}\right)$. **2**

(b) Express $z = i - 1$ in modulus-argument form. **1**

(c) Find, in modulus-argument form, all the roots of $z^3 = -8$. **2**

(d) Sketch on separate Argand diagrams the locus of a point $z = x + yi$ such that:

(i) $2|z| = z + \overline{z} + 4$ **2**

(ii) $\text{Im}(z^2) = -2$ **2**

(iii) $|\text{Re}(z)| > 1$ **2**

(e) A complex number z satisfies the equations $2|z-1| = |z|$ ***and*** $\arg(z-1) - \arg z = \dfrac{\pi}{3}$.

(i) Show that $\dfrac{z-1}{z} = \dfrac{1}{2} cis \dfrac{\pi}{3}$. **2**

(ii) Hence, or otherwise, solve for z. Leave your answer in Cartesian form. **2**

Question 12 (15 marks) **Begin a NEW answer booklet or answer sheet**

(a) Using the substitution $u = \ln x$, evaluate $\int_1^{e^3} \frac{(\ln x)^3}{x}\,dx$. **3**

(b) Find $\int \cos^3 x\,dx$. **2**

(c) (i) Show that $f(x) = x\sin x$ is an even function about the line $x = 0$. **2**

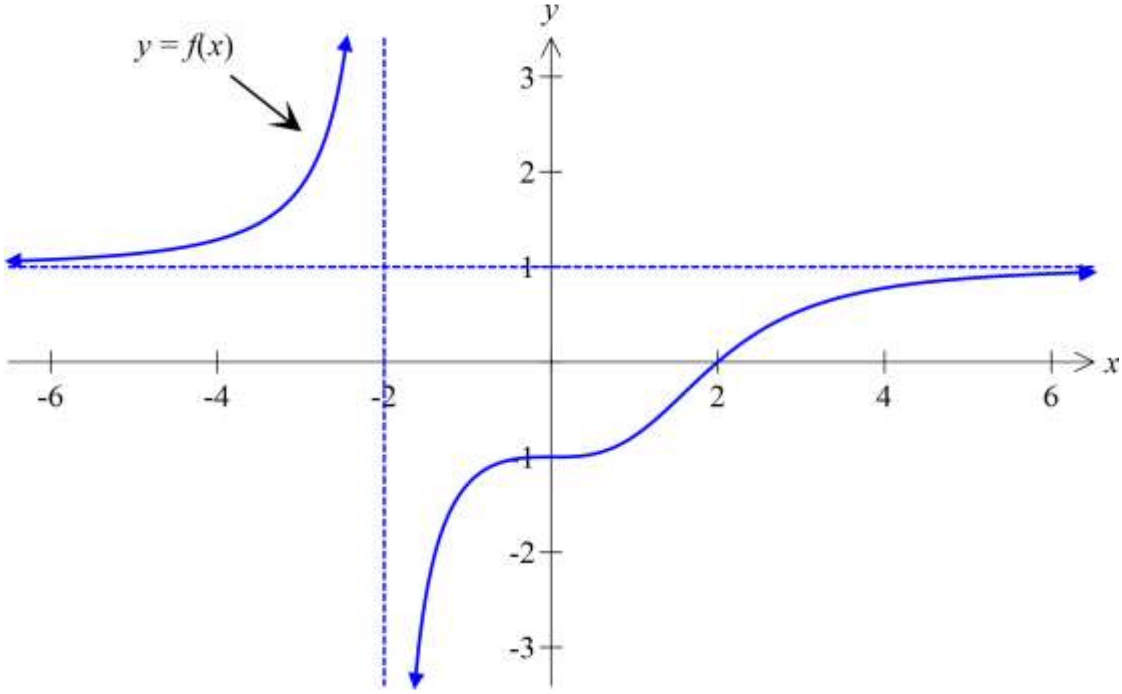
(ii) Find, using integration by parts, the area of the region bounded by $y = x\sin x$, $|\,x\,| = \frac{\pi}{2}$ and the x-axis. **3**

(d) (i) Find real values A, B and C such that: **3**

$$\frac{9 + x - 2x^2}{(1-x)(3+x^2)} = \frac{A}{1-x} + \frac{Bx + C}{3+x^2}$$

(ii) Hence find $\int \frac{9 + x - 2x^2}{(1-x)(3+x^2)}\,dx$. **2**

Question 13 (15 marks) **Begin a NEW answer booklet or answer sheet**



(a) The diagram above shows the graph of a function $y = f(x)$.

On the separate answer sheet provided, sketch the graphs of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(-x)$ **1**

(iii) $y = f\ |x|$ **1**

(iv) $|y| = f(x)$ **1**

(v) $y^2 = f(x)$ **2**

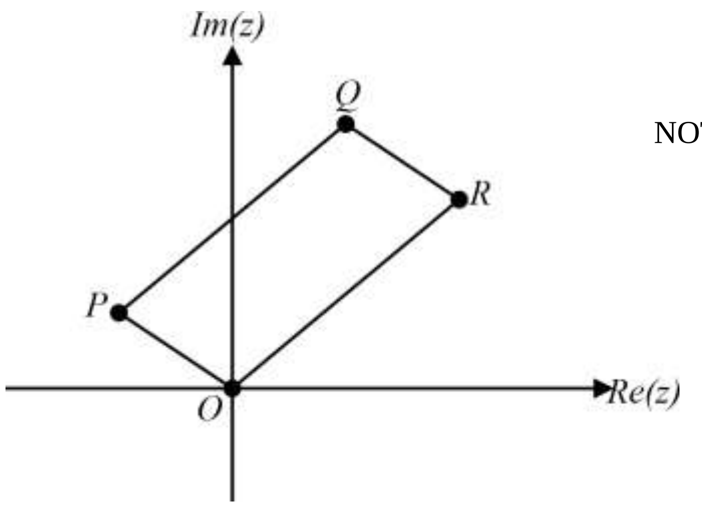
(vi) $y = \log_e f(x)$ **2**

(b) Use the principle of mathematical induction to show that $3^n > n^3$, for positive integers $n > 3$. **3**

Question 13 continues on the next page…

Question 13 continued:

(c)



NOT TO SCALE

In the diagram above, $OPQR$ is a parallelogram with $OP = \frac{1}{2}OR$.

The point P represents the complex number $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

If $\angle POR = 60°$, find in Cartesian form, the complex numbers representing

(i) R; **2**

(ii) Q. **1**

Question 14 commences on the next page

Question 14 (15 marks) **Begin a NEW answer booklet or answer sheet**

(a) Let $y=(x^2-1)\sqrt{x+1}$.

(i) State the domain of $y=(x^2-1)\sqrt{x+1}$. **1**

(ii) Find the x and y–intercept(s). **2**

(iii) Let $\dfrac{dy}{dx}=\dfrac{5x^2+4x-1}{2\sqrt{x+1}}$ and $\dfrac{d^2y}{dx^2}=\dfrac{3(5x^2+8x+3)}{4(x+1)\sqrt{x+1}}$ (DO NOT PROVE THIS). **3**

Find the coordinate(s) of any stationary point(s) and determine their nature.

(iv) Sketch the graph of $y=(x^2-1)\sqrt{x+1}$. **2**

(v) Find the area enclosed by $y=(x^2-1)\sqrt{x+1}$ and the x–axis. **4**

Give your final answer in the form $\dfrac{a\sqrt{b}}{c}$, where a, b and c are non-zero integers.

(b) (i) Expand and simplify $\cos(A+B)-\cos(A-B)$. **1**

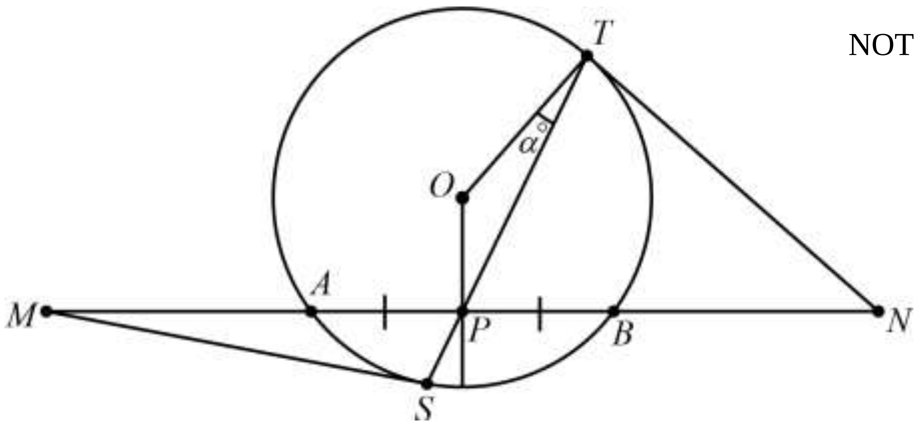
(ii) Hence or otherwise, solve $\cos x-\cos 3x=0$. **2**

Give your final answer(s) in general form.

Question 15 (15 marks) **Begin a NEW answer booklet or answer sheet**

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{1+\cos\theta+\sin\theta}\,d\theta$, using the substitution $t=\tan\frac{\theta}{2}$. **3**

(b)



In the diagram above, P is the midpoint of the chord AB in the circle with centre O. A second chord ST passes through P and the tangents at the endpoints meet AB produced at M and N respectively.

Copy or trace this diagram into your writing booklet.

(i) Explain why $OPNT$ is a cyclic quadrilateral. **2**

(ii) Explain why $OPSM$ is a cyclic quadrilateral. **2**

(iii) Let $\angle OTS=\alpha$. Show that $\angle ONP=\angle OMP=\alpha$. **2**

(iv) Prove that $AM = BN$. **2**

(c) Let $f(x)=2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}(1-x^2)$ for $0\le x\le 1$.

(i) Show that $f'(x)=0$. **2**

(ii) Hence evaluate $\int_0^1 f(x)\;dx$. **2**

Question 16 (15 marks) **Begin a NEW answer booklet or answer sheet**

(a) (i) Sketch the graphs of $f(x) = |3x+3| + |x-1|$ and $g(x) = 4x+3$ on the same number plane. **3**

(ii) Hence, or otherwise, solve for x where $f(x) \le g(x)$. **2**

(b) Let $I_n = \int_0^1 x^n e^{-x} \, dx$, where n is an integer such that $n \ge 0$.

(i) Evaluate I_0. **1**

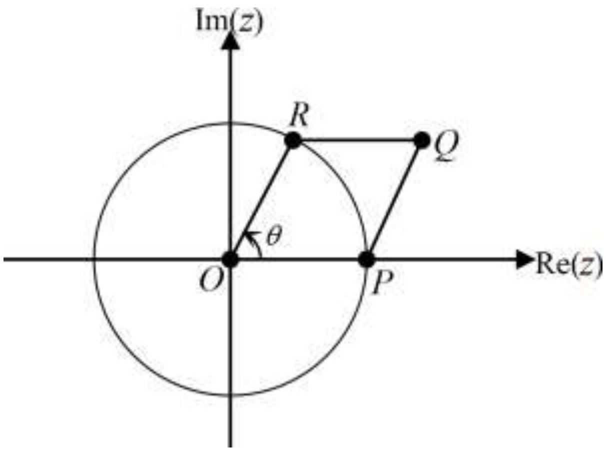
(ii) Show that $I_n = nI_{n-1} - \frac{1}{e}$, for $n > 0$. **2**

(iii) Hence, evaluate I_4. **2**

Question 16 continues on the next page…

Question 16 continued:

(c)



In the diagram above, R represents the complex number $z_1 = \cos\theta + i\sin\theta$.
P represents the complex number $1+0i$ and Q represents the complex number $z_2 = 1 + z_1$. Quadrilateral $OPQR$ is a rhombus.

