ABBOTSLEIGH

AUGUST 2004

YEAR 12
ASSESSMENT 4
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks (120)

- Attempt Questions 1-10.
- All questions are of equal value.

Marks

Question 1 (12 marks)
Use a SEPARATE writing booklet

(a) Simplify $\left(\sqrt{3}+1\right)^2$ **1**

(b) Simplify $\sqrt{m^3}\times m^2\sqrt{m}$ **2**

(c) Factorise fully $a^4 - ab^3$ **2**

(d) Solve $6x^2 - 5x = 1$ **2**

(e) Differentiate

(i) $\dfrac{1}{x-1}$ **1**

(ii) $\left(x^2-1\right)\left(x^2+1\right)$ **2**

(f) Solve $|x+1| = |2x-4|$ **2**

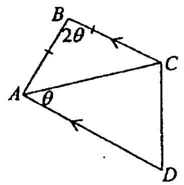
Question 2 (12 marks) Marks
Use a SEPARATE writing booklet

(a) Differentiate

(i) $y = \tan 3x$ 1

(ii) $y = x^2 e^x$ 2

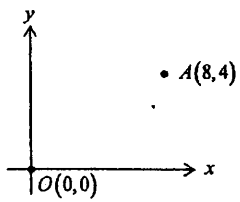
(b) $ABCD$ is a quadrilateral with $AB = BC$, $BC \parallel AD$, $\angle ABC = 2\theta$ and $\angle CAD = \theta$.



Not to Scale

Copy the diagram into your answer booklet and find the value of θ, giving reasons for your answer. 3

(c)



The diagram above shows the points $O(0,0)$ and $A(8,4)$ on the number plane. Copy the diagram into your writing booklet.

(i) Find the coordinates of M, the midpoint of OA. 1

(ii) Show that $2x + y - 10 = 0$ is the equation of the line l which passes through M and is perpendicular to OA. 2

(iii) Find the coordinates of the point B where the line l cuts the y-axis. 1

(iv) Show that the length of MB is twice the length of MA. 2

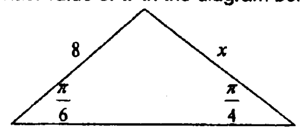
Question 3 (12 marks) Marks
Use a SEPARATE writing booklet

(a) Find a primitive function of:

(i) $\dfrac{1}{(2+x)^3}$ 1

(ii) $\dfrac{x^2}{x^3+2}$ 2

(b) Find the exact value of x in the diagram below. 2

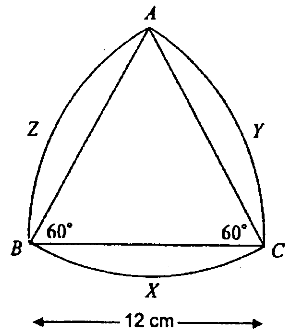


Not to Scale

(c) In a game of tennis, a fair coin is tossed to see who serves first. If Cleyton wins the toss he has an 80% chance of winning the first game, but if he loses the toss his chance of winning the first game is only 40%.

What is the probability that Cleyton wins the first game? 2

(d)



Not to Scale

In the figure, ABC is an equilateral triangle of side 12 cm. The circular arcs AZB, BXC and CYA are constructed with centres at C, A, B respectively. Find:

(i) The area of sector $CAZB$. 2

(ii) The total area enclosed by the figure $AZBXCYA$. 3

Marks

Question 4 (12 marks)
Use a SEPARATE writing booklet

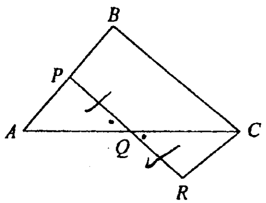
(a) Evaluate $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ **1**

(b) A polygon has 25 sides, the lengths of which form an arithmetic sequence. If the shortest side is 8 cm and the next longest side is 11 cm, find the perimeter of the polygon. **3**

(c) (i) Sketch the curve $y = e^x$ and shade on your diagram the region defined by $y \le e^x$, $0 \le x \le 1$ and $y \ge 0$. **2**

(ii) This shaded region is rotated about the x-axis. Find the exact volume of the solid of revolution formed. **2**

(d)



Not to Scale

P and Q are the midpoints of the sides AB and AC of triangle ABC. PQ is produced to R so that $PQ = QR$.

