Ascham School

Mathematics Form 6 - 2 Unit Trial Examination

2003

July 2003

Time allowed: 3 Hours
Plus 5 minutes reading time.

Instructions

1. Attempt ALL questions

2. All questions are of equal value.

3. All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.

4. Standard integrals are printed on page 10.

5. Board approved calculators may be used.

6. Answer each question in a *separate* writing booklet.

Question 1.

(a) Evaluate $\dfrac{x^3+y^4}{y^2}$ if $x=\left(\dfrac{3}{5}\right)^{\frac{1}{3}}$ and $y=\left(\dfrac{2}{5}\right)^{\frac{1}{2}}$ **[2]**

Give your answer in exact fractional form.

(b) Using the table of standard integrals, find $\int \dfrac{1}{\sqrt{x^2+16}}dx$ **[2]**

(c) Solve $|x-3|\geq 5$ **[2]**

(d) Find the exact value of $\sqrt{p^4-2p^2}$ when $p=2\sqrt{5}$ **[2]**

(e) Factorise $2x^2+x-6$ **[2]**

(f) If $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}=a+b\sqrt{15}$ find the values of a and b **[2]**

Question 2. Use a separate writing booklet.

(a) Differentiate with respect to x:

(i) $(5-3x^2)^6$ **[2]**

(ii) $3\tan(4x-\dfrac{\pi}{4})$ **[2]**

(iii) $\dfrac{\log_e 3x}{x}$ **[2]**

(b) Evaluate $\int_0^1 5x+\sin(5x)dx$ to 2 decimal places **[3]**

(c) Evaluate $\int_1^e \dfrac{2}{x}+\dfrac{x}{2}dx$ leaving your answer in exact form **[3]**

Please turn over to question 3

Question 3. Use a separate writing booklet.

(a) **(i)** On a number plane mark the origin O and A(-7, 0), B(-9, 3) and C(0, 9). Join A to B, B to C and C to A. **[1]**

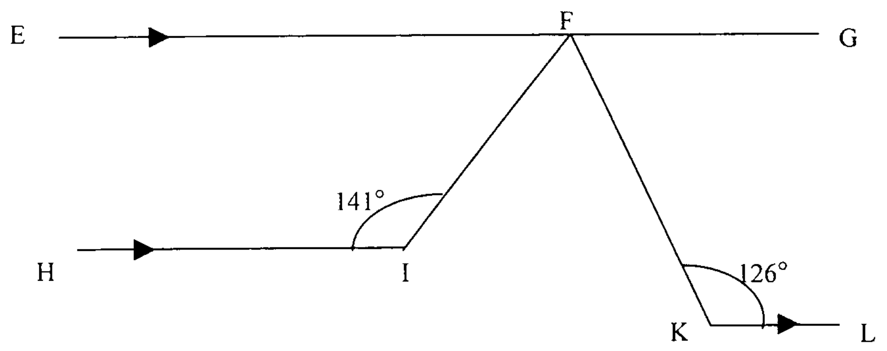
(ii) Show that AB and BC are perpendicular. **[2]**

(iii) Show that the line AB has equation $3x+2y+21=0$ **[2]**

(iv) Show that the length of AB is $\sqrt{13}$ **[2]**

(v) Find the perpendicular distance from O to AB **[2]**

(b)



EG // HI // KL, $\angle$HIF=141° and $\angle$FKL = 126°. Copy the diagram into your answer booklet and find $\angle$IFK giving reasons. **[3]**

Please turn over to question 4

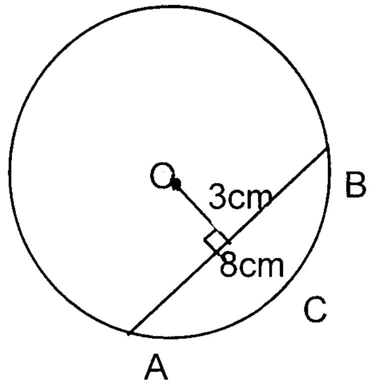
Question 4. Use a separate writing booklet.