(i) Show that $z_2 = 2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$. **3**

(ii) Hence, show that $\frac{1}{z_2} = \frac{1}{2} - \frac{i}{2}\tan\frac{\theta}{2}$. **2**

End of Section II

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \quad , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

Note $\ln x = \log_e x, \quad x > 0$

TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 2
Half Yearly Examination, HSC Assessment Task 3
SECTION I
ANSWER SHEET

Name: ……………………………………   **Class Teacher: ………………………..**

Be sure to write your answers for **Section I** on this answer sheet. After you have selected an answer, **CIRCLE** the correct answer. To change an answer, erase your previous mark completely, and then record your new answer. ***Mark only one answer for each question.***

Q1.	A	B	C	D
Q2.	A	B	C	D
Q3.	A	B	C	D
Q4.	A	B	C	D
Q5.	A	B	C	D
Q6.	A	B	C	D
Q7.	A	B	C	D
Q8.	A	B	C	D
Q9.	A	B	C	D
Q10.	A	B	C	D

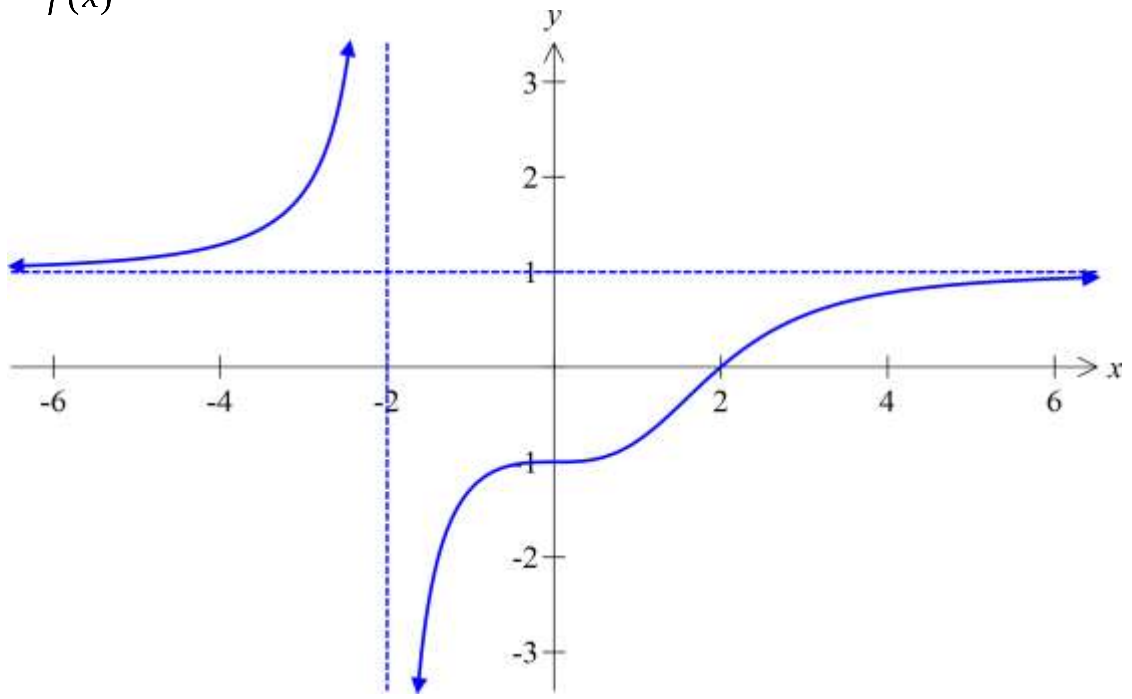
A table of Standard Integrals is printed on the reverse side

TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 2
Half Yearly Examination, HSC Assessment Task 3
SECTION II
QUESTION 13 (a) ANSWER SHEET

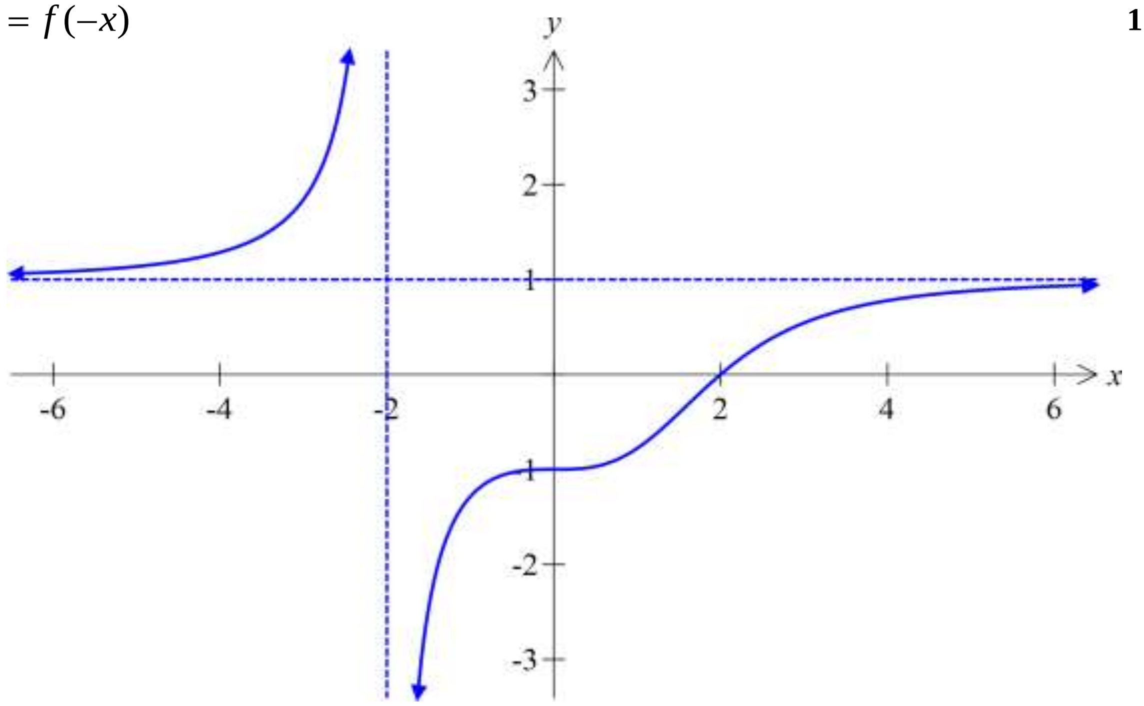
BOS Student Number: ………………………… **Class Teacher: ………………………..**

Each diagram below shows the graph of $y = f(x)$. On the number planes supplied below, sketch:

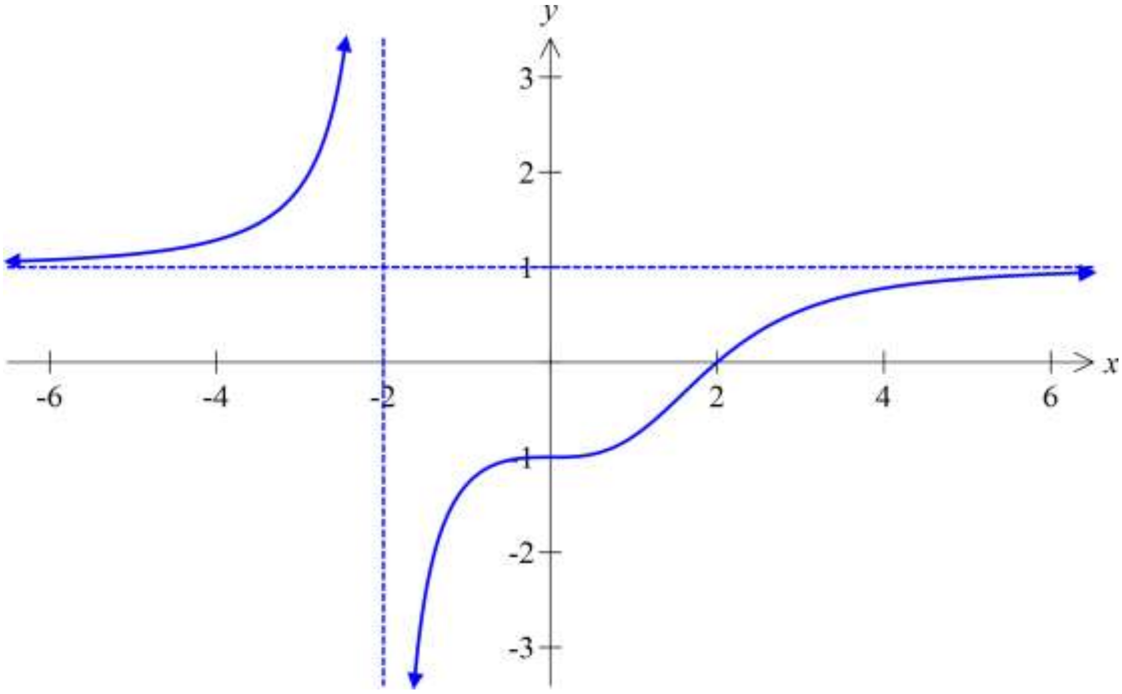
(i) $y = \dfrac{1}{f(x)}$ **2**



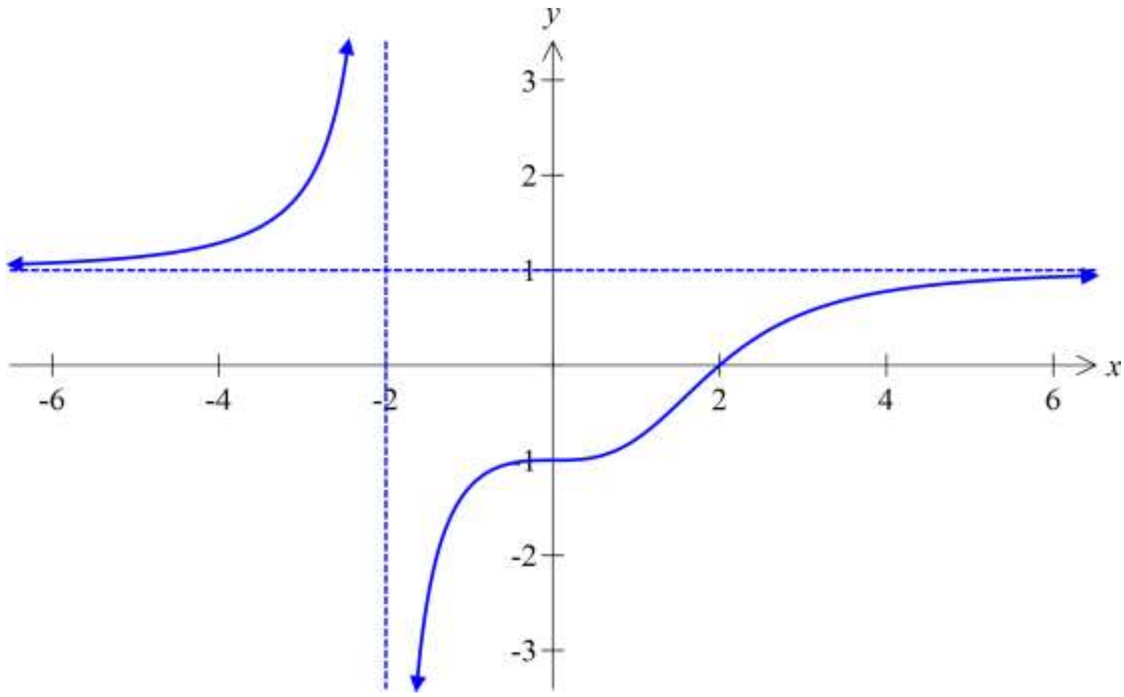
(ii) $y = f(-x)$ **1**



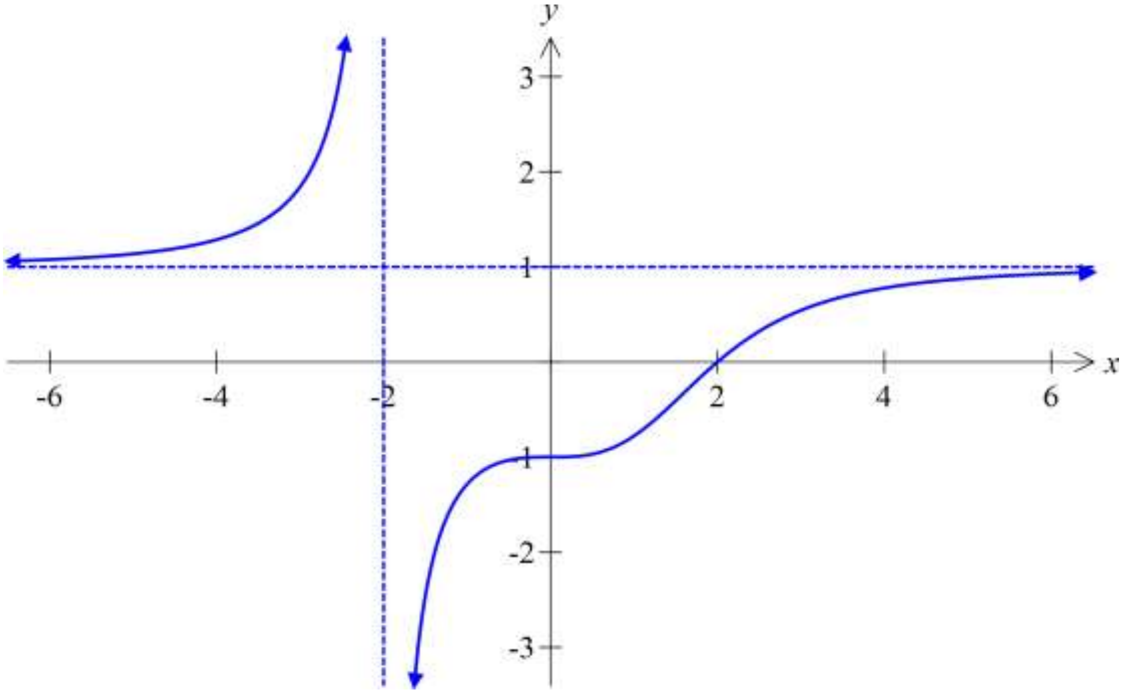
(iii) $y = f\left(|x|\right)$ **1**



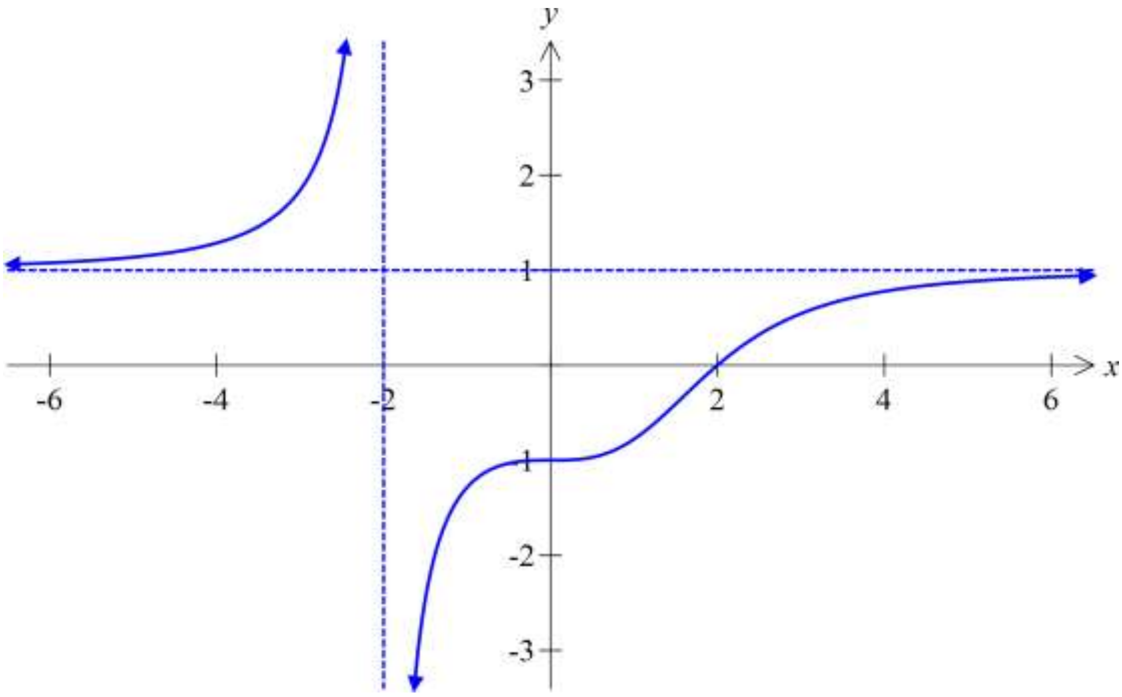
(iv) $|y| = f(x)$ **1**



(v) $y^2 = f(x)$ **2**



(vi) $y = \log_e f(x)$ **2**



End of Section II, Question 13(a)

