ABBOTSLEIGH

TRIAL HIGHER SCHOOL CERTIFICATE, 1991

MATHEMATICS 2/3 UNIT

Time allowed - Three hours

- All questions may be attempted.
- All questions are of equal value.
- All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.
- Standard integrals are printed on the back page of the examination paper.
- Each question attempted is to be returned in a separate book clearly marked Question 1, Question 2, etc. on the cover. Each booklet must show your Candidate Number.

Question 1 (Use a separate book)

(i) If $a\sqrt{3} = \sqrt{12}$, write down the value of a.

(ii) Express 2.5×10^{-3} as a fraction in simplified form.

(iii) If $v^2 = u^2 - 2as$, find s if $v = 7$, $u = 4$ and $a = -10$.

(iv) Solve $2^{x+1} = 128$

(v) Graph on a number line the solution of $|3x - 1| < 2$

(vi) Find the equation of a line perpendicular to $x - 3y + 1 = 0$ and passing through the point $(-1, 6)$

Question 2 (Use a separate book)

(i) A word is chosen from a paragraph containing 8 words of one letter, 12 words of two letters, 15 words of three letters, 25 words of four letters and 30 words of five or more letters. Find the probability that the word will have

(a) exactly four letters

(b) at least four letters

(ii) If $f(x) = x + \frac{1}{x}$

(a) Find $f\left(\frac{2}{3}\right)$

(b) Solve $f(x) = -2$

(c) Write down the domain of $f(x)$

(iii) Given that $g(x) = 2x^2 + x$, find the values of x for which $g(x) \geq 3$

(iv) The line $4x + 3y - 17 = 0$ just touches a circle centre $(1, -4)$. Find the radius of the circle.

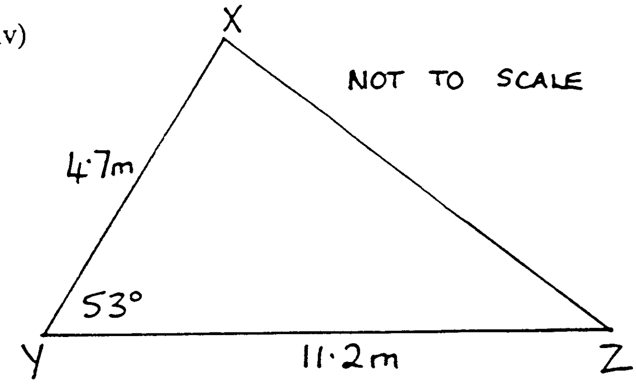
Question 3 (Use a separate book)

(i) On a diagram, show the region where $y \geq x^2$ and $y \geq x + 2$ hold simultaneously.

(ii) Find the exact value of tan 300°.

(iii) Find all values of x, $0° \leq x \leq 360°$, for which $\cos x = \frac{\sqrt{3}}{2}$

(iv)



From the diagram, find the length XZ correct to one decimal place.

Question 4 (Use a separate book)

(i) Differentiate with respect to x

(a) $(3x + 1)^7$

(b) $\frac{x}{1 - x}$

(c) $\log \sqrt{x}$

(ii) The tangent to the curve $y = x^3 - 3x^2 + 4x - 5$ at the point $(1, -3)$ crosses the x- and y- axes at points A and B respectively. Find

(a) the equation of the tangent

(b) the co-ordinates of A and B

(c) the length of AB (as a simplified surd)

Question 5 (Use a separate book)

(i) Find the primitives of

(a) $x^3 + 1$

(b) $(3x + 1)^7$

(c) $\cos \frac{x}{2}$

(ii) Find the exact values of

(a) $\int_1^2 x \, dx$

(b) $\int_1^2 \frac{dx}{x}$

(iii) The gradient at any point on a curve is given by $f'(x) = 3 - 2x$. Find the equation of the curve if it passes through $(-1, -6)$.

Question 6 (Use a separate book)

(i) If $\int_1^k \frac{dx}{x^2} = \frac{1}{4}$, find the value of k.

(ii) Find the stationary points and determine their nature for

$$y = (x - 1)(x^2 - 1)$$

Find any points of inflexion. Draw a sketch of the curve showing all important features.