(a) Consider the parabola with equation $0 = x^2 - 6y + 4x + 16$

(i) Express the equation in the form $(x-h)^2 = 4a(y-k)$ where a, h and k are constants. **[2]**

(ii) Write down the co-ordinates of the vertex and the equation of the directrix. **[2]**

(b) In the diagram, O is the centre of the circle. Chord AB is of length 8cm and is 3cm from the centre. Copy this diagram into your answer booklet and find the length of the arc ACB, giving your answer correct to 2 decimal places.

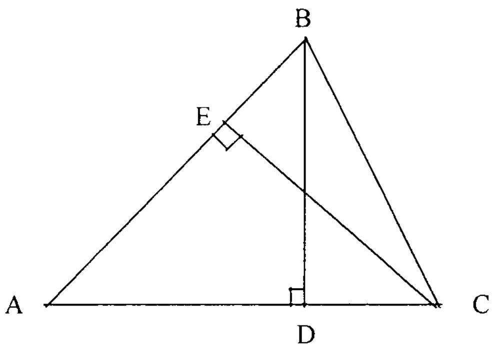


[3]

(c) In the diagram BD⊥AC and CE⊥AB

(i) Copy this diagram into your answer booklet and prove ΔECA ||| ΔDBA

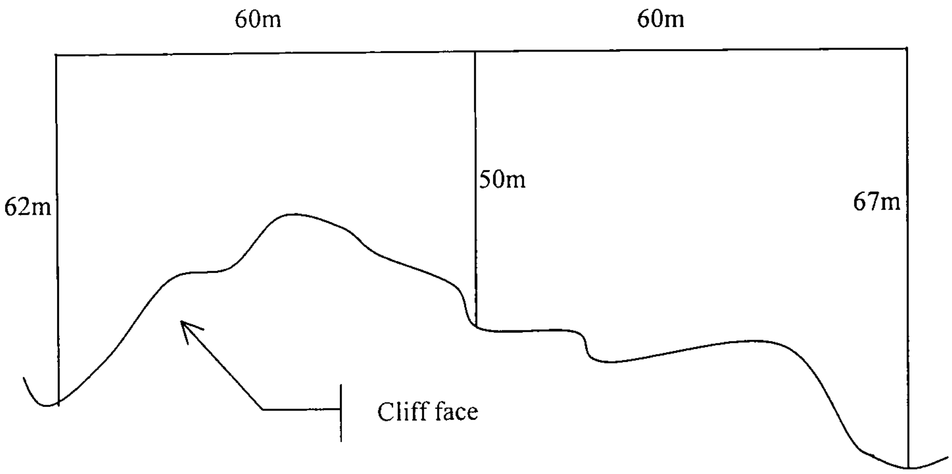
(ii) If AB = 10cm, BD = 7cm and AC = 6cm find the length of CE **[5]**



Please turn over to question 5

Question 5. Use a separate writing booklet.

(a) A farmer has a field that is bounded by fences on three straight sides. The fourth boundary is a steep cliff. He needs to determine the area for seed sowing. Distances within the field are given on the diagram. All distances are in metres and the diagram is not to scale.



(i) Use the trapezoidal rule to determine the approximate area of the field **[2]**

(ii) Is the estimate greater or less than the true area ? Justify your answer. **[1]**

(b) Let α and β be the roots of the equation $0 = 2x^2 - 5x + 1$. Find the values of: **[5]**