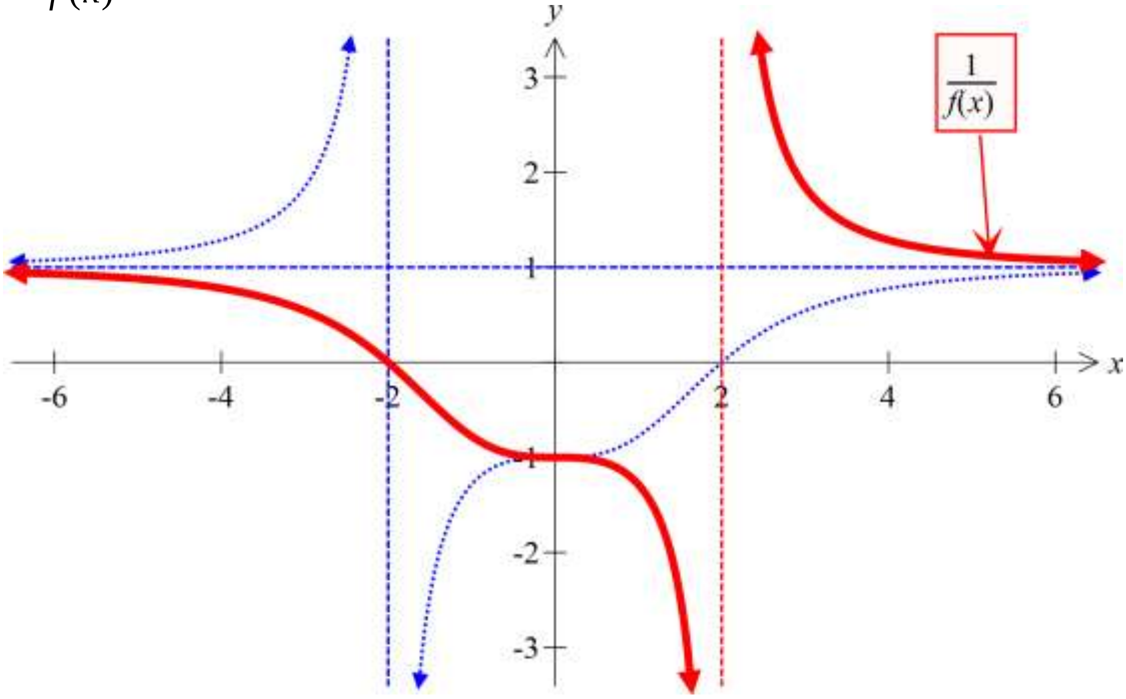
TRINITY GRAMMAR SCHOOL
2012, Year 12 Mathematics Extension 2
Half Yearly Examination, HSC Assessment Task 3
SECTION II
QUESTION 13 (a) ANSWER SHEET

ANSWERS

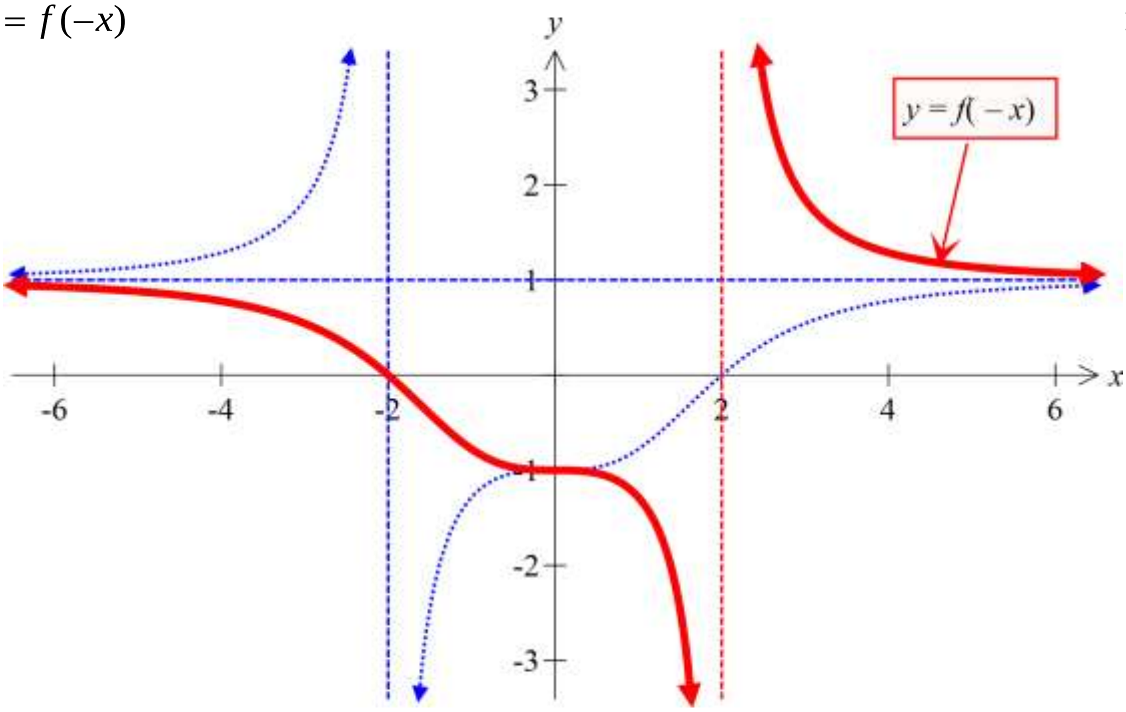
BOS Student Number: .... **Class Teacher:** ....

Each diagram below shows the graph of $y = f(x)$. On the number planes supplied below, sketch:

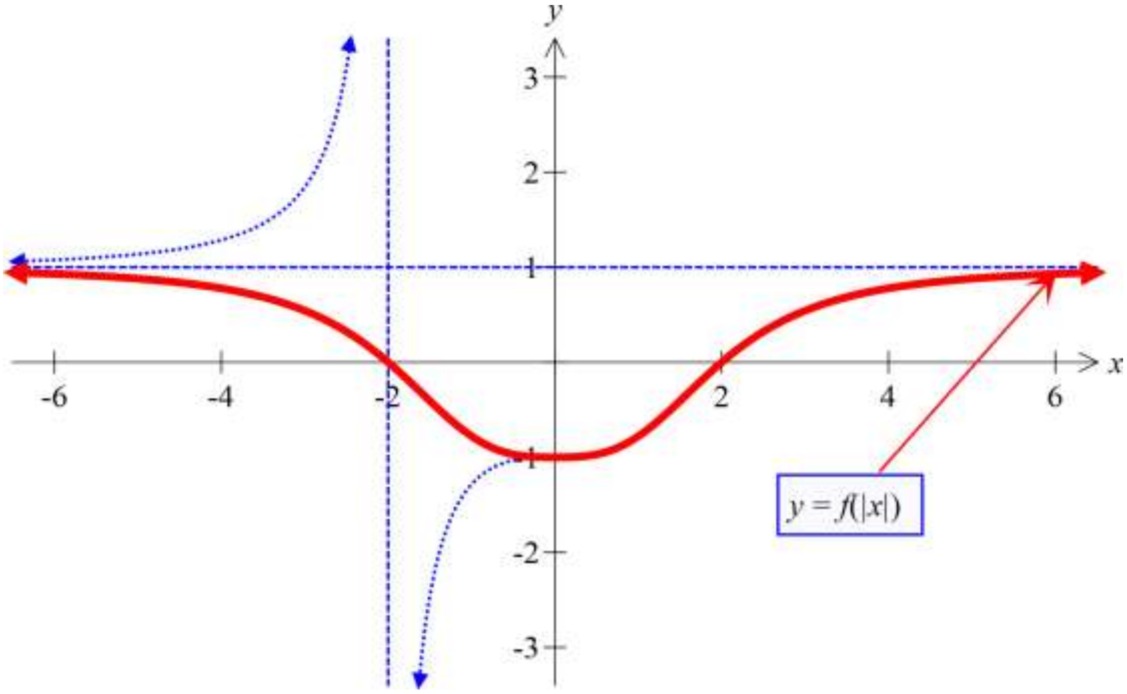
(i) $y = \dfrac{1}{f(x)}$ **2**



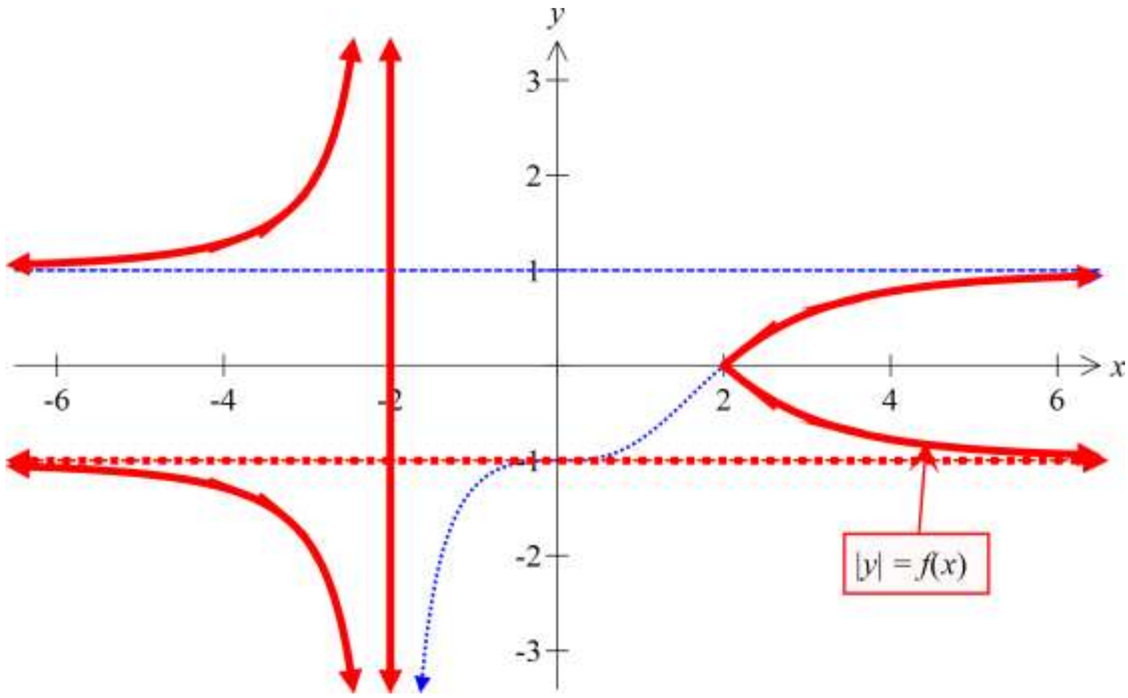
(ii) $y = f(-x)$ **1**



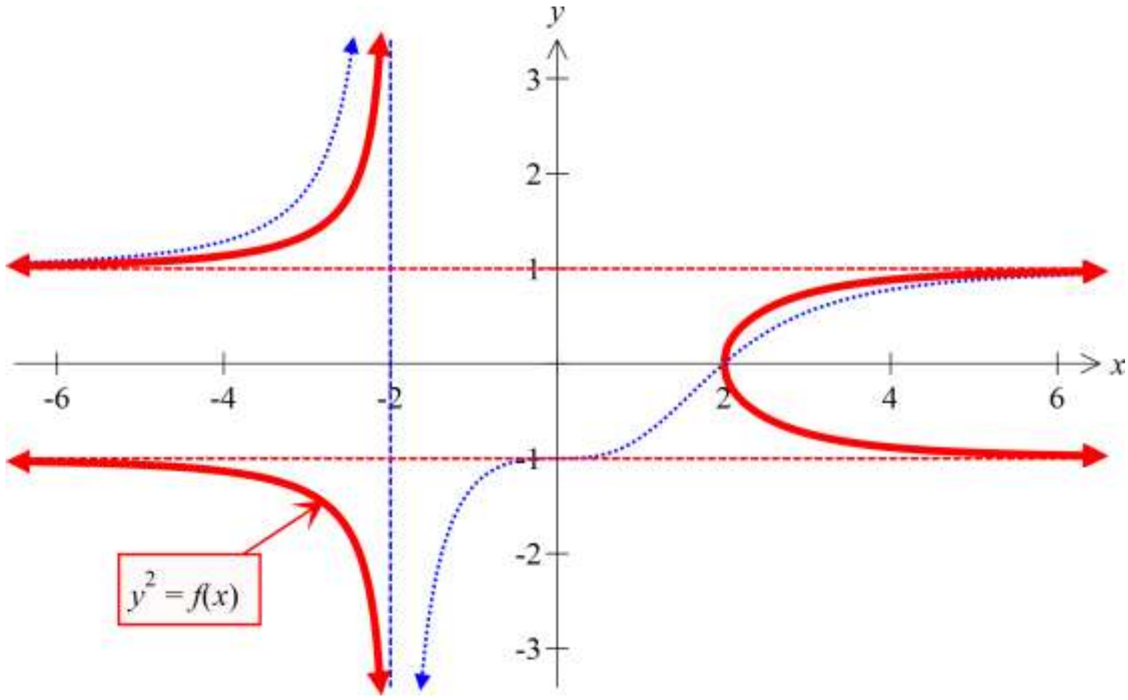
(iii) $y = f\left(|x|\right)$ **1**



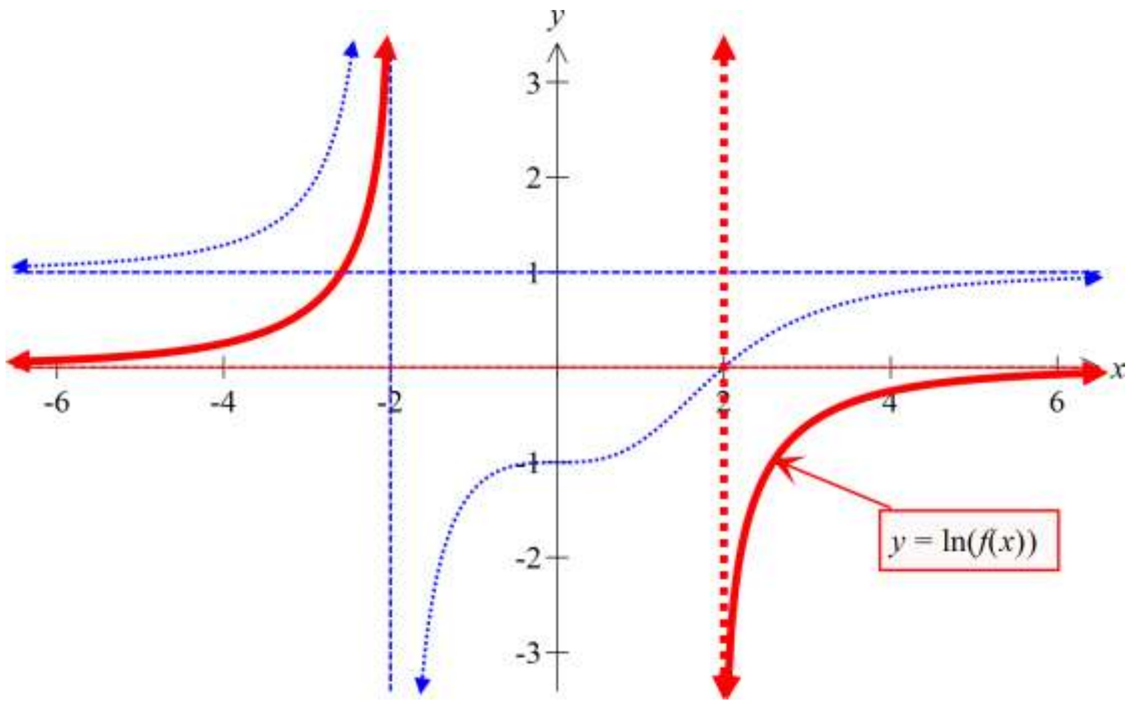
(iv) $|y| = f(x)$ **1**



(v) $y^2 = f(x)$ **2**

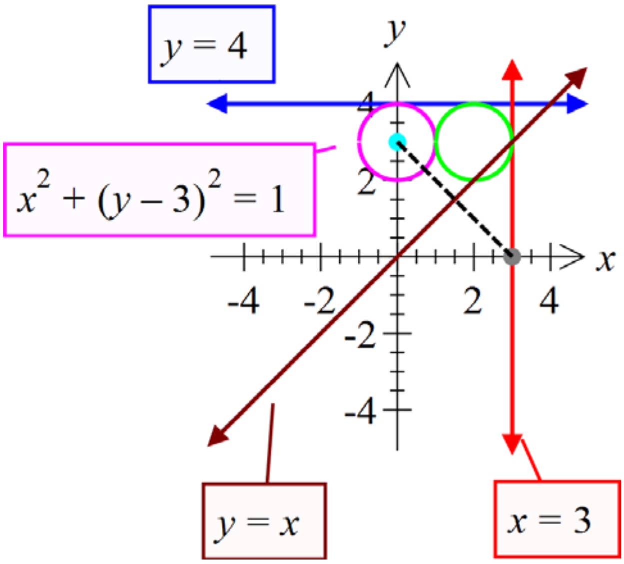


(vi) $y = \log_e f(x)$ **2**



End of Section II, Question 13(a)

SECTION I MULTIPLE CHOICE
QUESTIONS 1–10
MARKING GUIDELINES

	SUGGESTED SOLUTIONS	MARK
1.	Let $u=\tan 2x,\ \frac{du}{dx}=2\sec^2 2x \Rightarrow \frac{du}{2}=\sec^2 2x\,dx$ When $x=0, u=0$ $x=\frac{\pi}{24}, u=\tan\frac{\pi}{12}\quad =\tan\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$ $=\frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$ $=2-\sqrt{3}$ $\therefore \int_0^{\frac{\pi}{24}} \tan 2x\sec^2 2x\ \ dx=\int_0^{2-\sqrt{3}} \frac{u}{2}\ \ du$ **Answer:** D	1
2.	$\lvert z-(2+3i)\rvert=1$ is a circle of radius 1 and centre (2, 3). $\lvert z-3i\rvert=1$ is a circle of radius 1 and centre (0, 3). $\lvert z-3i\rvert=\lvert z-3\rvert$ is the perpendicular bisector of the line joining points (0,3) and (3, 0). $\text{Im}(z)=4$ represents the line $y=4$. $\text{Re}(z)=3$ represents the line $x=3$.  **Answer: B**	1

3.	Using implicit differentiation, $6x^2 - 2y\frac{dy}{dx} = 0$. Therefore $\frac{dy}{dx} = \frac{3x^2}{y}$. When $y = -3,\ 2x^3 - 9 = 7$ *i.e.* $x^3 = 8 \Rightarrow x = 2$ $\therefore \frac{dy}{dx} = \frac{3(2)^2}{-3} = -4$ **Answer: A**	**1**
4.	*A.* $\Rightarrow 2x = 0$ *B.* $\Rightarrow 3x = y$ *C.* $\Rightarrow x + yi = i(x - yi)$ $x + yi = ix + y$ $y(i-1) = x(i-1)$ $\therefore y = x$ *D.* $x + y = 1$ **Answer: D**	**1**
5.	$\lvert z \rvert > 1 \Rightarrow \text{mod}(z) > 1$ $z = \lvert z \rvert cis(-\theta)$ $\therefore$ $\overline{z} = \lvert z \rvert cis(\theta)$ $\Rightarrow \frac{1}{\overline{z}} = \frac{1}{\lvert z \rvert cis(\theta)}$ *i.e.* $\left\lvert \frac{1}{\overline{z}} \right\rvert = \frac{1}{\lvert z \rvert}$ $\arg\left(\frac{1}{\overline{z}}\right) = \arg 1 - \arg \overline{z} = 0 - \theta = -\theta$ $\therefore \frac{1}{\overline{z}} = \frac{1}{\lvert z \rvert} cis(-\theta)$ **Answer: C**	**1**
6.	*A.* $y = \frac{x^3 + a}{x} = x^2 + \frac{a}{x},\ a > 0$ *B.* $y = \frac{x^3 + a}{x} = x^2 + \frac{a}{x},\ a < 0$ *C.* $y = \frac{x^4 + a}{x^2} = x^2 + \frac{a}{x^2},\ a > 0$ *D.* $y = \frac{x^4 + a}{x^2} = x^2 + \frac{a}{x^2},\ a < 0$ **Answer: C**	**1**

7.	$\arg(z^7) = 7\arg z = \frac{7\pi}{5}$ Principal $\arg(z^7) = \frac{7\pi}{5} - 2\pi = -\frac{3\pi}{5}$ **Answer: B**	**1**
8.	$(1+i)^{2n+2} = [(1+i)^n]^2(1+i)^2$ $= [ai]^2(1+2i-1)$ $= -a^2(2i)$ $= -2a^2 i$ **Answer: D**	**1**
9.	$\frac{d}{dx}\cos^{-1}(\sin x) = -\frac{\cos x}{\sqrt{1-\sin^2 x}}$ $= -\frac{\cos x}{\lvert \cos x \rvert}$ $= 1, if\ \cos x < 0$ **Answer: D**	**1**
10.	$z^n + \frac{1}{z^n} = cisn\theta + cis(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ $= 2\cos n\theta$ **Answer: B**	**1**

D B A D C C B D D B

QUESTION 11
MARKING GUIDELINES

(a)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\text{Im}\left(\frac{3+4i}{1+2i}\right) = \text{Im}\left(\frac{3+4i}{1+2i}\times\frac{1-2i}{1-2i}\right)$ $= \text{Im}\left(\frac{3+4i}{1+2i}\times\frac{1-2i}{1-2i}\right)$ $= \text{Im}\left(\frac{11-2i}{5}\right)$ $= -\frac{2}{5}$	• Correct solution	**2**
	• Obtaining $\frac{11-2i}{5}$ or equivalent form • Attempt to realise the denominator e.g. writing anywhere $\frac{3+4i}{1+2i}\times\frac{1-2i}{1-2i}$	**1**

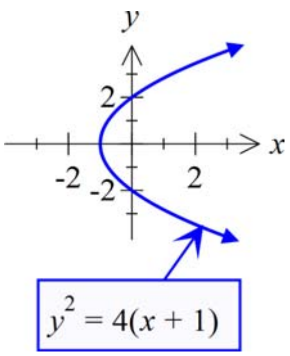
(b)

Suggested solution	CRITERIA/COMMENTS	MARKS
$z=-1+i \quad = \sqrt{(-1)^2+(1)^2}\,cis\left(\pi-\tan^{-1}(\tfrac{1}{1})\right)$ $= \sqrt{2}cis(\pi-\tfrac{\pi}{4})$ $= \sqrt{2}cis\frac{3\pi}{4}$	• Correct answer	**1**

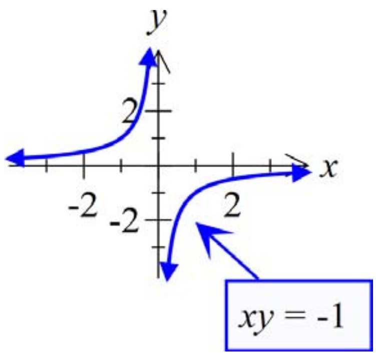
(c)

Suggested solution	CRITERIA/COMMENTS	MARKS
$z^3 = -8cis\pi$ $\therefore \quad z = (-8)^{\frac{1}{3}}cis\left(\frac{\pi+2k\pi}{3}\right)$ *i.e.* $z_1 = -2cis\pi$ $z_2 = -2cis\left(-\frac{\pi}{3}\right)$ $z_3 = -2cis\left(\frac{\pi}{3}\right)$	• Correct solution with three correctly **listed** roots in mod-argument form	**2**
	• Correct solution with three correct roots in Cartesian form • At least one correct root in Cartesian or mod-arg form • Expressing -8 in mod-arg form • Correctly factorising z^3+8 into one linear and quadratic factor • Correctly factorising z^3+8 into three linear factors (one real and two complex factors) • Correctly identifying the roots represent points equally spaced on a circle of radius 2 , centre (0, 0) e.g. argument between each complex number is given by $\frac{2\pi}{n}$, where n = 3, $120°$, *or* $\frac{2\pi}{3}$	**1**

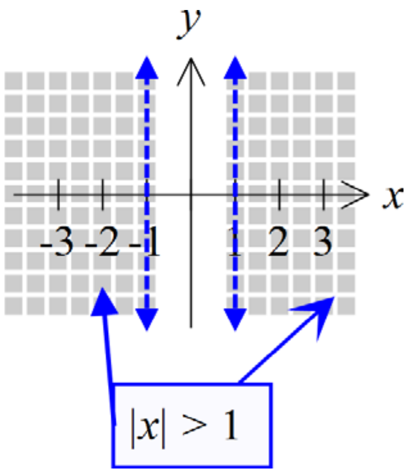
(d) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
Let $z = x + yi$ $2\sqrt{x^2+y^2} = 2x+4$ $\sqrt{x^2+y^2} = x+2$ $x^2+y^2 = x^2+4x+4$ *i.e.* $y^2 = 4(x+1)$  	• Correctly positioned (& directed) graph (parabola with vertex at (-1, 0), focus at (0, 0), *y*-intercepts at *y* = 2 **and** -2) NB: labelled axes are sufficient	2
	• Attempt to find the Cartesian equation of the locus e.g. replacing *z* with *x* + *yi* into both sides of the given complex equation • Correctly positioned (& directed) parabola but unlabelled axes at key features (such as vertex, focus, intercepts)	1

(d) (ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\text{Im}(z^2) = -2$ $\Rightarrow$ *Let* $z = x + yi$ $\text{Im}(z^2) = 2xy$ $\therefore \quad 2xy = -2$ $xy = -1$  	• Correctly positioned hyperbola with vertex at (0, 0) and branch in Q2 and Q4, and asymptotes	2
	• Attempt to find the Cartesian equation of the locus e.g. replacing *z* with *x* + *yi* into the LHS of the given complex equation	1

