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BARKER COLLEGE

FORM VI

MATHEMATICS

2/3 UNIT

AM WEDNESDAY 17 AUGUST

TRIAL H.S.C. 1988.

TIME; 3 HOURS

200 copies

INSTRUCTIONS:

* ALL questions may be attempted, and are of equal value.
* ALL necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.
* Standard integrals are printed on the last page which may be removed.
* Silent, non-programmable calculators may be used.
* Each question attempted is to be returned in a separate booklet clearly marked Question 1, Question 2, etc. on the cover. Each booklet must show your Candidate's Number.
* If you do not attempt a question, you must still hand in a booklet for that question, with NOT ATTEMPTED written clearly on the front.

* * * * * * * * *

QUESTION 1: (use a separate book)

(i) Express 493.461

a) in scientific notation

b) correct to the nearest tenth

(ii) Simplify $2(2x - 5) - (4 - x)$

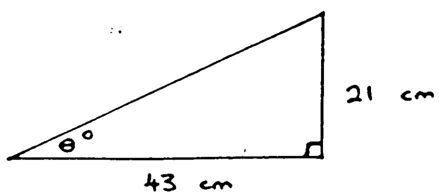
(iii) Solve for x:

a) $\dfrac{x - 1}{3} = \dfrac{2x + 5}{2}$

b) $5 - 3x < 8$

(iv) If $s = ut + \frac{1}{2}at^2$, find s if $u = 15.2$, $t = 3$, and $a = 9.8$

(v) Find the value of θ in the diagram (correct to the nearest degree).



QUESTION 2: (use a separate book)

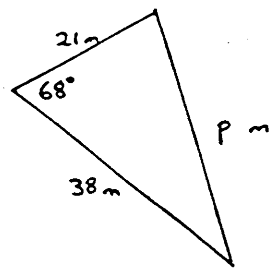
(i) Solve $|3x - 5| = 7$

(ii) Find the product of 4×10^{-3} and 3.2×10^5, expressing your answer in scientific notation

(iii) Rationalise the denominator: $\dfrac{1}{2 - \sqrt{3}}$

(iv) Factorise fully: $5m^3 + 40$

(v) Find the value of p in the diagram, correct to 2 decimal places



(vi) Solve $3^x = 11$, correct to 3 decimal places.

QUESTION 3: (use a separate book)

(i) Find the domain of the function $F(x) = \sqrt{x - 5}$

(ii) The numbers $x + 3$, $6x$, 18 are in geometric progression. Find the value(s) of x .

(iii) Differentiate the following expressions with respect to x:

a) $x^3 - 3x$

b) $\dfrac{2}{\sqrt{x}}$

c) $\cos(3x - 2)$

d) $\dfrac{\log_e x}{x}$

QUESTION 4: (use a separate book)

(i) Factorise fully: $x^3 - x^2 - x + 1$

(ii) Find primitives of the following functions (with respect to x)

a) $\sec^2 3x$

b) $\frac{1}{\sqrt{3x - 5}}$

(iii) The third and seventh term of an arithmetic progression are 15 and 59 respectively. Find:

a) the first term and common difference

b) the sum of the first 18 terms

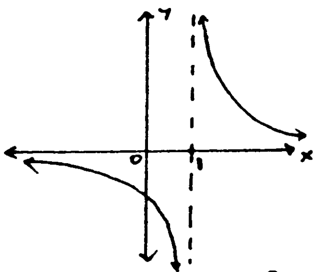
(iv) Comment on the validity of the following statement:

"The first term of a geometric progression is 5 and its sum to infinity is 2".

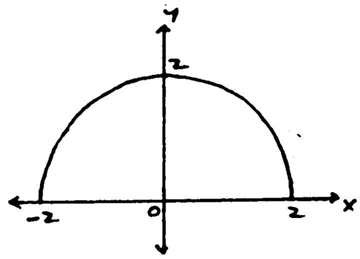
QUESTION 5: (use a separate book)

(i) Write down possible equations for the following curves:

a)