(i) Copy the diagram into your answer booklet and prove that $\Delta PQA \equiv \Delta RQC$. **2**

(ii) Prove $CR = \frac{1}{2}AB$ **2**

Marks

Question 5 (12 marks)
Use a SEPARATE writing booklet

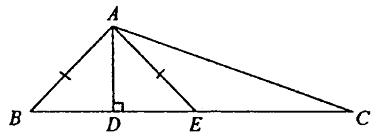
(a) A box contains ten chocolates, all of identical appearance. Three of the chocolates have caramel centres and the other seven have mint centres. Anna randomly selects and eats three chocolates from the box, picking one chocolate at a time.

Find the probability that:

(i) Anna eats three mint chocolates, **2**

(ii) Anna eats exactly one caramel chocolate. **2**

(b)



Not to Scale

In the diagram, $AD = 12\text{cm}$, $BE = 18\text{cm}$, $AC = 20\text{cm}$, $AB = AE$ and $AD \perp BE$.

(i) Copy the diagram into your answer booklet, marking the above information on it and show that the length of DC is 16cm. **2**

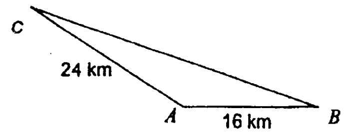
(ii) Prove $\angle BAC = 90°$. **3**

(c) If $y = e^{kx}$ is a solution of the equation $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 14y = 0$, find the value(s) of k. **3**

Question 6 (12 marks) **Marks**
Use a SEPARATE writing booklet

(a) Differentiate $y = \log_e(2x+1)$ **1**

(b)



Not to Scale

In the diagram above A, B and C represent the locations of three towns. The town A is due west of B, and the bearing of C from A is 340°. Copy the diagram into your answer booklet.

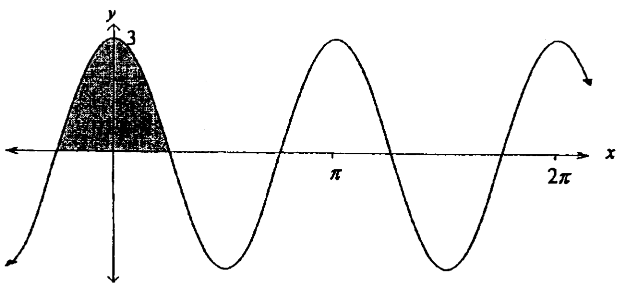
(i) Find the size of $\angle CAB$. **1**

(ii) Find the distance BC correct to the nearest kilometre. **2**

(iii) What is the bearing of A from C? **1**

(c) Show that the sum of n consecutive odd integers beginning with 1 is always a perfect square. **3**

(d)



(i) The curve $y = A\cos nx$ is sketched above. Find the values of A and n. **2**

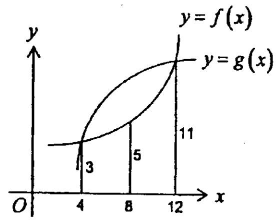
(ii) Find the value of the shaded area. **2**

Question 7 (12 marks) **Marks**
Use a SEPARATE writing booklet

(a) (i) Find the equation of the tangent at the point $(-1, 11)$ on the curve $y = 2x^3 + 6x^2 - 3x + 4$. **2**

(ii) Prove that $(-1, 11)$ is a point of inflexion. **2**

(b)



Not to Scale

(i) Use the Trapezoidal Rule with three function values to approximate the value of $\int_4^{12} f(x)\,dx$ **2**

(ii) Given that the area between $y = g(x)$ and the x-axis from $x = 4$ to $x = 12$ is 67 square units, calculate the approximate area between the two curves. **1**

(c) Given the function $f(x) = x\sqrt{4-x^2}$.