(d) (iii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\lvert \text{Re}(z) \rvert > 1$ $\Rightarrow Let\ z = x + yi$ $i.e.\quad \lvert x \rvert > 1$ 	• Correctly positioned 'broken' vertical straight lines through $x = 1$ **and** $x = -1$ **and** the correct shaded region	**2**
	• Attempt to find the Cartesian equation of the locus e.g. replacing z with $x + yi$ into the LHS of the given complex equation • Correctly positioned 'broken' or 'unbroken' vertical straight lines through $x = 1$ **and** $x = -1$ • A correctly shaded region (that is extending to the right and left of their x-intercept) but axes unlabelled at $x = 1$ or -1 or their intercept. Must have a pair of vertical straight lines though.	**1**

(e) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$Given\quad 2\lvert z-1 \rvert = \lvert z \rvert$ ....(1) & $\arg(z-1) - \arg z = \dfrac{\pi}{3}$ ....(2) $From\ (1)\ \&\ (2):$ $\dfrac{\lvert z-1 \rvert}{\lvert z \rvert} = \dfrac{1}{2}$ $i.e.\quad \left\lvert \dfrac{z-1}{z} \right\rvert = \dfrac{1}{2}$ $\&\quad \arg\left(\dfrac{z-1}{z}\right) = \dfrac{\pi}{3}$ $Hence:\quad \dfrac{z-1}{z} = \left\lvert \dfrac{z-1}{z} \right\rvert cis\left[\arg\left(\dfrac{z-1}{z}\right)\right]$ $= \dfrac{1}{2} cis \dfrac{\pi}{3}$	• A correct solution, must refer to the fact (or clearly implied from working) that $\left\lvert \dfrac{z_1}{z_2} \right\rvert = \dfrac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$ **and** $\arg(z_1) - \arg(z_2) \pm 2n\pi = \arg\left(\dfrac{z_1}{z_2}\right)$ Answer is already given check solution carefully for 'fudged' attempts • A correct solution by changing the LHS to Cartesian form after solving for z in Cartesian or mod-arg form the complex equations & then establishing the required answer(long-winded approach)	**2**
	• Correctly indicating $\left\lvert \dfrac{z-1}{z} \right\rvert = \dfrac{\lvert z-1 \rvert}{\lvert z \rvert} = \dfrac{1}{2}$ • Correctly indicating $\arg\left(\dfrac{z-1}{z}\right) = \arg(z-1) - \arg(z)$	**1**

(e) (ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$From(i) \quad \dfrac{z-1}{z}=\dfrac{1}{2}cis\dfrac{\pi}{3}$ $i.e. \quad z-1=z\left(\dfrac{1}{2}cis\dfrac{\pi}{3}\right)$ $z\left(1-\dfrac{1}{2}cis\dfrac{\pi}{3}\right)=1$ $z\left(\dfrac{3}{4}-\dfrac{\sqrt{3}}{4}i\right)=1$ $\therefore \quad z=\dfrac{4}{3-\sqrt{3}i}$ $=1+\dfrac{\sqrt{3}}{3}i$	• Correct solution in $x+yi$ form only	2
	• Correct answer in mod-arg form • Reasonable attempt to find z e.g. showing $z=\dfrac{4}{3-\sqrt{3}i}$ or equivalent but not in simplest Cartesian form $x+yi$	1

QUESTION 12
MARKING GUIDELINES

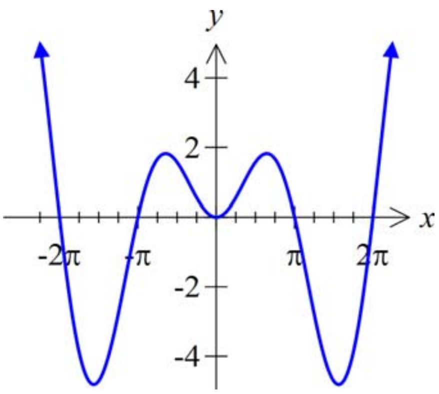
(a)

Suggested solution	CRITERIA/COMMENTS	MARKS
$Let\ u=\ln x \ \therefore \dfrac{du}{dx}=\dfrac{1}{x}$ $when\ x=1, u=\ln 1=0$ $when\ x=e^3, u=\ln e^3=3$ $\therefore \displaystyle\int_1^{e^3}\frac{(\ln x)^3}{x}dx=\int_1^{e^3}\frac{(\ln x)^3}{x}\frac{dx}{du}du$ $=\displaystyle\int_0^3 u^3\ du$ $=\left.\dfrac{u^4}{4}\right]_0^3$ $=\dfrac{81}{4}-0$ $=\dfrac{81}{4}(or\ 20\tfrac{1}{4}\ or\ 20.25)$	• Correct solution using the substitution given (& specified technique) only	3
	• Correct solution using a modified primitive and substitution or equivalent **and/or** other appropriate method • Correctly finding the new limits (in terms of u) **and** $\dfrac{du}{dx}$ **and** establishing the integrand of u^3 (seen anywhere) **but** the answer is incorrect.	2
	• Correctly finding the new limits in terms of u • Correctly finding their $\dfrac{du}{dx}$ • Reasonable attempt to use the substitution technique (any substitution) • Find a correct primitive of their integrand • Correct answer from their incorrect integral statement	1

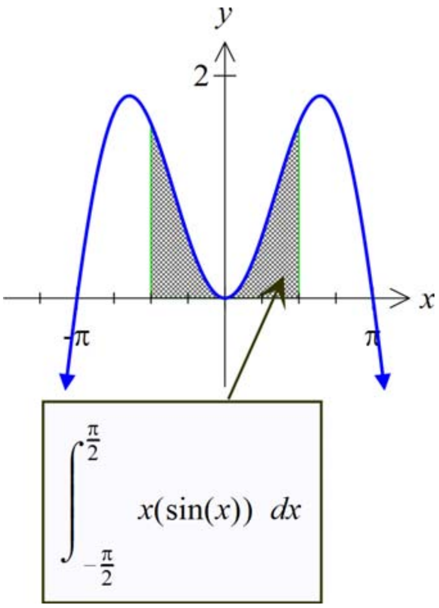
(b)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\int \cos^3 x \; dx = \int \cos^2 x \cos x \; dx$ $= \int \left(1-\sin^2 x\right)\cos x \; dx$ $= \int \cos x - \cos x(\sin x)^2 \; dx$ $= \sin x - \frac{\sin^3 x}{3} + c$ *or* $\int \cos^3 x \, dx = \frac{1}{4}\left(\frac{\sin 3x}{3} + 3\sin x\right) + c$	• Correct solution in terms of x using any method such as a modified integrand by substitution or integration by parts (+c is not required)	**2**
	• Reasonable attempt to find the primitive of the given expression (+c is not required)	**1**

(c) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$f(x) = x\sin x$ $f(-x) = (-x)\sin(-x)$ $= -x(-\sin x)$ $= x\sin x$ $= f(x)$ *Since* $f(-x) = f(x)$ *then* $f(x)$ *is an even function* 	• **Algebraically:** Correct solution clearly showing a substitution of x with $-x$ **and** the property $\sin(-x) = -\sin x$ (an odd function) • **Graphically:** A sketch graph of $y = f(x)$ would suffice showing symmetry about $x = 0$.	**2**
	• Reasonable attempt to sketch the graph of $y = f(x)$ or substituting x with $-x$ in the given expression • Identifying both x and $\sin x$ are odd functions by perhaps drawing their individual graphs and then attempting to use multiplication of ordinates to deduce $y = f(x)$ is an even function	**1**

(c) (ii)

<table>
<tr><th>Suggested solution</th><th>CRITERIA/COMMENTS</th><th>MARKS</th></tr>
<tr><td rowspan="3">

$A = 2\int_0^{\frac{\pi}{2}} x\sin x \ \ dx$

$Let \quad u = x \quad dv = \sin x$

$du = 1 \quad v = -\cos x$

$\therefore A = 2\left[uv - \int v \ \ du\right]$

$= 2\left[\left(-x\cos x\right)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \ \ dx\right]$

$= 2\left[0 + \int_0^{\frac{\pi}{2}} \cos x \ \ dx\right]$

$= 2\int_0^{\frac{\pi}{2}} \cos x \ \ dx$

$= 2\times(\sin x)\Big|_0^{\frac{\pi}{2}}$

$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 2\times 1$

$= 2 \ sq.units$</td><td>• Correct solution using only the method of integration by parts</td><td>3</td></tr>
<tr><td>• Setting up a correct integral statement with correct limits leading to the requested area and uses integration by parts to find a correct primitive but makes an error along the way</td><td>2</td></tr>
<tr><td>• Correct integral statement
• Correct primitive based on their integral statement
• Correct answer using their incorrect integral or primitive statement
• Attempts to use integration by parts e.g. attempts to label u, dv, v, and du seen anywhere</td><td>1</td></tr>
</table>

(d) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$9+x-2x^2 \equiv A(3+x^2)+(Bx+C)(1-x)$ $Let\ x=1,$ $\therefore \quad 8=4A$ $i.e. \quad A=2$ $Let\ x=\sqrt{3}\,i$ $\therefore 9+\sqrt{3}\,i-2(-3)=(B\sqrt{3}\,i+C)(1-\sqrt{3}\,i)$ $=B\sqrt{3}\,i+3B+C-C\sqrt{3}\,i$ $i.e. \quad \begin{cases} 15=3B+C \\ 1=B-C \end{cases}$ $Solving,\ B=4, C=3$ $\therefore (A,B,C)=(2,4,3)$ Note that other values for *x* can be chosen such as *x* = -1, and *x* = 0.	• Correct solution	**3**
	• Correct values for *A*, *B* **and** *C* without working • Correct values for **any two** of *A*, *B* or *C* **with** supporting working	**2**
	• A correct value for *A*, or *B* or *C* with/without appropriate working • Writing the statement or equivalent $9+x-2x^2 \equiv A(3+x^2)+(Bx+C)(1-x)$ • Substituting up to 3 different values of *x* and attempting to solve their simultaneous equations to find a value for *A*, *B*, or *C*	**1**

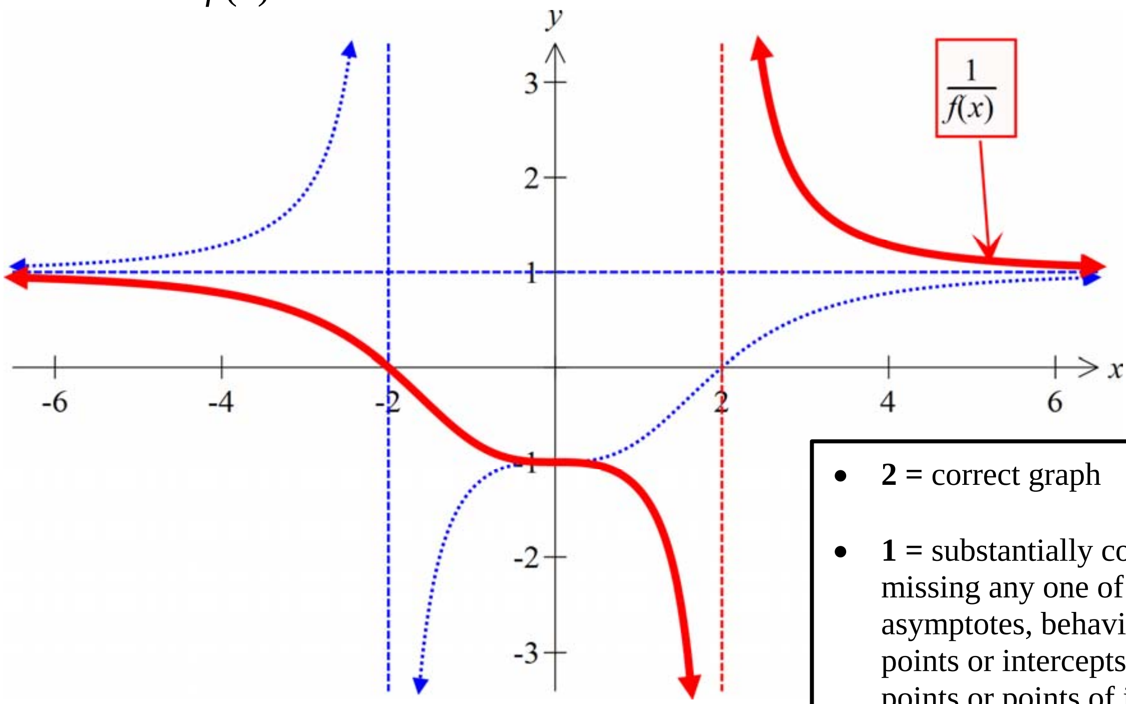
(d) (ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\int \frac{9+x-2x^2}{(1-x)(3+x^2)}\,dx = \int \frac{2}{1-x}+\frac{4x+3}{3+x^2}\,dx$ $=\int \frac{2}{1-x}+\frac{4x}{3+x^2}+\frac{3}{3+x^2}\,dx$ $=-2\ln\lvert 1-x\rvert+2\ln(3+x^2)+\frac{3}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+c$	• Correct solution of their integrand in terms of *x* using a modified integrand by partial fractions from (i) (provided the integral has not been made easier than intended) (+*c* is not required)	**2**
	• Reasonable attempt to find a primitive of the given expression e.g. Splitting the integrand into partial fractions • Correct solution of their integrand in terms of *x* using any method other than by partial fractions (+*c* is not required)	**1**