Question 7 (Use a separate book)

(i) The first four terms of a geometric sequence are 1, 3, 9 and 27 respectively. Find the ninth term.

(ii) A pendulum is set swinging. It covers 18cm in the first swing, 12cm in the second swing, 8cm in the third swing, and so on. Find the total distance it swings through before coming to rest.

(iii) The lengths of the sides of a right-angled triangle are consecutive terms of an arithmetic sequence. The sum of their lengths is 18cm. Find the lengths of the three sides.
Hint: Let the sides be a - d, a, a + d.

Question 8 (Use a separate book)

(i) Find the set of values of k such that $(k + 3)x^2 + kx + 1 = 0$ has real roots.

(ii) Find the area bounded by the curve $y = 3e^{-x}$, the x-axis, and the ordinates $x = 1$ and $x = 4$.
Give your answer correct to three significant figures.

(iii) Find the volume of solid of revolution obtained by revolving the area between $y = \frac{1}{\sqrt{x}}$ and the x- axis between $x = 1$ and $x = 4$ about the x- axis.
Give your answer exactly.

Question 9 (Use a separate book)

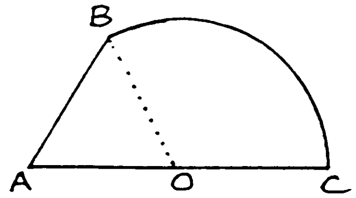
(i) $f(x) = \log_e x$

(a) Draw up a table of values of f(x) (correct to two decimal places) for $x = 1,\ 2.5,\ 4,\ 5.5,\ 7$.

(b) Use Simpson's Rule with these five function values to estimate

$$\int_1^7 \log_e x \, dx$$

(ii)



Δ AOB is an equilateral triangle of side 12cm. AC is a straight line. BC is an arc of the circle, centre 0.

Write down in terms of π the size of

(a) $\angle BAC$

(b) $\angle BOC$

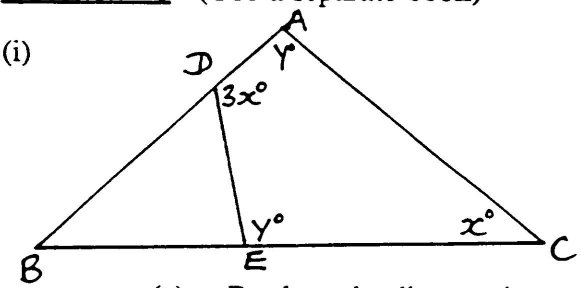
Giving answers in exact form, find

(c) the perimeter of the shape.

(d) the area of the shape.

Question 10 (Use a separate book)

(i)

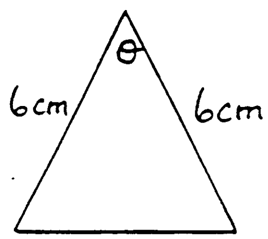


ABC is a triangle.
$\angle BAC = \angle DEC$
$\angle ADE = 3$ times $\angle ACE$

(a) Re-draw the diagram in your answer booklet.

(b) Prove that AB = AC.

(ii)



The two equal sides of an isosceles triangles are 6cm and the angle between these two sides is θ.

(a) Write down an expression for the area of the triangle, A, in terms of θ.

(b) (i) Find the value of θ which will make this area maximum.

(ii) Hence find the value of the third side of the triangle.