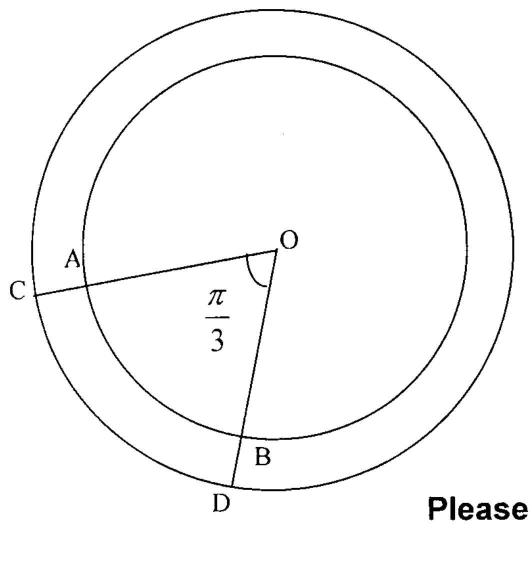
(i) $\frac{2}{\alpha} + \frac{2}{\beta}$

(ii) $\beta^2 + \alpha^2$

(c) The diagram, which is not to scale, shows two concentric circles with centre O. The radius of the larger circle is 9.4 cm

(i) The area of sector AOB is 18.4cm^2 Calculate the radius of this sector AOB, correct to 2 decimal places **[2]**

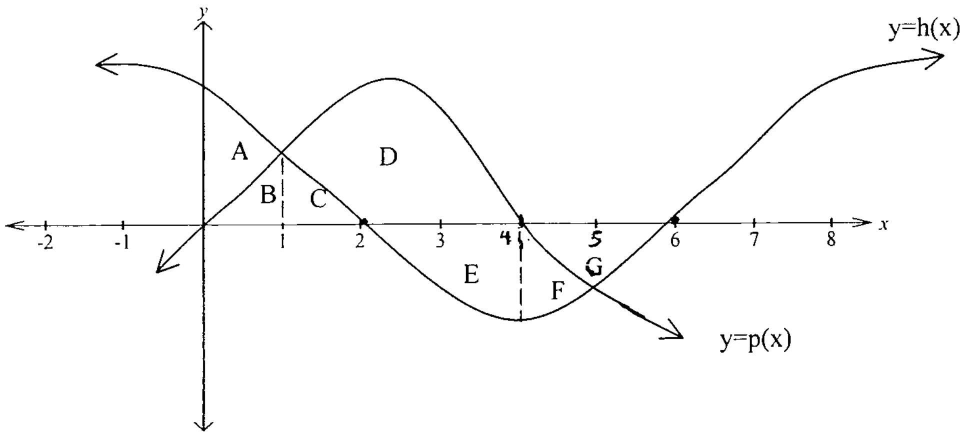
(ii) Calculate the area of triangle COB, correct to 1 decimal place. **[2]**



Please turn over to question 6

Question 6. Use a separate writing booklet.

(a) The two curves $y = p(x)$ and $y = h(x)$ are sketched below. Different areas enclosed by the curves and the axes are labelled A to G.



The integrals below represent the sum of which areas. **[3]**

(i) $\int_0^4 p(x)dx$

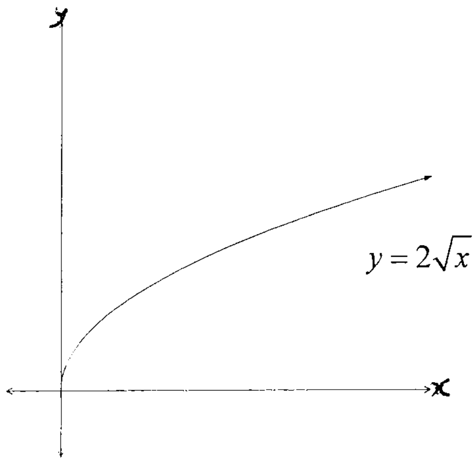
(ii) $\int_1^5 p(x) - h(x)dx$

(iii) Give an integral which would define the area denoted by letter G

(b) **(i)** Find the points of intersection of the curve $y = x^2 - 2x - 3$ and the line $y = 3x - 3$ **[2]**

(ii) Calculate the area of the region enclosed by the graphs **[4]**

(c) The region enclosed by the curve $y = 2\sqrt{x}$ and the y axis between y=1 and y=3 is rotated about the y axis. Find the exact volume of the solid of revolution formed. **[3]**



Please turn over to question 7

Question 7. Use a separate writing booklet.