QUESTION 13
MARKING GUIDELINES

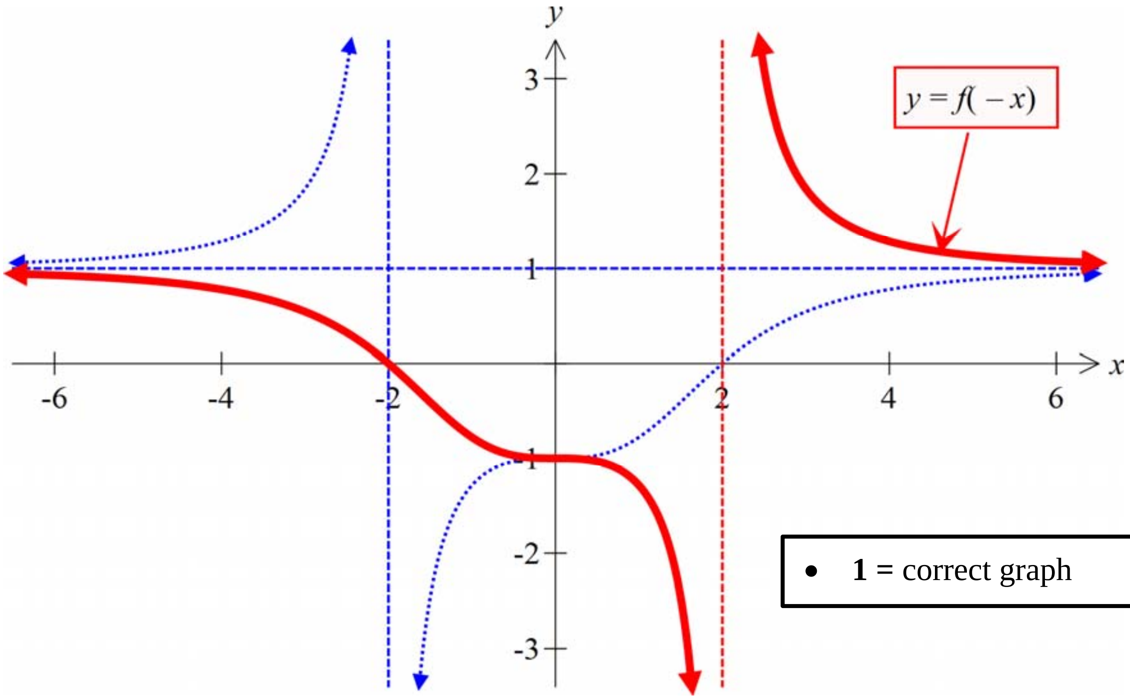
(a)

(i) $y = \dfrac{1}{f(x)}$ **2**



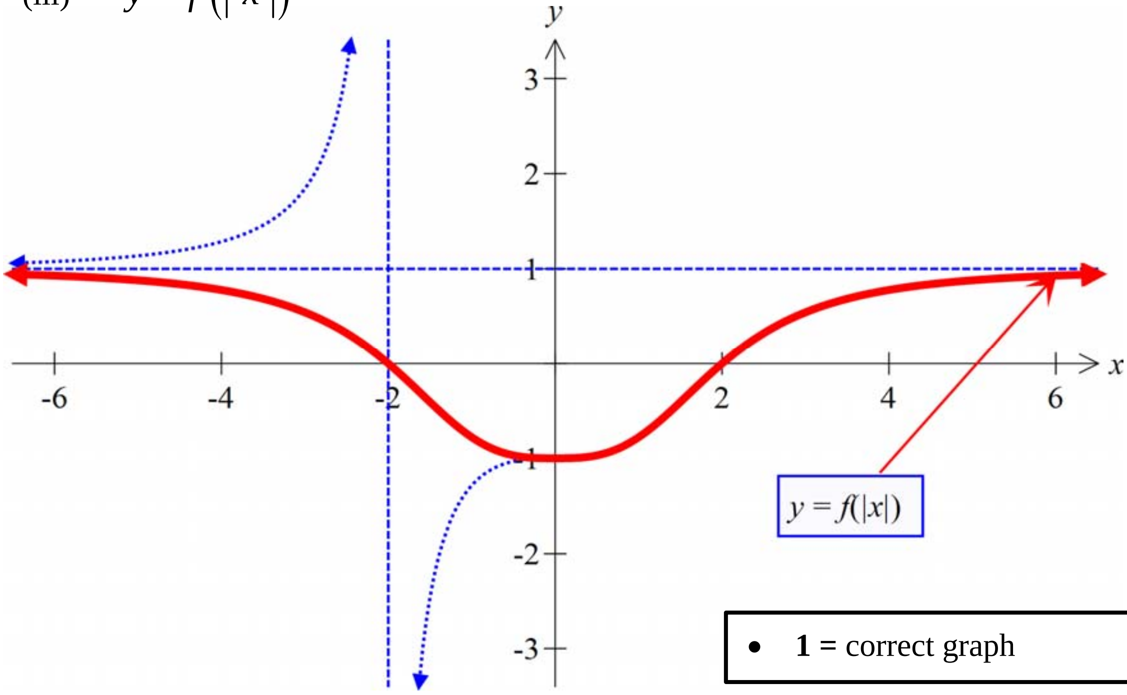
- **2** = correct graph
- **1** = substantially correct graph but missing any one of intercepts, asymptotes, behaviour at end points or intercepts, or stationary points or points of inflexion

(ii) $y = f(-x)$ **1**



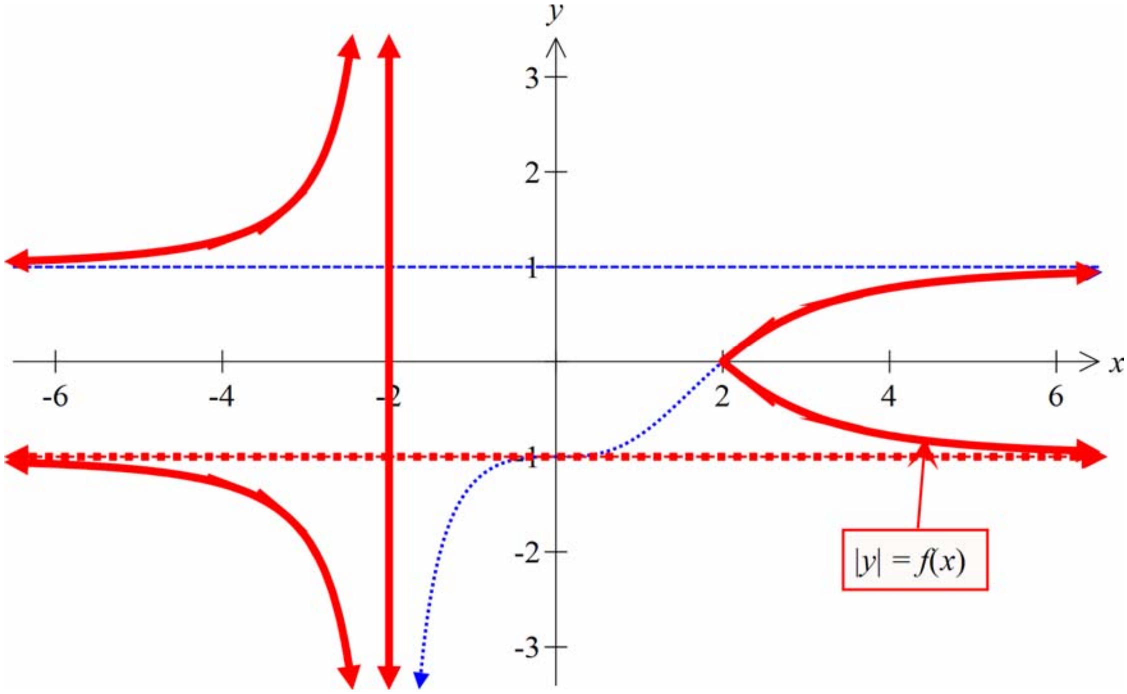
- **1** = correct graph

(iii) $y = f\left(\left| x \right|\right)$ **1**



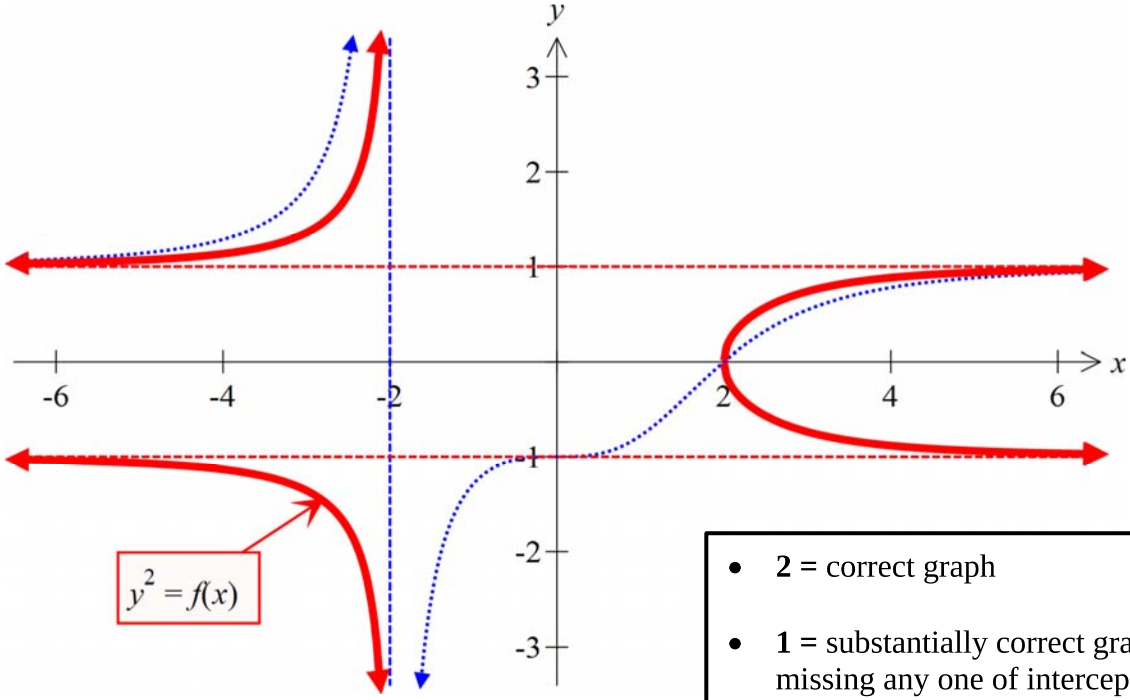
- **1 =** correct graph

(iv) $\left| y \right| = f(x)$ **1**



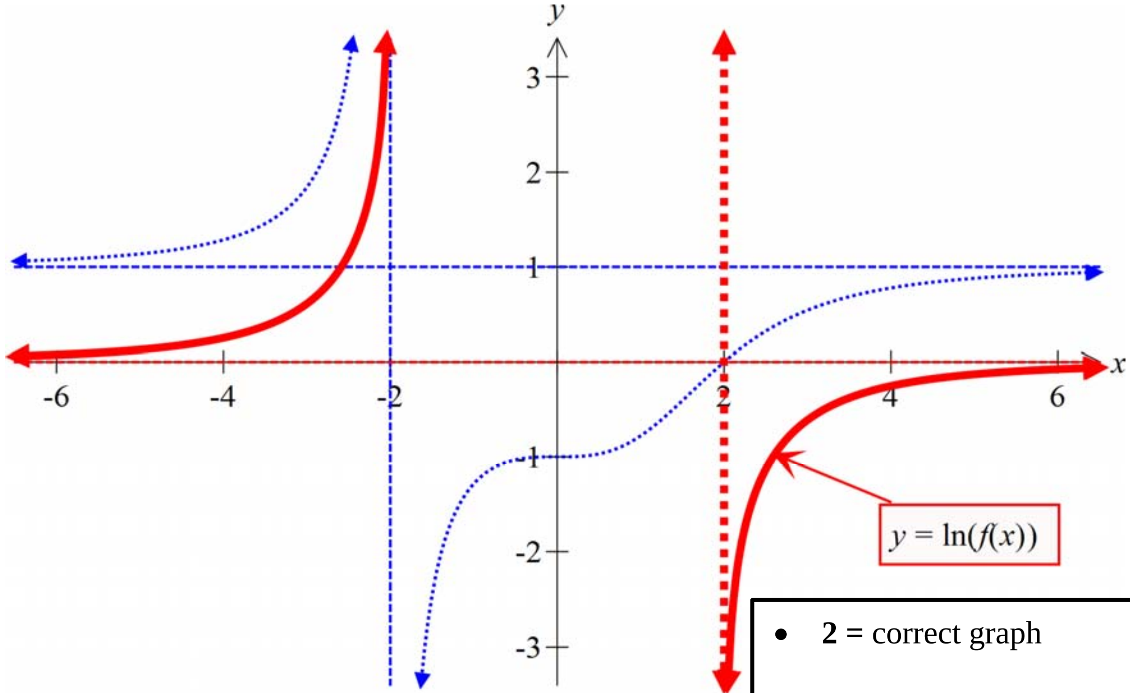
- **1 =** correct graph

(v) $y^2 = f(x)$ **2**



- **2 =** correct graph
- **1 =** substantially correct graph but missing any one of intercepts, asymptotes, behaviour at end points or intercepts, or stationary points or points of inflexion

(vi) $y = \log_e f(x)$ **2**



- **2 =** correct graph
- **1 =** substantially correct graph but missing any one of intercepts, asymptotes, behaviour at end points or intercepts, or stationary points or points of inflexion