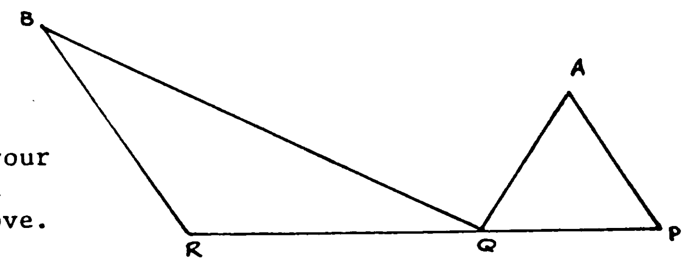
b)



(ii) Evaluate $\lim_{x \to 3} \left(\frac{x^2 - 2x - 3}{x - 3} \right)$

(iii) In the diagram PQR is a straight line,

PA = PQ, RB = RQ and AP||BR



a) Reproduce the diagram in your answer book, showing on it the features mentioned above.

b) Prove (giving reasons) that $A\hat{Q}B = 90°$

(iv) ABCD is a quadrilateral whose diagonals bisect each other. Use congruent triangles to prove that AB||DC

QUESTION 6: (use a separate book)

(i) Find the equation of the normal to the curve $y = \sqrt{x}$ at the point where $x = 4$ (give your answer in general form).

(ii) Consider the expression $p(x) = 2x^2 + 2x + k + 3$

a) For what values of k does this expression have real roots?

b) For what value(s) of k is the sum of the roots twice the product of the roots?

(iii) Find the centre and radius of the circle $x^2 + 4x + y^2 - 2y = 11$

(iv) If $\cos x = \tan x$, find the values of $\sin x$

QUESTION 7: (use a separate book)

(i) For the parabola $(y - 2)^2 = 2x + 6$ find:

a) the vertex

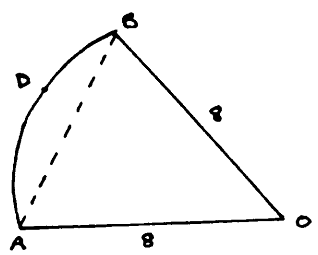
b) the focal length

c) the equation of the directrix

(ii) The gradient function of a curve is given by $y' = 3x^2 - 4$. If the curve passes through the point $(-1,5)$, find the equation of the curve.

(iii) Find the equation of the locus of all points equidistant from the points $(-2,1)$ and $(2,3)$.

(iv)



The diagram shows a sector of a circle, centre O, whose radius is 8 cm. If the area of the triangle AOB is 25 cm^2, and $A\hat{O}B$ is acute, find:

a) the size of $A\hat{O}B$ (in radians, correct to two decimal places)

b) the length of the arc ADB, correct to the nearest millimetre.

QUESTION 8: (use a separate book)

(i) Find the EXACT value of $\int_3^4 \frac{x}{x^2 - 8}\,dx$ in its simplest form

(ii) a) Find the co-ordinates of the points of intersection of the two curves $y = 4x - x^2 - 3$ and $y = x^2 - 2x + 1$

b) Calculate the area contained by the two curves between the points of intersection.

(iii) Frank decides he wants to retire in 25 years time and collect superannuation totalling \$500 000. He decides to invest a set sum of money in a fund which offers an interest rate of 14% per annum (compounded quarterly). If he makes his first payment now, and his last payment 3 months before he retires, what sum of money will he have to invest each quarter to realise his ambition (give your answer to the nearest dollar)?

QUESTION 9: (use a separate book)

(i) a) Expand and simplify $(x + \frac{1}{x})^2$

b) The area between the curve $y = x + \frac{1}{x}$, the x-axis and the ordinates $x = 1$ and $x = 3$ is rotated about the x-axis. Find the exact volume of the solid formed.

c) What is the size of the error if this volume is calculated using the trapezoidal rule with 3 ordinate values (give your answer as a fraction of π)?

ii) A 4 metre piece of wire is cut into 3 pieces, which are bent to form a square and two congruent circles.

a) If the radius of each circle is r metres, show that the total area (A square metres) of the 3 figures is given by

$$A = 2\pi r^2 + (1 - \pi r)^2$$

b) Find the value of r and the length of the side of the square which will make this total area a minimum (give your answers in EXACT form).