(i) Show that $f(x)$ is an odd function. **1**

(ii) Find the x and y intercepts of this function. **2**

(iii) Hence sketch this function in the domain $-2 \le x \le 2$, given that there is a maximum turning point at $(\sqrt{2}, 2)$. **2**

Marks

Question 8 (12 marks)
Use a SEPARATE writing booklet

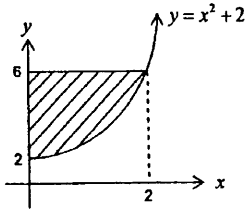
(a) Consider the geometric series $1+(3x-2)+(3x-2)^2+........$

(i) For what values of x does this series have a limiting sum? **2**

(ii) Find the limiting sum. Simplify your answer. **1**

(iii) Find the value of x if this limiting sum has a value of $\frac{2}{3}$. **2**

(b)



Find the value of the area shaded in the diagram. **3**

(c) Solve the equation $2\sin x=-1$ for $0\le x\le 2\pi$ **2**

(d) Prove that $\dfrac{1+\tan^2 x}{1+\cot^2 x}=\tan^2 x$ **2**

Marks

Question 9 (12 marks)
Use a SEPARATE writing booklet

(a) Sketch a possible graph of $y=f(x)$ in the interval $0\le x\le 1$ if $f(x)>0$, $f'(x)<0$ and $f''(x)>0$. **2**

(b) The curve $y=f(x)$ has second derivative $f''(x)=1$. If the curve has a minimum turning point at the point (2,3), find its equation. **3**

(c) Jane invests \$400 in a bank account at the beginning of each month for 8 years. Interest is to be paid at a rate of 6% per annum compounded monthly.

(i) Find the amount in the account at the end of the first month. **1**

(ii) Show that the total value of her investment at the end of n years is given by $\$400\left(1.005+1.005^2+1.005^3+\ldots 1.005^{12n}\right)$ **1**

(iii) Find the final value of Jane's investment at the end of the 8 years. Give your answer correct to the nearest dollar. **3**

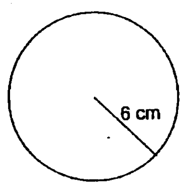
(iv) What single investment at the beginning of the 8 years, with interest compounded monthly, would achieve the same final value? Give your answer to the nearest dollar. **2**

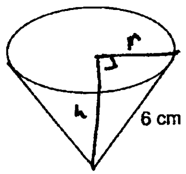
Marks

Question 10 (12 marks)
Use a SEPARATE writing booklet

(a) Prove that the function $y = \dfrac{\sin x}{1+\cos x}$ does not have a turning point. **3**

(b) A circular filter paper of radius 6cm is cut and folded to make a conical filter.





(i) Show that the volume, V, of the cone is $\frac{1}{3}\pi r^2\sqrt{36-r^2}$ where r is the base radius. **2**

(ii) Show that $\dfrac{dV}{dr} = \dfrac{2\pi r\sqrt{36-r^2}}{3} - \dfrac{\pi r^3}{3\sqrt{36-r^2}}$ **3**

(iii) Find the maximum volume of the cone and the corresponding radius in exact form. **4**

END OF PAPER

2004 Trial Mathematics solutions

1 (a) $4+2\sqrt{3}$

(b) $m^{3/2}\times m^2\times m^{1/2}$

$=m^4$

(c) $a(a^3-b^3)$

$=a(a-b)(a^2+ab+b^2)$

(d) $6x^2-5x-1=0$

$(6x+1)(x-1)=0$

$x=-\frac{1}{6}$ or 1

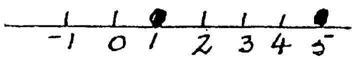
(e)(i) $f(x)=(x-1)^{-1}$

$f'(x)=-(x-1)^{-2}$

(ii) $f(x)=x^4-1$

$f'(x)=4x^3$

(f) $|x+1|=2|x-2|$



$x=1$ or 5

2 (a)(i) $f'(x)=3\sec^2 3x$

(ii) $f'(x)=e^x.2x+x^2e^x$

$=xe^x(2+x)$

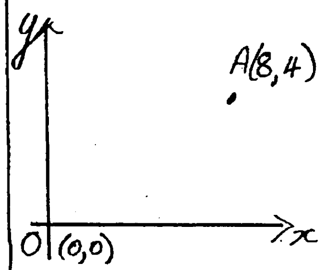
(b) $\angle BCA=\angle CAD=\theta$ (alt $\angle$s, $BC\parallel AD$)

$\angle BAC=\angle BCA=\theta$ (isos. $\triangle$)

$\angle ABC+\angle BCA+\angle BAC=180^0$ ($\angle$ sum of $\triangle$)

$\therefore 4\theta=180$ ...