(a) Consider the curve $y = x^3 + 3x^2 - 9x - 5$

(i) Find $\frac{dy}{dx}$ **[1]**

(ii) Find the co-ordinates of the two stationary points. **[3]**

(iii) Determine their nature. **[2]**

(iv) Sketch the curve for the domain $-5 \le x \le 3$ **[2]**

(v) By drawing an appropriate line on your graph or otherwise, solve **[2]**

$$0 = x^3 + 3x^2 - 9x + 5$$

(b) Find the limiting sum of the series $\frac{9}{8}, \frac{3}{4}, \frac{1}{2} .....$. **[2]**

Please turn over to question 8

Question 8. Use a separate writing booklet.

(a) Find a number n which when added to each of 2, 5 and 9 will give a set of three numbers in geometric progression. **[3]**

(b) A couple decide to borrow $350 000 to buy a home. Interest is calculated monthly on the balance still owing, at a rate of 7.2% per annum. The loan is to be repaid at the end of 20 years with equal monthly repayments of $M.
Let the amount left owing after the *nth* repayment be A_n.

(i) Derive an expression for A_1, the amount owing at the end of the first month, just after the first repayment has been made. **[1]**

(ii) Write an expression involving M for A_{240}, the last repayment **[2]**

(iii) Show that M= $\frac{2100(1.006)^{240}}{1.06^{240}-1}$ **[2]**

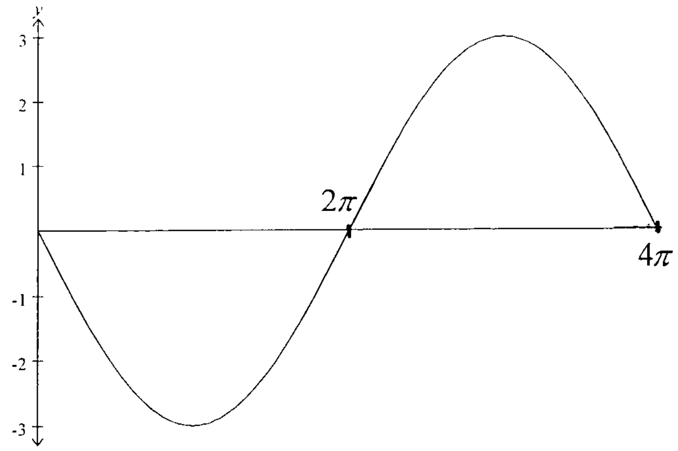
(iv) Find the value of M to the nearest cent. **[1]**

(c) The diagram below represents a possible sine or cosine curve.

(i) Give the amplitude **[1]**

(ii) Give the period. **[1]**

(iii) Write down a possible equation for the curve. **[1]**



Please turn over to question 9

Question 9. Use a separate writing booklet.

(a) The electrical charge Q retained by a capacitor t minutes after charging is given by $Q = Q_0 e^{-kt}$ where Q_0 and k are constants. The charge after 20 minutes is one half of the initial charge.

(i) Show that $k = \frac{1}{20}\log_e 2$ **[3]**

(ii) How long will it be before only one tenth of the original charge is retained ? (Answer to the nearest minute) **[3]**

(b) A particle moves in a straight line so that its distance x, in metres, from a fixed point O at time t, in seconds, is given by $x = 5t + \log_e(1-2t)$, $0 \le t \le \frac{1}{2}$.

(i) Find the particles initial velocity and acceleration. **[4]**

(iii) When does the particle come to rest ? **[2]**

Question 10. Use a separate writing booklet.