(b) (ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
Let $S(n)$ be the statement $3^n > n^3$ Step 1: Show $S(4)$ is true $LHS = 3^4 = 81$ $RHS = 3^3 = 27$ SINCE $81 > 27$ then $S(4)$ is true Step 2: Assume $S(k)$ is true for $3 < k \le n$, $(k,n) \in \mathbb{Z}^+$ i.e. $3^k > k^3$ (1) Aim to prove $S(k+1)$ is true i.e. $3^{k+1} > (k+1)^3$ (2) The statement (2) can be shown if we can show that for $k > 3$, $3^{k+1} - (k+1)^3 > 0$. $3^{k+1} - (k+1)^3 = 3(3^k) - (k+1)^3$ $> 3k^3 - (k+1)^3$ from (1) $= 3 \times 4^3 - 5^3, \quad k = 4$ $= 67$ > 0, if $k > 3$ Since $S(1)$ and if $S(k)$ is true then $S(k+1)$ is also true. So by the principle of mathematical induction $3^n > n^3$, for integral $n > 3$.	• Correct solution including verification of the initial case	3
	• Establishes the truth of $S(k+1)$ *by correctly using the assumption* $S(k)$ **but** ignores the initial case or equivalent method	2
	• Verifies at least *the statement* is true for $n = 4$ • Writes down a correct assumption e.g. Assume $3^k > k^3$ **or** $3^k - k^3 > 0$ No penalty for omission of a conclusion Other methods are available such as Calculus or consider: $3^{\mathbf{k}} > \mathbf{k}^3$ (**assumptn**) $\therefore 3.3^{\mathbf{k}} > 3\mathbf{k}^3$ $3^{\mathbf{k}+1} > 3\mathbf{k}^3$ *Is* $3\mathbf{k}^3 > (\mathbf{k}+1)^3$ **Least value k** = 4. $3 \times 64 > 4^3$ This suggests that $(k+1)^3 > 64$ (at least)	1

(c) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\lvert \overrightarrow{OR} \rvert = 2\lvert \overrightarrow{OP} \rvert$ $= 2\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$ $= 2 \times 1$ $= 2$ $\arg(\overrightarrow{OP}) = 180° - \tan^{-1}(\sqrt{3})$ $= 60°$ $\therefore \arg(\overrightarrow{OR}) = (120 - 60)° = 60°$ i.e. $\overrightarrow{OR} = 2cis60°$ $= 1 + \sqrt{3}i$	• A correct solution in correct modulus and argument form **or** in correct Cartesian form	2
	• Numeric form of either one of $\lvert \overrightarrow{OP} \rvert, \lvert \overrightarrow{OR} \rvert = 2\lvert \overrightarrow{OP} \rvert, \arg(\overrightarrow{OP})$, *or* $\arg(\overrightarrow{OR})$ • Recognising that $\overrightarrow{OR}$ is a rotation of $\overrightarrow{OP}$ clockwise by 60° but twice the modulus of $\overrightarrow{OP}$.	1

ALTERNATIVE: $Given \quad \angle POR = 60°$ To rotate $\overrightarrow{OP}$ clockwise by $60°$ we need to mutliply $\overrightarrow{OP}$ by $cis(-60°)$. $i.e. \quad \overrightarrow{OR} = 2\times\overrightarrow{OP}\times cis(-60°)$ $= 2cis120°cis(-60°)$ $= 2cis60°$		

(c) (ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\overrightarrow{OQ} = \overrightarrow{OR}+\overrightarrow{OP}$ $=1+\sqrt{3}i+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$ $=\frac{1}{2}+\left(\sqrt{3}+\frac{\sqrt{3}}{2}\right)i$ $\left(\cup\frac{1}{2}+\frac{3\sqrt{3}}{2}\mathbf{i}\right)$	• Correct answer in Cartesian form only	**1**

QUESTION 14
MARKING GUIDELINES

(a) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$Want\ x+1\geq 0$ $i.e. \quad D: x\geq -1$	• Correct answer	**1**

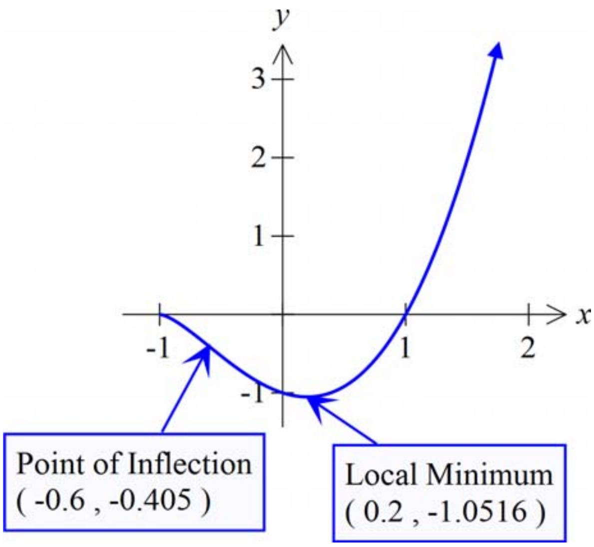
(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$Y-INTERCEPT: \quad x=0,\ y=-1$ $X-INTERCEPT: \quad y=0,\ x^2-1=0\ or$ $x+1=0$ $i.e.\ x=\pm 1$	• Correct answers **one** y-intercept and **two** x-intercepts	**2**
	• One x-intercept **or** y-intercept	**1**

(iii)

Suggested solution	CRITERIA/COMMENTS	MARKS
Given $\frac{dy}{dx}=\frac{5x^2+4x-1}{2\sqrt{x+1}}$ and $\frac{d^2y}{dx^2}=\frac{3(5x^2+8x+3)}{4(x+1)\sqrt{x+1}}$ $SP's\ when\ \frac{dy}{dx}=0$	• Correct solution/justification **and** establishing the nature **and/or** feature at $\left(\frac{1}{5},-\frac{24\sqrt{30}}{125}\right)$ using Calculus **or** equivalent method	**3**
i.e. *when* $5x^2+4x-1=0$ $(5x-1)(x+1)=0$ $x=\frac{1}{5}$ *or* $x=-1$ *When* $x=\frac{1}{5},\ y=-\frac{24\sqrt{30}}{125}\left(\approx -1.05\right)$	• Correctly solves $\frac{dy}{dx}=0$ giving the **two** *x*-values • Correctly finds the coordinates $\left(\frac{1}{5},-\frac{24\sqrt{30}}{125}\right)$ or equivalent	**2**
When $x=-1,\ y=0$ *NATURE USING* y''. *At* $x=\frac{1}{5}, y''=\frac{3(5(\frac{1}{5})^2+8(\frac{1}{5})+3)}{4(\frac{1}{5}+1)\sqrt{\frac{1}{5}+1}}$ >0 $\therefore \left(\frac{1}{5},-\frac{24\sqrt{30}}{125}\right)$ *is a rel.* min. *SP* *At* $x=-1, y''$& y' *are inconclusive tests.* To establish what is happening at the point $(-1,\ 0),$ note that there is a possible point of inflexion when $5x^2+8x+3=0.$ *That is when* $(x+1)(5x+3)=0$ *or at* $x=-1$ & $x=-\frac{3}{5}$. This suggests that the curve has greatest negative slope at $x=-\frac{3}{5}$, *but the slope at* $x=-1$ is undefined because it is an endpoint. Given $x\geq -1$,we suspect that the nature to the right of (-1,0) is a 'half' concave down feature to where $x=-\frac{3}{5}$. For $-1<x<-\frac{3}{5}, y''<0.$ *Verify this by substitution of* $x=-\frac{3}{5}$ into y''.	• Correctly solves $\frac{dy}{dx}=0$ giving one *x*-value (either $x=-\frac{1}{5}$ *or* $x=-1$)	**1**

(iv)

Suggested solution	CRITERIA/COMMENTS	MARKS
	• Correct graph showing all important features such as the behaviour at x = -1, intercepts and SP's	2
	• Correct graph showing all important features but makes an error in at least one of behaviour at x = -1, intercepts, SP's or POI's	1

(v)

Suggested solution	CRITERIA/COMMENTS	MARKS
$A = \left\lvert \int_{-1}^{1} (x^2-1)\sqrt{x+1}\ \ dx \right\rvert$ $Let\ u = x+1, \frac{du}{dx} = 1,$ $when\ x=-1, u=0$ $when\ x=1, u=2$ $\therefore A = \left\lvert \int_{0}^{2} (u^2-2u)\sqrt{u}\ \ du \right\rvert$ $= \left\lvert \int_{0}^{2} u^2\sqrt{u} - 2u\sqrt{u}\ \ du \right\rvert$ $= \left\lvert \int_{0}^{2} u^{\frac{5}{2}} - 2u^{\frac{3}{2}}\ \ du \right\rvert$ $= \left\lvert \frac{2}{7}u^3\sqrt{u} - \frac{4}{5}u^2\sqrt{u} \right\rvert_0^2$ $= \left\lvert \frac{16}{7}\sqrt{2} - \frac{16}{5}\sqrt{2} \right\rvert$ $= \frac{32}{35}\sqrt{2}\ sq.units$	• Correct solution in the form $\frac{a\sqrt{b}}{c}$	4
	• A correct primitive of $(x^2-1)\sqrt{x+1}$, using a substitution such as $u = x+1$ $or\ u^2 = x+1$ **or** any appropriate method leading to the correct answer (if $u^2 = x+1$ is chosen then $x^2 - 1 = (u^2-1)^2 - 1$ • A correct primitive of **their** incorrect definite integral **and** an attempt to evaluate this definite integral but makes an error along the way (the integral expression must not be easier than intended, otherwise award a maximum of 2 marks)	3
	• A correct definite integral expression **with/without** the absolute value sign, for example $A = \left\lvert \int_{-1}^{1} (x^2-1)\sqrt{x+1}\ \ dx \right\rvert$, $\int_{0}^{2} (u^2-2u)\sqrt{u}\ \ du$	2
	• An attempt to use integration to find the area of the region specified	1

(b) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$	• Correct answer	**1**

(ii)

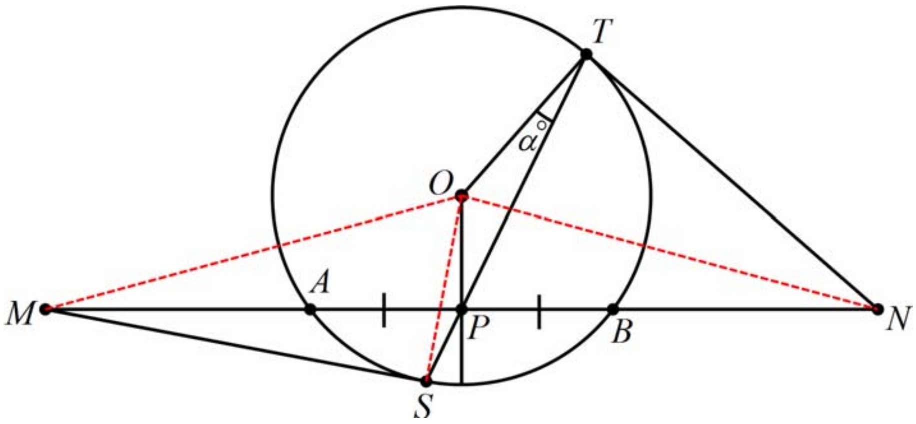
Suggested solution	CRITERIA/COMMENTS	MARKS
$\cos x-\cos 3x=-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)$ $=-2\sin 2x\sin(-x)$ $=2\sin 2x\sin x$ $\therefore 2\sin 2x\sin x=0$ *i.e.* $\sin 2x=0$ *or* $\sin x=0$ $\therefore x=\frac{n\pi}{2}$ $\quad x=n\pi$ Since the solution $x=n\pi$ is generated in the solution $x=\frac{n\pi}{2}$, for some integer value of *n*, *then* $x=\frac{n\pi}{2}$ *only*.	• Correct solution using any appropriate method; answer must be in general form (but it is not unique)	**2**
	• Simplifying $\cos x-\cos 3x$ *to* $2\sin x\sin 2x$ or equivalent. • Stating without working $x=\frac{n\pi}{2}$ *or* $x=n\pi$ • An attempt to solve the given equation	**1**

QUESTION 15
MARKING GUIDELINES

(a)

Suggested solution	CRITERIA/COMMENTS	MARKS
Let $t=\tan\frac{\theta}{2},\ \frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}\ \&\ d\theta=\frac{2dt}{1+t^2}$ *Also,* $\cos\theta=\frac{1-t^2}{1+t^2},\sin\theta=\frac{2t}{1+t^2}$ $\therefore\int_0^{\frac{\pi}{2}}\frac{2}{1+\cos\theta+\sin\theta}\,d\theta$ $=\int_0^1\frac{2}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\frac{2}{1+t^2}\,dt$ $=2\int_0^1\frac{1}{1+t}\,dt$ $=2\ln\lvert 1+t\rvert_0^1$ $=2\ln 2-2\ln 1$ $=2\ln 2$ (*or* $\ln 4$)	• Correct solution (must use the given substitution up to 2nd last step at least)	**3**
	• Substantial evidence of using the substitution method on a correct integrand but made an error along the way (if limits are incorrect or if no change of limits or if answer is left as an indefinite answer award a maximum of 1 mark)	**2**
	• Correct change of limits or variable(s) or finding in terms of *t*, *du,* $\frac{d\theta}{dt}$ • Attempt to use the given substitution	**1**

(b) (i)