QUESTION 10: (use a separate book)

(i) a) Sketch the curve $y = 3 \sin 2x$, $0 \le x \le 2\pi$

b) Solve the equation $\sin 2x = \frac{1}{3}$, $0 \le x \le 2\pi$, given that the smallest solution in this domain is 0.17 (give all answers correct to two decimal places) {HINT: use your sketch in part (a)}

(ii) a) Find the stationary point(s) for the curve $y = xe^x$ and determine its (their) nature.

b) Sketch the curve $y = xe^x$, showing stationary point (s), intercepts and asymptotes (there is no need to find points of inflexion)

c) Hence or otherwise find the value(s) of k for which the equation $xe^x = k$ has

α) 2 solutions

β) 1 solution

E N D O F P A P E R

1 i) a) 4.93461×10^2

b) 493.5

ii) $2(2x-5)-(4-x) = 4x-10-4+x$
$= 5x-14$

iii) a) $\frac{x-1}{3} = \frac{2x+5}{2}$

$6x+15 = 2x-2$

$4x = -17$

$x = -\frac{17}{4}$

b) $5-3x < 8$

$-3 < 3x$

$-\frac{3}{3} < x$

ie $x > -1$

iv) $s = (15.2)(3) + \frac{1}{2}(9.8)(3)^2$

$= 89.7$

v) $\tan\theta = \frac{21}{43}$

$\theta = 26°$

2 i) $|3x-5| = 7$

$3x-5 = 7$ or $3x-5 = -7$

$3x = 12$ or $3x = -2$

$x = 4$ or $x = -\frac{2}{3}$

ii) $4 \times 10^{-3} \times 3.2 \times 10^5 = 12.8 \times 10^2$

$= 1.28 \times 10^3$

iii) $\frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3}$

$= 2+\sqrt{3}$

iv) $5m^3+40 = 5(m^3+2^3)$

$= 5(m+2)(m^2-2m+4)$

v) $p^2 = 38^2+21^2 - 2\times21\times38\cos 68$

$p = 35.88$

vi) $3^x = 11$

$\log_3 11 = x$

$x = \frac{\log_{10} 11}{\log_{10} 3} = 2.183$

3 i) $x-5 \geq 0$, $x \geq 5$

ii) $\frac{6x}{x+3} = \frac{18}{6x} = \frac{3}{x}$

$\therefore 6x^2 = 3x+9$

$2x^2-x-3 = 0$

$2x^2-3x+2x-3 = 0$

$x(2x-3)+1(2x-3) = 0$

$(x+1)(2x-3) = 0$

$x = -1$ or $\frac{3}{2}$

iii) a) $y = x^3-3x$

$y' = 3x^2-3$

b) $y = 2x^{-1/2}$

$y' = -x^{-3/2}$

$= \frac{-1}{x\sqrt{x}}$

c) $y = \cos(3x-2)$

$y' = -3\sin(3x-2)$

d) $y = \frac{\ln x}{x}$

$y' = \frac{x\frac{1}{x} - \ln x}{x^2}$

$= \frac{1-\ln x}{x^2}$

4 i) $x^3-x^2-x+1 = x^2(x-1)-1(x-1)$

$= (x^2-1)(x-1)$

$= (x+1)(x-1)(x-1)$

ii) a) $\int \sec^2 3x\, dx = \frac{1}{3}\tan 3x + c$

b) $\int (3x-5)^{-\frac{1}{2}} dx = \frac{(3x-5)^{\frac{1}{2}}}{\frac{1}{2}\times 3} + c$

$= \frac{2\sqrt{3x-5}}{3} + c$

iii) a) $\left.\begin{array}{l} u_3 = 15 \\ u_7 = 59 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a+2d = 15 \\ a+6d = 59 \end{array}\right\} \Rightarrow 4d = 44$

$\therefore d = 11$, $a = -7$

b) $S_{18} = \frac{18}{2}\{2\times-7 + 17\times11\} = 9\times173 = 1557$

iv) $a = 5$, $\frac{a}{1-r} = 2$

$\therefore \frac{5}{1-r} = 2$ $\therefore 5 = 2-2r$

$2r = -3$

$r = -1\frac{1}{2}$ ie $|r| > 1$

but S_∞ cant exist if $|r| > 1$

$\therefore$ there is no such sequence.