(a) A rectangle has two of its vertices on the curve $y = 4x - x^2$ and the other two on the x axis in the interval $0 \le x \le 4$

(i) If the height of the rectangle is c cm, show that its area is $2c\sqrt{4-c}$ square cm **[4]**

(ii) Show that the greatest value of this area is $\left(\frac{32\sqrt{3}}{9}\right) cm^2$ **[5]**

(b) Consider the quadratic equation in x:

$$(p^2+q^2)x^2 + 2q(p+r)x + (q^2+r^2) = 0$$ **[3]**

Find a relation, in simplest form, between p, q and r such that this quadratic has two equal roots.

End of test.

FORM 6 MATHEMATICS (2 UNIT) TRIAL 2003

QUESTION 1

a) $\dfrac{x^3+y^4}{y^2} = \dfrac{\frac{3}{5}+\left(\frac{2}{5}\right)^2}{\frac{2}{5}}$

$= \dfrac{19}{10}$

b) $\displaystyle\int \frac{1}{\sqrt{x^2+16}}\,dx = \ln\left(x+\sqrt{x^2+16}\right)+c$

c) $|x-3| \geqslant 5$

$x-3 \geqslant 5$ or $x-3 \leqslant -5$

$x \geqslant 8$ or $x \leqslant -2$

d) $\sqrt{p^4-2p^2} = \sqrt{16\times 25 - 2\times 4\times 5}$

$= \sqrt{360}$

$= 6\sqrt{10}$

e) $2x^2+x-6 = (2x-3)(x+2)$

f) $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = a+b\sqrt{15}$

$lhs = \dfrac{(\sqrt{5}-\sqrt{3})(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$

$= \dfrac{5-2\sqrt{15}+3}{5-3}$

$= \dfrac{8-2\sqrt{15}}{2}$

$= 4-\sqrt{15}$

$\therefore a = 4$

$b = -1$

QUESTION 2

a) i) $\dfrac{d}{dx}(5-3x^2)^6 = -36x(5-3x^2)^5$

ii) $\dfrac{d}{dx}\, 3\tan\left(4x-\frac{\pi}{4}\right) = 12\sec^2\left(4x-\frac{\pi}{4}\right)$

iii) $\dfrac{d}{dx}\,\dfrac{\log_e 3x}{x} = \dfrac{x.\frac{3}{3x}-\log 3x}{x^2}$

$= \dfrac{1-\log 3x}{x^2}$

b) $\displaystyle\int_0^1 5x+\sin 5x\,dx = \left[\frac{5x^2}{2}-\frac{\cos 5x}{5}\right]_0^1$

$= \dfrac{5}{2}-\dfrac{\cos 5}{5}-\left(0-\dfrac{\cos 0}{5}\right)$

$= 2\frac{1}{2}-\frac{1}{5}\cos 5+\frac{1}{5}$

$= 2.64$ (2dp)

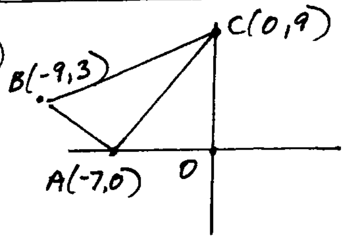
c) $\displaystyle\int_1^e \frac{2}{x}+\frac{x}{2}\,dx = \left[2\ln x+\frac{x^2}{4}\right]_1^e$

$= 2\ln e+\dfrac{e^2}{4}-2\ln 1-\dfrac{1}{4}$

$= 1\frac{3}{4}+\dfrac{e^2}{4}$ or $\dfrac{7+e^2}{4}$

QUESTION 3

a) i)