Abbotsleigh

trial HSC 1991 2/3 unit Mathematics Solution

Q1

i) $\sqrt{12} = 2\sqrt{3}$

so $a = 2$

ii) $2.5 \times 10^{-3} = \frac{2.5}{1000}$

$= \frac{1}{400}$

iii) $v^2 = u^2 - 2as$

$7^2 = 4^2 - 2(-10)s$

$49 = 16 + 20s$

$\frac{33}{20}$ or $1\frac{13}{20} = s$

iv) $2^{x+1} = 128$

$= 2^7$

so $x + 1 = 7$

$x = 6$

v) $-2 < 3x - 1 < 2$

$-1 < 3x$ and $3x < 3$

$-\frac{1}{3} < x < 1$

vi) $x - 3y + 1 = 0$

$y = \frac{1}{3}x + \frac{1}{3}$

so for $\perp$ $m = -3$

$\therefore$ equation $y - 6 = -3(x + 1)$

$3x + y - 3 = 0$

Q2

i) $N(S) = 90$

a) $P(4) = \frac{25}{90}$

b) $P(4)$ or $P(5)$

$= \frac{25}{90} + \frac{30}{90}$

$= \frac{55}{90}$

$= \frac{11}{18}$

ii) $f(x) = x + \frac{1}{x}$

a) $f(\frac{2}{3}) = \frac{2}{3} + \frac{3}{2}$

$= \frac{13}{6}$ or $2\frac{1}{6}$

b) $x + \frac{1}{x} = -2$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

c) $D: x \neq 0$

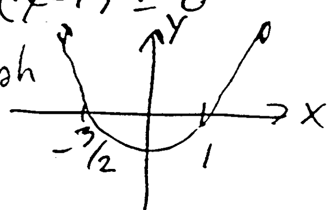
iii) $g(x) = 2x^2 + x$

$2x^2 + x \geq 3$

$2x^2 + x - 3 \geq 0$

$(2x+3)(x-1) \geq 0$

from graph



Solution $x \geq 1$ or $x \leq -\frac{3}{2}$

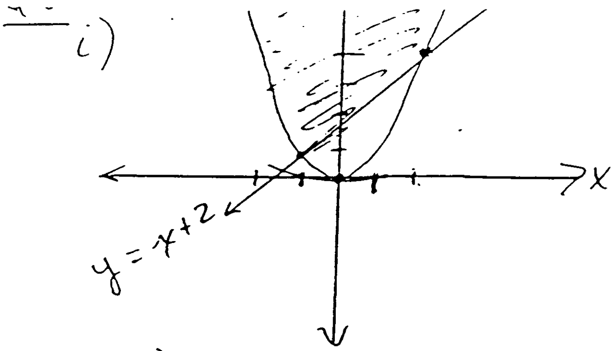
iv) using distance from a point to a line

$d = \left| \frac{4(1) + 3(-4) - 17}{\sqrt{25}} \right|$

$= \left| \frac{-25}{5} \right|$

$= 5$

i)



ii). $\tan 300^\circ = -\tan(360-$
$= -\tan 60$
$= -\sqrt{3}$

iii) $\cos x = \frac{\sqrt{3}}{2}$
so $x = 30^\circ, 330^\circ$

iv) using Cosine rule
$x^2 = 4.7^2 + 11.2^2 - 2(4.7)(11.2)\cos 55$
$= 22.09 + 125.44 - 63.3591$
$= 84.1709$
so $x = 9.2$ m to one decimal place

Q4

i) a) $\frac{d}{dx}(3x+1)^7$
$= 7(3x+1)^6 \times 3$
$= 21(3x+1)^6$

b) $\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(1-x)+x}{(1-x)^2}$
$= \frac{1}{(1-x)^2}$

c) $\frac{d}{dx} \ln x^{1/2} = \frac{d}{dx} \frac{1}{2}\ln x$
$= \frac{1}{2x}$

ii) a) $y' = 3x^2 - 6x + 4$
$y'(1) = 1$
so t: $y + 3 = 1(x-1)$
$y = x - 4$

b) for A $y = 0$
so $0 = x - 4$
$4 = x \quad \therefore A : (4,0)$
for B $x = 0$
so $y = -4 \therefore B : (0,-4)$

c) $AB = \sqrt{16+16}$
$= \sqrt{32}$
$= 4\sqrt{2}$

Q5

i) a) $\int (x^3+1)dx = \frac{x^4}{4} + x + C$

b) $\int (3x+1)^7 dx = \frac{(3x+1)^8}{3 \times 8} + C$
$= \frac{(3x+1)^8}{24} + C$