2(c).



(i) $(4,2)$

(ii) $m_{OA}=\frac{1}{2}$

$m_l=-2$

Equ. of l is

$y-2=-2(x-4)$

$y=-2x+10$

$\therefore 2x+y-10=0$

(iii) $x=0\ \therefore y=10$

$\therefore B$ is $(0,10)$

(iv) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$MB=\sqrt{(0-4)^2+(10-2)^2}$

$=\sqrt{16+64}$

$=\sqrt{80}$

$=4\sqrt{5}$

$MA=\sqrt{(8-4)^2+(4-2)^2}$

$=\sqrt{16+4}$

$=\sqrt{20}$

$=2\sqrt{5}$

$\therefore MB=2MA$

3 (a) $\int(2+x)^{-3}dx=\frac{-1}{(2+x)^2}+C$

(ii) $\int\frac{x^2}{x^3+2}dx=\frac{1}{3}\int\frac{3x^2}{x^3+2}dx$

$=\frac{1}{3}\ln(x^3+2)+C$

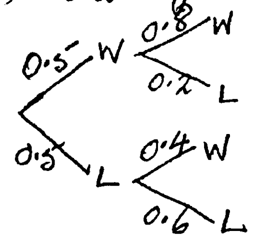
(b) $\frac{x}{\sin\frac{\pi}{6}}=\frac{8}{\sin\frac{\pi}{4}}$

$\frac{x}{\frac{1}{2}}=\frac{8}{\frac{1}{\sqrt{2}}}$

$x=\frac{1}{2}\times 8\times\frac{\sqrt{2}}{1}$

$x=4\sqrt{2}$

(c) toss game



$P=0.5\times0.8+0.5\times0.4$

$=0.6$ (60%)

(d)(i) $\frac{1}{2}r^2\theta=\frac{1}{2}\times12^2\times\frac{\pi}{3}$

$=24\pi\ cm^2$

(ii) $A_{\triangle ABC}=\frac{1}{2}\times12\times12\sin60^0$

$=\frac{1}{2}\times12\times12\times\frac{\sqrt{3}}{2}$

$=36\sqrt{3}$

Asegment $AZB=24\pi-36\sqrt{3}$

Total area $=3(24\pi-36\sqrt{3})+36\sqrt{3}$

$=72(\pi-\sqrt{3})\ cm^2$

4 (a) $\lim_{x\to2}\frac{x^2-4}{x-2}=\lim_{x\to2}\frac{(x-2)(x+2)}{x-2},\ x\neq2$

$=4$

(b) $a=8,\ d=3$

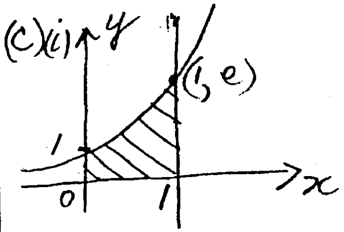
$S_n=\frac{n}{2}[2a+(n-1)d]$

$S_{25}=\frac{25}{2}[2\times8+24\times3]$

$=1100$

$\therefore$ Perimeter is 1100 cm

(c)(i)



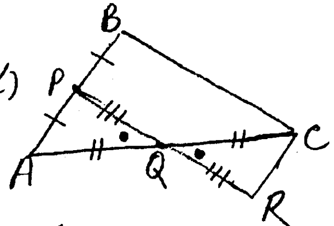
(ii) $V=\pi\int_a^b y^2dx$

$=\pi\int_0^1 e^{2x}dx$

$=\pi\left[\frac{1}{2}e^{2x}\right]_0^1$

$=\frac{\pi}{2}(e^2-1)\ u^3$

(d)



(i) In $\triangle$s PQA, RQC

$PQ=RQ$ (given)

$\angle PQA=\angle RQC$ (vertically opp.)