ii) $m_{AB} = \dfrac{3-0}{-9+7} = -\dfrac{3}{2}$

$m_{BC} = \dfrac{9-3}{0+9} = \dfrac{6}{9} = \dfrac{2}{3}$

$m_{AB}\times m_{BC} = -\frac{3}{2}\times\frac{2}{3} = -1$

$\therefore AB \perp BC$

iii) $y-0 = -\frac{3}{2}(x+7)$

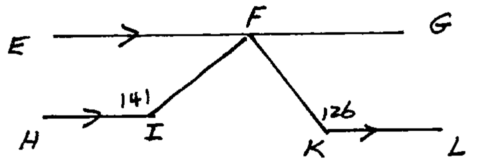
$y = -\frac{3}{2}x-\frac{21}{2}$

$2y = -3x-21$

$3x+2y+21 = 0$

iv) $d_{AB} = \sqrt{3^2 + (-2)^2}$
$= \sqrt{13}$

v) $pd = \left| \frac{0+0+21}{\sqrt{9+4}} \right|$
$= \frac{21}{\sqrt{13}}$
$= \frac{21\sqrt{13}}{13}$

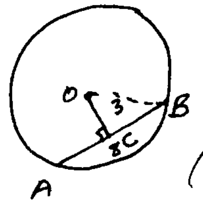
b) 

$\angle EFI = 39°$ (co-int ∠s $EF \parallel HI$)
$\angle GFK = 54°$ (co-int ∠s $FG \parallel KL$)
$\therefore \angle IFK = 87°$ (st line)

QUESTION 4

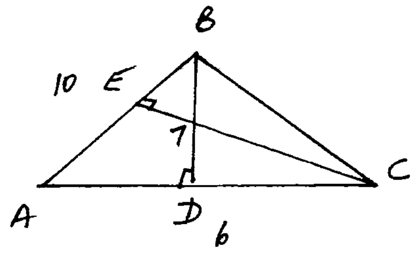
a) $0 = x^2 - 6y + 4x + 16$
i) $x^2 + 4x = 6y - 16$
$x^2 + 4x + 4 = 6y - 12$
$(x+2)^2 = 6(y-2)$
$(x+2)^2 = 4 \times \frac{3}{2}(y-2)$

ii) Vertex $= (-2, 2)$
Directrix: $y = \frac{1}{2}$

b) 

OB = 5 (Pythag)
b/c CB = 4
(line fr centre ⊥ chord)

To find ∠COB: $\tan \angle COB = \frac{4}{3}$
$\angle COB \doteqdot 0.927\ldots^c$
$l = r\theta$
$= 5 \times 0.927\ldots \times 2$
$= 9.27$ (2dp) cm

c) 

i) In Δs ECA, DBA
∠CEA = ∠BDA (rt ∠s given)
∠A is common
∴ ΔECA ||| ΔDBA (equiangular)

ii) $\frac{EC}{DB} = \frac{CA}{BA} = \frac{EA}{DA}$ (corr sides sim Δs)
$\frac{EC}{7} = \frac{6}{10}$
CE = 4.2 cm

QUESTION 5

a) i) Area $\doteqdot \frac{h}{2}(y_0 + y_2 + 2y_1)$
$\doteqdot \frac{60}{2}(62 + 67 + 2 \times 50)$
$\doteqdot 6870 \text{ m}^2$

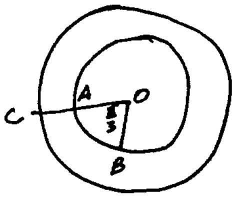
ii) Estimate is greater because the two trapezia formed are greater than the actual area.

b) $2x^2 - 5x + 1 = 0$
$\alpha + \beta = \frac{5}{2}$, $\alpha\beta = \frac{1}{2}$

i) $\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2\beta + 2\alpha}{\alpha\beta}$
$= \frac{2(5/2)}{1/2}$
$= 10$

ii) $\beta^2 + \alpha^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= \frac{25}{4} - 1$
$= 5\frac{1}{4}$

b) 9.4 → C



i) Area $= \frac{1}{2} r^2 \theta$

$18.4 = \frac{1}{2} r^2 \pi/3$

$r = \sqrt{\frac{18.4 \times 6}{\pi}}$

$= 5.93$cm (2dp)

ii) Area $\Delta COB = \frac{1}{2} 9.4 \times 5.9\ldots \sin\frac{\pi}{3}$

$= 24.1$ cm^2 (1dp)

c) $y = 2\sqrt{x}$

$y/2 = \sqrt{x}$

$x^2 = \left(\frac{y^2}{4}\right)^2$

$= \frac{y^4}{16}$

$Vol = \pi \int_1^3 \frac{y^4}{16} dy = \pi \left[\frac{y^5}{16 \times 5}\right]_1^3$