c) $\int \cos x/2 \, dx = \frac{\sin x/2}{1/2} + C$
$= 2\sin x/2 + C$

ii) a) $\int_1^2 x\,dx = \left|\frac{x^2}{2}\right|_1^2$
$= 4/2 - 1/2$
$= 3/2$

b) $\int_1^2 \frac{dx}{x} = \left|\ln x\right|_1^2$
$= \ln 2 - \ln 1$
$= \ln 2$

Q5 (cont)

iii) $f'(x) = 3 - 2x$

$f(x) = 3x - x^2 + c$

$f(x) = -6$ when $x = -1$

so $-6 = -3 - 1 + c$

$-2 = c$

$\therefore f(x) = 3x - x^2 - 2$

Q6

i) $\int_1^k \frac{dx}{x^2} = 1/4$

so $\left[-1/x\right]_1^k = 1/4$

$-1/k + 1/1 = 1/4$

$-1/k = -3/4$

$k = 4/3.$

ii) $y' = (x^2-1) + 2x(x-1)$

$= 3x^2 - 2x - 1$

for stationary points $y' = 0$

so $0 = 3x^2 - 2x - 1$

$0 = (3x+1)(x-1)$

$x = -1/3$ or $x = 1$

Now $y'' = 6x - 2$

$y''(-1/3) = -4 < 0$ $\quad y''(1) = 4 > 0.$

$\therefore (-1/3, 32/27)$ a maxima

and $(1, 0)$ a minima.

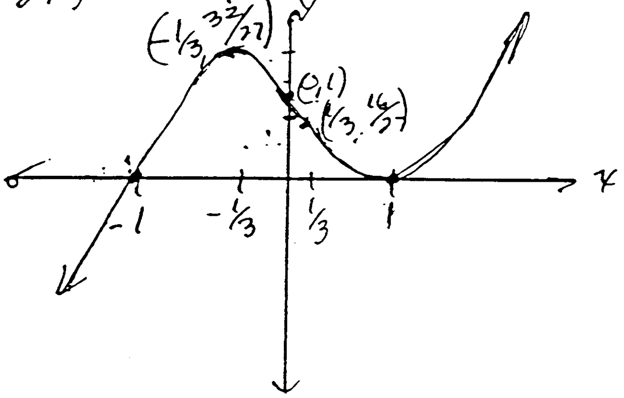
for a pt. of inflexion $y'' = 0$ i.e. $0 = 6x - 2$

$1/3 = x$

$y''(1/3 + \varepsilon) > 0, \quad y''(1/3 - \varepsilon) < 0$

$\therefore (1/3, 16/27)$ is a point of inflexion.

Graph.:



Q7 i) G.P.: 1, 3, 9, 27

$t_9 = ar^8 \qquad r = 3$

$= (1)(3)^8$

$= 3^8$

$= 6561$

ii) distances are 18, 12, 8 ...

$\frac{12}{18} = \frac{8}{12} = \frac{2}{3}$ so a G.P.

$|\frac{2}{3}| < 1$ so a sum to infinity

$S_\infty = \frac{a}{1-r}$

$= \frac{18}{\frac{1}{3}}$

$= 54$ cm

iii) for sides $a-d, a, a+d$

then $a + (a-d) + (a+d) = 3a = 18$

so $a = 6$.

Now $6^2 + (6-d)^2 = (6+d)^2$

$36 + 36 - 12d + d^2 = 36 + 12d + d^2$

$36 = 24d$

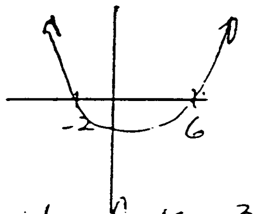
$\frac{3}{2} = d$

so sides are $4\frac{1}{2}$ cm, 6 cm and $7\frac{1}{2}$ cm.

Q8 i) for real roots $b^2 - 4ac \geq 0$

$K^2 - 4K - 12 \geq 0$

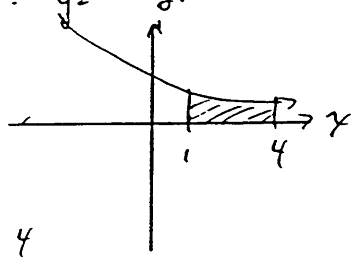
$(K-6)(K+2) \geq 0$



from graph.