$AQ=CQ$ (given)

$\therefore\triangle PQA\equiv\triangle RQC$ (SAS)

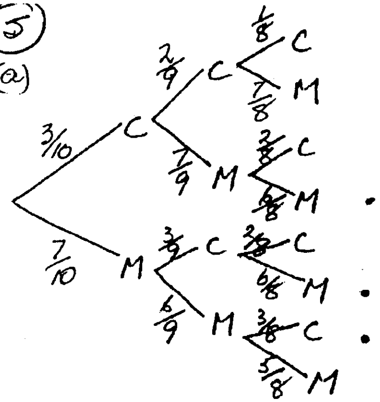
(ii) $CR=AP$ (matching sides in cong $\triangle$s)

$AP=\frac{1}{2}AB$ (given)

$\therefore CR=\frac{1}{2}AB$

(5)

(a)



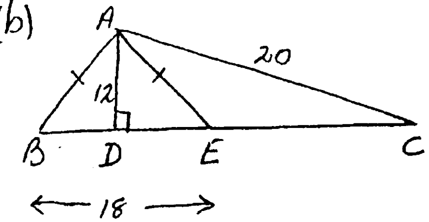
(i) $\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} = \underline{\underline{\frac{7}{24}}}$

(ii) $\frac{3}{10} \times \frac{7}{9} \times \frac{6}{8}$

$+ \frac{7}{10} \times \frac{3}{9} \times \frac{6}{8}$

$+ \frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} = \underline{\underline{\frac{21}{40}}}$

(b)



(i) In $\triangle ADC$,

$DC^2 + 12^2 = 20^2$ (Pythag.)

$DC = \sqrt{20^2 - 12^2}$

$DC = 16$ cm

(ii) $BD = DE = 9$ cm (perp. to base of isos $\triangle$ bisects base)

$\therefore BC = 9 + 16 = 25$ cm

$BC^2 = 25^2 = 625$

In $\triangle ABD$, $AB^2 = 9^2 + 12^2 = 225$

$AB^2 + AC^2 = 225 + 20^2 = 625$

$\therefore AC^2 = AB^2 + BC^2$

5(c)

$y = e^{kx}$, $\frac{dy}{dx} = ke^{kx}$, $\frac{d^2y}{dx^2} = k^2e^{kx}$

$\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 14y = 0$

$\therefore k^2e^{kx} - 9ke^{kx} + 14e^{kx} = 0$

$\therefore e^{kx}(k^2 - 9k + 14) = 0$

$e^{kx} \neq 0$, $(k-2)(k-7) = 0$

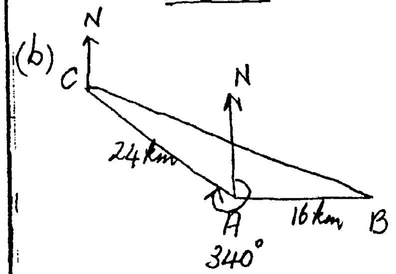
$\therefore \underline{\underline{k = 2 \text{ or } 7}}$

(6)

(a) $y = \log_e(2x+1)$

$\frac{dy}{dx} = \underline{\underline{\frac{2}{2x+1}}}$

(b)



(i) $\angle CAB = 20 + 90 = \underline{\underline{110°}}$

(ii) $BC^2 = 24^2 + 16^2 - 2 \times 24 \times 16 \cos 110°$

$BC = \sqrt{1094.6715}$

$\doteqdot 33$ km

(iii) $180 - 20 = \underline{\underline{160°}}$

(c) $1 + 3 + 5 + \cdots + (2n-1)$

$a = 1$, $d = 2$, $T_n = 2n - 1$

$S_n = \frac{n}{2}(a + l)$

$= \frac{n}{2}(1 + 2n - 1)$

$= n^2$ which is a perfect square

(d)(i) $A = 3$, $n = 2$

(ii)