$= \pi \left[\frac{243}{80} - \frac{1}{80}\right]$

$= \frac{121\pi}{40} u^3$

=

QUESTION 6

a) i) $\int_0^4 p(x)\,dx = B + C + D$

ii) $\int_1^5 p(x) - h(x)\,dx = D + E + F$

iii) $\left|\int_4^5 p(x)\,dx\right| + \left|\int_5^6 h(x)\,dx\right|$

b) i) $y = x^2 - 2x - 3$ & $y = 3x - 3$

$x^2 - 2x - 3 = 3x - 3$

$x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0$ or 5

∴ pts of int $(0,-3)$ & $(5,12)$

ii) Area $= \int_0^5 3x - 3 - x^2 + 2x + 3\,dx$

$= \int_0^5 5x - x^2\,dx$

$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$

$= \frac{125}{2} - \frac{125}{3} - 0$

$= 20\frac{5}{6}$ sq units

QUESTION 7

a) $y = x^3 + 3x^2 - 9x - 5$

i) $y' = 3x^2 + 6x - 9$

ii) for st pts $y' = 0$

$3(x^2 + 2x - 3) = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

$y = 22, -10$

∴ st pts are $(-3, 22)$ & $(1, -10)$

iii) $y'' = 6x + 6$

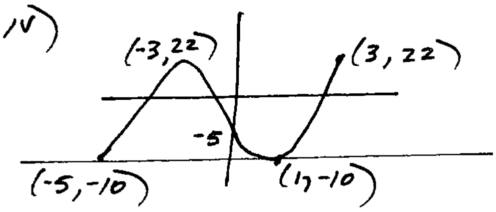
If $x = -3$ $y'' = -18 + 6 < 0$

∴ $(-3, 22)$ max t/pt

If $x = 1$, $y'' = 6 + 6 > 0$

∴ $(1, -10)$ min t/pt

iv)



v) $x^3 + 3x^2 - 9x + 5 = 0$

$x^3 + 3x^2 - 9x - 5 = -10$

$y = -10$ drawn on graph

∴ $x = -5$ or 1

b) $\frac{9}{8}, \frac{3}{4}, \frac{1}{2}$ $\quad r = \frac{3/4}{9/8} = \frac{2}{3}$

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{\frac{9}{8}}{1-2/3}$$
$$= \frac{27}{8}$$

QUESTION 8

a) Let no. be x

$2+x,\ 5+x,\ 9+x$

$$\frac{5+x}{2+x} = \frac{9+x}{5+x}$$

$$25+10x+x^2 = 18+11x+x^2$$
$$-x = -7$$
$$x = 7$$

$\therefore$ 7 must be added

b) i) $A_1 = 350000(1.006) - M$

ii) $A_2 = 350000(1.006)^2 - M(1+1.006)$

$$A_{240} = 350000(1.006)^{240} - M(1+1.006+\cdots 1.006^{239})$$
$$= 350000(1.006)^{240} - M\left(\frac{1.006^{240}-1}{0.006}\right)$$