$K \leq -2$ or $K \geq 6$. Note: for $K = -3$ we have one real root

ii)



$A = \int_1^4 3e^{-x}\,dx$

$= -3\Big|_1^4 e^{-x}$

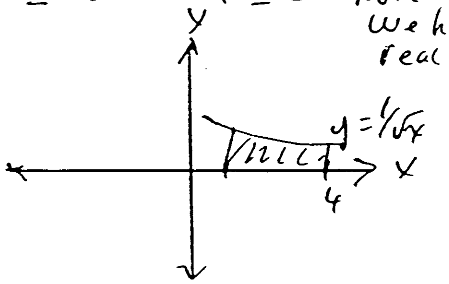
$= -3(e^{-4} - e^{-1})$

$\doteqdot -3(-0.34956)$

$\doteqdot 1.04869$

$\doteqdot 1.05$ units^2 to three significant figures.

iii)



$V = \pi \int_1^4 y^2\,dx$

$= \pi \int_1^4 \frac{1}{x}\,dx$

$= \pi \Big|_1^4 \ln x$

$= \pi(\ln 4 - \ln 1)$

$= \pi \ln 4$ units^3.

Q9 i) $f(x) = \ln x$

a)

x	1	2.5	4	5.5	7
$f(x)$	0	0.92	1.39	1.70	1.95

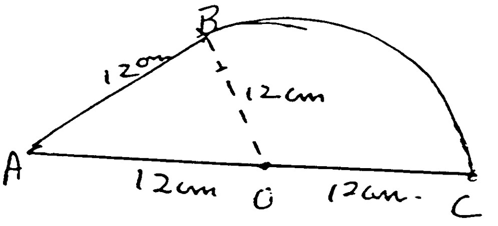
b) $\int_1^7 \ln x \, dx \doteq \frac{3/2}{3}\left[y_0 + y_4 + 2(y_2) + 4(y_1 + y_3)\right]$

$\doteq \frac{1}{2}\left[0 + 1.95 + 2(1.39) + 4(0.92 + 1.70)\right]$

$\doteq \frac{1}{2}(1.95 + 2.78 + 10.48]$

$\doteq 7.605$

ii)



a) $\angle BAC = \pi/3$

b) $\angle BOC = \pi - \pi/3$

$= 2\pi/3$

c) $P = 12 + 24 + \overset{\frown}{BC}$

$= 36 + r\theta$

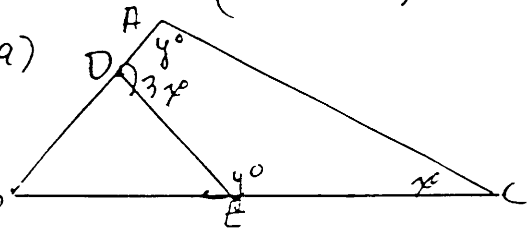
$= 36 + 12(2\pi/3)$

$= (36 + 8\pi)$ cm.

d) $A = \frac{1}{2}(144)\frac{\sqrt{3}}{2} + \frac{1}{2}(12)^2(\frac{2}{\ldots}$

$= (36\sqrt{3} + 48\pi)$ cm^2

Q10) i) a)



b) Prove $AB = AC$

1. $4x + 2y = 360$ $\angle$'s sum of a quad.

so $2x + y = 180$

2. $\angle B + x + y = 180$ $\angle$'s sum of a Δ.

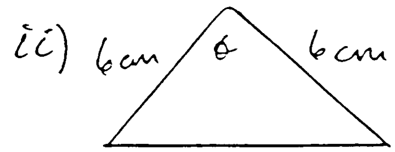
so. $2x + y = \angle B + x + y$ = to =

$\therefore x = \angle B$

and so ΔABC is isosceles - base $\angle$'s are

i.e. $AB = AC$.

ii)



a) $A = \frac{1}{2}(36)\sin\theta$

$= 18\sin\theta$

b)(i) maximum occurs when $\sin\theta = 1$

so $\theta = \pi/2$

(ii) $x^2 = 36 + 36 - 2(36)\cos\frac{\pi}{2}$

$= 72$

$x = 6\sqrt{2}$ cm