$A = 2\int_0^{\frac{\pi}{4}} 3\cos 2x \, dx$

$= 2\left[\frac{3}{2}\sin 2x\right]_0^{\pi/4}$

$= 3\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \underline{\underline{3 \text{ unit}^2}}$

7

(a)(i) $y = 2x^3 + 6x^2 - 3x + 4$

$y' = 6x^2 + 12x - 3$

at $(-1, 11)$, $y' = 6 - 12 - 3$

$= -9$

Equ of tangt is

$y - y_1 = m(x - x_1)$

$y - 11 = -9(x + 1)$

$y = -9x + 2$

(ii) $y'' = 12x + 12$

$y'' = 0$ when $x = -1$

test

x	-2	-1	0
y''	-12	0	12
	CD		CU

∴ changes concavity

∴ $(-1, 11)$ is an inflex.

(b)(i) $\int_a^b f(x)\,dx \doteqdot \frac{b-a}{2}[f(a) + f(b)]$

∴ $\int_4^{12} f(x)\,dx$

$\doteqdot \frac{8-4}{2}[3+5] + \frac{12-8}{2}[5+11]$

$= 48$

(ii) Area $= 67 - 48$

$= 19$ units^2

(c)(i) $f(x) = x\sqrt{4-x^2}$

$f(-x) = -x\sqrt{4-(-x)^2}$

$= -x\sqrt{4-x^2}$

$= -f(x)$

∴ odd

(ii) $x = 0$, $f(x) = 0$

∴ y-intercept is 0

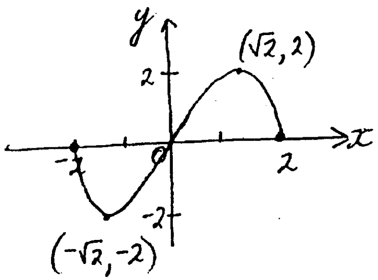
$y = 0$, $0 = x\sqrt{4-x^2}$

∴ $x = 0$ or $4 - x^2 = 0$

$x = 0$ or 2 or -2

∴ x-intercepts are

$-2, 0, 2$



8

(a)(i) $r = 3x - 2$

$-1 < r < 1$

∴ $-1 < 3x - 2 < 1$

$1 < 3x < 3$

$\frac{1}{3} < x < 1$

(ii) $S = \frac{a}{1-r}$

$= \frac{1}{1-(3x-2)}$

$= \frac{1}{3-3x}$

(iii) $\frac{1}{3-3x} = \frac{2}{3}$

$3 = 6 - 6x$

$6x = 3$

$x = \frac{1}{2}$

(b)

$A = 6 \times 2 - \int_0^2 (x^2 + 2)\,dx$

$= 12 - \left[\frac{x^3}{3} + 2x\right]_0^2$

$= 12 - \left[\frac{8}{3} + 4 - 0\right]$

$= 5\frac{1}{3}$ units^2

(c) $2\sin x = -1$ $\quad 0 \le x \le 2\pi$

$\sin x = -\frac{1}{2}$ $\quad$ 3rd + 4th Quad

basic angle $\frac{\pi}{6}$

$x = \pi + \frac{\pi}{6}$, $2\pi - \frac{\pi}{6}$

$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

(d) Prove $\frac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

LHS $= \dfrac{1 + \frac{\sin^2 x}{\cos^2 x}}{1 + \frac{\cos^2 x}{\sin^2 x}}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \div \frac{\sin^2 x + \cos^2 x}{\sin^2 x}$

$= \frac{1}{\cos^2 x} \times \frac{\sin^2 x}{1}$

$= \left(\frac{\sin x}{\cos x}\right)^2$

$= \tan^2 x$

$= $ RHS

∴ $\frac{1 + \tan^2 x}{1 + \cot^2 x} = \tan^2 x$

OR

$\cos^2 x + \sin^2 x = 1$

$\div \cos^2 x$) $\quad 1 + \tan^2 x = \frac{1}{\cos^2 x}$

$\div \sin^2 x$ $\quad \cot^2 x + 1 = \frac{1}{\sin^2 x}$

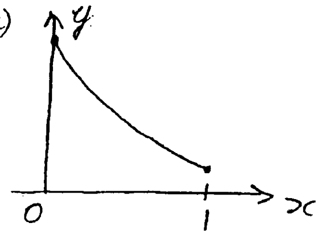
∴ LHS $= \dfrac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}$

$= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$

$= $ RHS

(9)

(a)



(b) $y'' = 1$

$y' = \int 1\, dx$

$= x + c$

$x = 2,\ y' = 0$

$\therefore 0 = 2 + c$

$c = -2$

$\therefore y' = x - 2$

$y = \int (x-2)\, dx$

$= \frac{x^2}{2} - 2x + k$

$x = 2,\ y = 3$

$\therefore 3 = 2 - 4 + k$

$\therefore k = 5$

$\therefore$ Eqn is $y = \frac{x^2}{2} - 2x + 5$

(c)(i) $A = P(1+r)^n$

$\therefore r = 0.06 \div 12$

$= 0.005$

$A_1 = 400(1.005)^1$

$= 400(1.005)$

(ii) $A_2 = 400 \times 1.005^2 + 400 \times 1.005$

n years = 12n months

$\therefore A_n = 400 \times 1.005^{12n} + \cdots\cdots$

$+ 400 \times 1.005^3 + 400 \times 1.005^2$

$+ 400 \times 1.005$

(iii) $A_8 = 400 \times 1.005 + \cdots + 400 \times 1.005^{96}$

$= 400(1.005 + \cdots + 1.005^{96})$

G.P, $a = 1.005$, $r = 1.005$,

$n = 96$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$\therefore A_8 = \frac{400 \times 1.005(1.005^{96} - 1)}{1.005 - 1}$

$\doteqdot \$49377$

(iv) $A = P(1+r)^n$

$49377 = P(1.005)^{96}$

$P = 49377 \div 1.005^{96}$

$= \$30590$

$\therefore$ Single amount is $\$30590$

(10)

(a) $y = \frac{\sin x}{1 + \cos x}$, $1 + \cos x \neq 0$

$y' = \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$

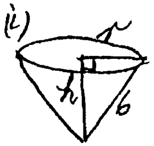
$= \frac{1 + \cos x}{(1+\cos x)^2}$

$= \frac{1}{1 + \cos x}$

$\neq 0$

$\therefore$ no turning point.

(b)(i)



$h^2 + r^2 = 6^2$

$h = \sqrt{36 - r^2}$

$V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2 \sqrt{36 - r^2}$

(ii) $V = \frac{1}{3}\pi r^2 (36 - r^2)^{\frac{1}{2}}$

$u \quad v$

$\frac{dV}{dr} = (36 - r^2)^{\frac{1}{2}} \times \frac{2}{3}\pi r$

$+ \frac{1}{3}\pi r^2 \times \frac{1}{2}(36 - r^2)^{-\frac{1}{2}} \times -2r$

$= \frac{2\pi r\sqrt{36 - r^2}}{3} - \frac{\pi r^3}{3\sqrt{36 - r^2}}$

(b)(iii) $\frac{dV}{dr} = 0$

$\therefore \frac{2\pi r(36 - r^2) - \pi r^3}{3\sqrt{36 - r^2}} = 0$

$\therefore \pi r[72 - 2r^2 - r^2] = 0$

$\therefore \pi r(72 - 3r^2) = 0$

$\therefore r = 0$ discard

or $72 - 3r^2 = 0$

$r^2 = 24$

$r = \pm\sqrt{24}$

$\therefore$ as $r > 0$, $r = 2\sqrt{6}$

Test

r	2	$2\sqrt{6}$	5
$\frac{dV}{dr}$	22.2	0	-4.7
	/	—	\

$\therefore$ Max when $r = 2\sqrt{6}$

$V_{max} = \frac{1}{3}\pi \times 24\sqrt{36 - 24}$

$= 8\pi\sqrt{12}$

$= 16\sqrt{3}\pi \text{ cm}^3$

$r = 2\sqrt{6}$ cm