Construct *OM, ON and OS* as shown in the diagram above

Suggested solution	CRITERIA/COMMENTS	MARKS
$\angle OTN = 90°$(NT is a tangent to circle at T) $\angle OPN = 90°$(OP is $\perp$ bisector of chord AB) $\Rightarrow \angle OTN + \angle OPN = 180°$ $\therefore$ *OPNT is a cyclic quadrilateral* (opposite angles of a quadrilateral are supp.) (or both are angles in a semi-circle on ON)	• Correct solution including complete reasons	2
	• $\angle OTN = 90°$ *or* $\angle OPN = 90°$ without reasons • $\angle MPO = \angle OTN = 90°$ exterior angle of cyclic quad equals int. opp. angle	1

(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
Construct OM and OS $\angle OSM = 90°$(MS is a tangent to circle at S) $\angle OPM = 90°$(OP is $\perp$ bisector of chord AB) $\Rightarrow \angle OSM = \angle OPM$ (equal angles standing on same line OM) $\therefore$ *OPSM is a cyclic quadrilateral* (opposite angles of a quadrilateral are supp.) (*i.e.* OM is a diameter)	• Correct solution including complete reasons	2
	• $\angle OSM = 90°$ *or* $\angle OPM = 90°$ without reasons	1

(iii)

Suggested solution	CRITERIA/COMMENTS	MARKS
OPNT is a cyclic quadrilateral $\angle ONP = \angle OTP = \alpha$ (*angles on same chord OP*) $\angle OSP = \angle OTS = \alpha$ (*base angles in isosceles* ΔOST) *OPSM is a cyclic quadrilateral* $\therefore \angle OMP = \angle OSP = \alpha$ (*angles on same chord OP*) $\therefore \angle OMP = \angle ONP$	• Correct solution including complete reasons (must refer to the cyclic quads. established in in (i) and (ii) in some way and a link made) Paraphrased reasons or in-concise reasons are unacceptable.	2
	• Noting at least one other angle equivalent to $\angle OTS$ with or without reasons	1

(iv)

Suggested solution	CRITERIA/COMMENTS	MARKS
ΔOMN *is isosceles* (*base angles equal*) OP *is an altitude of* ΔOMN $\therefore MP = PN$ $AM = MP - AP$ $= PN - PB$ $= BN$	• Correct solution including complete reasons • Proving $\Delta OPN \equiv \Delta OPM$	**2**
	• OP is an altitude of isosceles triangle OMN	**1**

(c) (i)

Suggested solution	CRITERIA/COMMENTS	MARKS
$f(x) = 2\cos^{-1}\frac{x}{\sqrt{2}} - \sin^{-1}(1-x^2)$ $f'(x) = -2\left[\frac{\frac{d}{dx}\left(\frac{x}{\sqrt{2}}\right)}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\right] - \frac{\frac{d}{dx}(1-x^2)}{\sqrt{1-(1-x^2)^2}}$ $= \frac{-2\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}}\sqrt{2-x^2}} + \frac{2x}{\sqrt{1-(1-2x^2+x^4)}}$ $= -\frac{2}{\sqrt{2-x^2}} + \frac{2x}{\sqrt{2x^2-x^4}}$ $= -\frac{2}{\sqrt{2-x^2}} + \frac{2x}{x\sqrt{2-x^2}}$ $= 0$	• Correct simplified derivatives of $\cos^{-1}\frac{x}{\sqrt{2}}$ **and** $\sin^{-1}(1-x^2)$	**2**
	• Correct derivative of either $\cos^{-1}\frac{x}{\sqrt{2}}$ **or** $\sin^{-1}(1-x^2)$ unsimplified	**1**

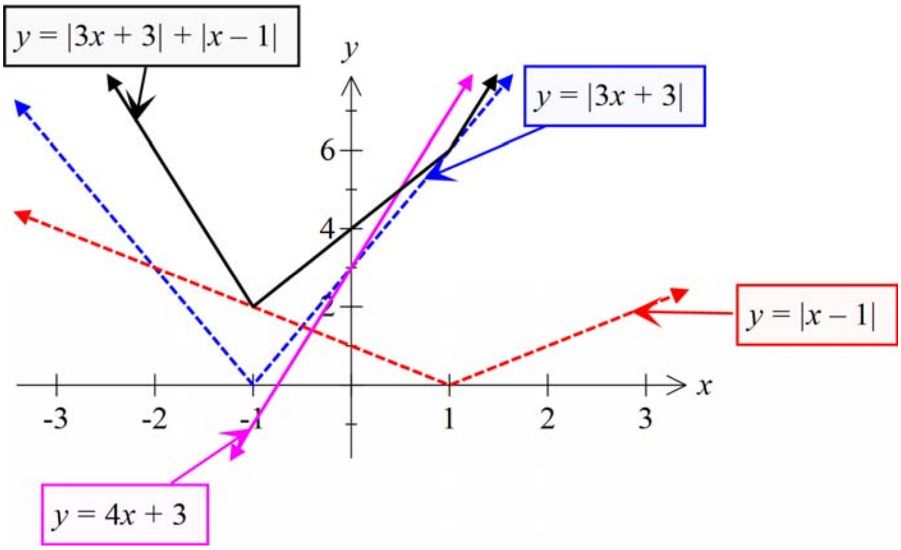
(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
$f'(x) = 0, \therefore f(x)$ is a constant function. For $0 \le x \le 1$, *let* $x = 1$, $f(1) = 2\cos^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}(1-1)$ $= \frac{\pi}{2}$ $\therefore \int_0^1 f(x)\ dx = \int_0^1 \frac{\pi}{2}\ dx$ $= \left.\frac{\pi x}{2}\right]_0^1$ $= \frac{\pi}{2}$	• Correct solution (may or may not include a graph) but including establishment that $f(x) = \frac{\pi}{2}, in\ 0 \le x \le 1$ • Integration by parts is acceptable but not necessary.	**2**
	• Explanation that $f(x)$ is a constant function • $f(x) = \frac{\pi}{2}, in\ 0 \le x \le 1$ • $\int_0^1 f(x)\ dx = \int_0^1 \frac{\pi}{2}\ dx$	**1**

QUESTION 16
MARKING GUIDELINES

(a)

(i)



3 = correctly drawn graphs of $y = |3x+3| + |x-1|$ & $y = 4x+3$
2 = correctly drawn graphs of $y = 4x+3$ *and* $y = |3x+3|$ *or* $y = |x-1|$
1 = correctly drawn graphs of $y = 4x+3$ *or* $y = |3x+3|$ *or* $y = |x-1|$

(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
correctly drawn graphs of $\lvert 3x+3\rvert + \lvert x-1\rvert \le 4x+3$ Consider the intersection of $y = 4x+3$ *and* $y = (3x+3)-(x-1) \Rightarrow$ *see graph* $= 2x+4$ *Solving* $2x+4 = 4x+3$ $x = \frac{1}{2}$ $\therefore x \ge \frac{1}{2}$ *only*	• Correct solution from their graphs in (i) provided they are well labelled and distinguished from one another and the problem has not been made easier than intended	**2**
	• Establishing the graphs of $y = 4x+3$ and $y = (3x+3)-(x-1)$ intersect at $x = ½$. • Solving appropriate equations for the following cases, $x \ge 1, -1 \le x \le 1, x < -1$	**1**

(b) **(i)**

Suggested solution	CRITERIA/COMMENTS	MARKS
$I_0 = \int_0^1 e^{-x}\, dx = -\left[e^{-x}\right]_0^1 = 1 - \frac{1}{e}$	• Correct answer only	**1**

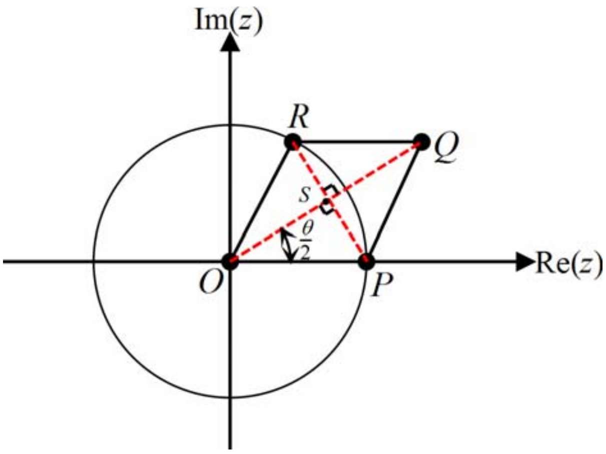
(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
Let $I_n = \int_0^1 x^n e^{-x} \; dx$ *Let* $u = x^n \quad dv = e$ $du = nx^{n-1} \quad v = -e^{-x}$ $I_n = uv - \int v \; du$ $= -\left[x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1}e^{-x} \; dx$ *i.e.* $I_n = -[e^{-1}] + nI_{n-1}$ $= nI_{n-1} - \frac{1}{e}$	• Correct proof	**2**
	• Correct primitive of their *dv* • Correct derivative of their *u* • Correct evaluation of either $-\left[x^n e^{-x}\right]_0^1$ or $n\int_0^1 x^{n-1}e^{-x} \; dx$	**1**

(iii)

Suggested solution	CRITERIA/COMMENTS	MARKS
Let $I_4 = \int_0^1 x^4 e^{-x} \; dx$ $I_4 = 4I_3 - \frac{1}{e}$ $= 4\left[3I_2 - \frac{1}{e}\right] - \frac{1}{e}$ $= 4\left[3\left(2I_1 - \frac{1}{e}\right) - \frac{1}{e}\right] - \frac{1}{e}$ $= 4\left[3\left(2\left\{I_0 - \frac{1}{e}\right\} - \frac{1}{e}\right) - \frac{1}{e}\right] - \frac{1}{e}$ $= 4\left[3\left(\left\{2I_0 - \frac{2}{e}\right\} - \frac{1}{e}\right) - \frac{1}{e}\right] - \frac{1}{e}$ $= 4\left[\left(\left\{6I_0 - \frac{6}{e}\right\} - \frac{3}{e}\right) - \frac{1}{e}\right] - \frac{1}{e}$ $= 24I_0 - \frac{24}{e} - \frac{12}{e} - \frac{4}{e} - \frac{1}{e}$ $= 24\left(1 - \frac{1}{e}\right) - \frac{41}{e}$ $= 24 - \frac{65}{e}$	• Correct solution	**2**
	• Statement $I_4 = \int_0^1 x^4 e^{-x} \; dx$ • Statement $I_4 = 4I_3 - \frac{1}{e}$	**1**

(c) (i)



Construct diagonals *OQ and PR of rhombus OPQR. They intersect at right-angles (property of a rhombus) at S (see diagram)*

Suggested solution	CRITERIA/COMMENTS	MARKS
$\angle QOP = \frac{\theta}{2}$ (diagonals of a rhombus bisect the angles through which they pass) $\therefore z_2 = \overrightarrow{OQ} = \left\|\overrightarrow{OQ}\right\| cis\frac{\theta}{2}$ $But \left\|\overrightarrow{OQ}\right\| = 2\left\|\overrightarrow{OS}\right\|$ & $\left\|\overrightarrow{OP}\right\| = 1(given)$ $In\ right-\Delta SOP, \frac{\left\|\overrightarrow{OS}\right\|}{\left\|\overrightarrow{OP}\right\|} = \cos\frac{\theta}{2}$ $\therefore \left\|\overrightarrow{OS}\right\| = \cos\frac{\theta}{2}$ $\& \left\|\overrightarrow{OQ}\right\| = 2\cos\frac{\theta}{2}$ $Hence,\ z_2 = 2\cos\frac{\theta}{2} cis\frac{\theta}{2}$ $= 2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$	• Correct proof including all three of $\angle QOP = \frac{\theta}{2}$ & $z_2 = \overrightarrow{OQ} = \left\|\overrightarrow{OQ}\right\| cis\frac{\theta}{2}$ & $\left\|\overrightarrow{OS}\right\| = \cos\frac{\theta}{2}$ or equivalent notation and fact(s)	**3**
	• Identifying a minimum of two facts from $\angle QOP = \frac{\theta}{2}$ or $z_2 = \overrightarrow{OQ} = \left\|\overrightarrow{OQ}\right\| cis\frac{\theta}{2}$ or $\left\|\overrightarrow{OS}\right\| = \cos\frac{\theta}{2}$ or equivalent notation and fact(s)	**2**
Can use the facts that $1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$ $\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ and just simplify the given expression	• Identifying only one of $\angle QOP = \frac{\theta}{2}$ or $z_2 = \overrightarrow{OQ} = \left\|\overrightarrow{OQ}\right\| cis\frac{\theta}{2}$ or $\left\|\overrightarrow{OS}\right\| = \cos\frac{\theta}{2}$ or equivalent notation and fact(s)	**1**

(ii)

Suggested solution	CRITERIA/COMMENTS	MARKS
From (i) $\quad z_2 = 2\cos\frac{\theta}{2}\, cis\frac{\theta}{2}$ $\therefore \quad \frac{1}{z_2} = \frac{\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]^{-1}}{2\cos\frac{\theta}{2}}$ $= \frac{\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]}{2\cos\frac{\theta}{2}} \quad (de\ Moivre's\ Th)$ $= \frac{1}{2}\left[\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} - i\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right]$ $= \frac{1}{2} - \frac{i}{2}\tan\frac{\theta}{2}$	• Correct proof including a clear indication of the splitting of terms (see second last line of suggested solution) • Can realise the denominator	**2**
	• Statement $\frac{1}{z_2} = \frac{\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]^{-1}}{2\cos\frac{\theta}{2}}$ or $\frac{\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right]}{2\cos\frac{\theta}{2}}$ from de Moivre's Theorem	**1**

End of Marking Guidelines, Criteria and Solutions