But $A_{240} = 0$

iii) $$M = \frac{350000(1.006)^{240} \times 0.006}{1.006^{240}-1}$$
$$= \frac{2100(1.006)^{240}}{1.006^{240}-1}$$

iv) $M = \$2755.72$

c) i) amp $= 3$

ii) period $= 4\pi$

iii) $y = -3\sin\frac{x}{2}$

QUESTION 9

a) i) $Q = Q_0 e^{-kt}$

$$\tfrac{1}{2}Q_0 = Q_0 e^{-k\times 20}$$
$$\tfrac{1}{2} = e^{-20k}$$
$$\ln\tfrac{1}{2} = -20k$$
$$k = \frac{-\ln 2}{-20}$$
$$= \frac{\ln 2}{20}$$

ii) $$\frac{1}{10} = e^{-\frac{\ln 2}{20}t}$$
$$\ln\frac{1}{10} = -\frac{\ln 2}{20}t$$
$$t = \frac{-20\ln\frac{1}{10}}{\ln 2}$$
$$\doteqdot 66.438\ldots$$
$$= 66 \text{ min (to n min)}$$

b) $x = 5t + \log(1-2t)$ $\quad 0 \le t \le \frac{1}{2}$

i) $$v = 5 + \frac{1}{1-2t} \times -2$$
$$= 5 - \frac{2}{1-2t}$$

$t=0 \quad v = 3$ m/s

$$a = \frac{-4}{(1-2t)^2}$$

$t=0 \quad a = -4\,\text{m/s}^2$

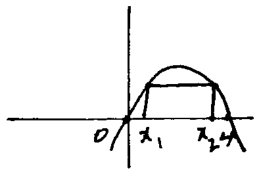
ii) $v = 0$

$$5 = \frac{2}{1-2t}$$
$$5 - 10t = 2$$
$$10t = 3$$
$$t = 3/10$$

At rest after $\frac{3}{10}$ sec

QUESTION 10

a)



$$y = 4x - x^2$$
$$= x(4-x)$$

i) $y = c$: $c = 4x - x^2$

$$x^2 - 4x + c = 0$$
$$x = \frac{4 \pm \sqrt{16-4c}}{2}$$
$$= 2 \pm \sqrt{4-c}$$
$$x_2 = 2 + \sqrt{4-c}$$
$$x_1 = 2 - \sqrt{4-c}$$

length of rec $= x_2 - x_1$
$$= 2\sqrt{4-c}$$

Area $= c \times 2\sqrt{4-c}$
$$= 2c\sqrt{4-c} \text{ cm}^2$$

ii) $A' = 2c \cdot \frac{1}{2}(4-c)^{-1/2} \cdot -1 + \sqrt{4-c} \times 2$
$$= \frac{-c}{\sqrt{4-c}} + 2\sqrt{4-c}$$

Max A when $A' = 0$, ~~$A'' < 0$~~

$$\frac{c}{\sqrt{4-c}} = 2\sqrt{4-c}$$
$$c = 8 - 2c$$
$$3c = 8$$
$$c = \frac{8}{3}$$

Test A'

c	2	$8/3$	3
A'	1.4…	0	−1

$\therefore$ max A when $c = 8/3$

Max Area $= 2 \times \frac{8}{3}\sqrt{4 - 8/3}$
$$= \frac{16}{3}\sqrt{4/3}$$
$$= \frac{16 \times 2}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{32\sqrt{3}}{9} \text{ cm}^2$$

b) $(p^2+q^2)x^2 + 2q(p+r)x + (q^2+r^2) = 0$

$$\Delta = 4q^2(p+r)^2 - 4(p^2+q^2)(q^2+r^2)$$
$$= 4q^2(p^2+2pr+r^2) - 4(p^2q^2+p^2r^2+q^4+q^2r^2)$$
$$= 4p^2q^2 + 8q^2pr + 4q^2r^2 - 4p^2q^2 - 4p^2r^2 - 4q^4 - 4q^2r^2$$
$$= 8q^2pr - 4p^2r^2 - 4q^4$$

For equal rts $\Delta = 0$

$$4(2q^2pr - p^2r^2 - q^4) = 0$$
$$p^2r^2 - 2q^2pr + q^4 = 0$$
$$(pr - q^2)^2 = 0$$
$$pr = q^2$$