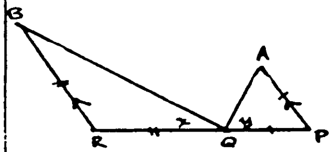
5 i) a) $y = \frac{1}{x-1}$ b) $y = \sqrt{4-x^2}$

ii) $\lim_{x\to3} \frac{x^2-2x-3}{x-3} = \lim_{x\to3} \frac{(x-3)(x+1)}{x-3}$

$= \lim_{x\to3} (x+1)$

$= 4$

iii)



b) Label $x = B\hat{Q}R$, $y = P\hat{Q}A$ as shown.

Then $B\hat{R}Q = 180-2x$ (∠ sum of isos △BRQ)

and $A\hat{P}Q = 180-2y$ (" " " △APQ)

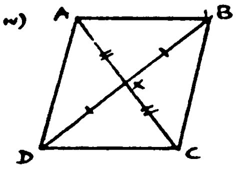
These are coint. on BR ∥ AP

$\therefore 180-2x+180-2y = 180$

$\therefore x+y = 90$

$\therefore A\hat{Q}B = 90°$ (str. ∠)

iv)



Let the diagonals intersect at X.

In △'s ABX, CDX

BX = DX (AC bisects BD)

AX = CX (BD bisects AC)

$A\hat{X}B = C\hat{X}D$ (vert. opp)

$\therefore$ △ABX ≡ △CDX (SAS)

$\therefore A\hat{B}X = C\hat{D}X$ (corr. ∠'s of ≡ △'s)

$\therefore$ AB ∥ DC (alt. ∠'s equal)

6 i) $y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

when $x=4$ $y' = \frac{1}{4}$

$\therefore m = -4$

when $x = 4$ $y = 2$

$\therefore$ point is $(4,2)$

$\therefore$ equation is $y-2 = -4(x-4)$

$y-2 = -4x+16$

$4x+y-18 = 0$

ii) a) $\Delta \geq 0$ $\Delta = 4 - 4\times2(k+3)$

$= 4 - 8(k+3)$

≥ 0

$8(k+3) \leq 4$

$k+3 \leq \frac{1}{2}$

$k \leq -2\frac{1}{2}$

b) $\alpha+\beta = -\frac{2}{2} = -1$

$\alpha\beta = \frac{k+3}{2}$

$\alpha+\beta = 2\alpha\beta$

$-1 = k+3$

$k = -4$

iii) $x^2+4x+y^2-2y = 11$

$x^2+4x+4+y^2-2y+1 = 16$

$(x+2)^2+(y-1)^2 = 4^2$

$\therefore$ centre $= (-2,1)$ radius $= 4$

iv) $\cos x = \frac{\sin x}{\cos x}$

$\cos^2 x = \sin x$

$1-\sin^2 x = \sin x$

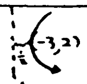
$\sin^2 x + \sin x - 1 = 0$

$\sin x = \frac{-1 \pm \sqrt{1-4\times1\times-1}}{2}$

$= \frac{-1\pm\sqrt{5}}{2}$

⑦ i) $(y-2)^2 = 2(x+3)$

$(y-2)^2 = 4 \times \frac{1}{2}(x+3)$



a) V (-3, 2) b) $\frac{1}{2}$ c) $x = -3\frac{1}{2}$

ii) $y' = 3x^2 - 4$

$y = x^3 - 4x + c$

$5 = (-1)^3 - 4(-1) + c$

$5 = -1 + 4 + c$

$c = 2 \quad \therefore y = x^3 - 4x + 2$

iii) $(x+2)^2 + (y-1)^2 = (x-2)^2 + (y-3)^2$

$x^2 + 4x + 4 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2 - 6y + 9$

$8x + 4y - 8 = 0$

$2x + y - 2 = 0$

iv) a) $\frac{1}{2}(8)^2 \sin\theta = 25$

$\sin\theta = \frac{25}{32}$

$\theta = 0.90$

b) $l = r\theta$

$= 8 \times 0.90$

$= 7.2$ cm (nearest mm)

⑧ i) $\int_3^4 \frac{x}{x^2-8}\,dx = \frac{1}{2}\int_3^4 \frac{2x}{x^2-8}\,dx$

$= \frac{1}{2}\left[\ln(x^2-8)\right]_3^4$

$= \frac{1}{2}\{\ln 8 - \ln 1\}$

$= \frac{1}{2}\ln 8$

$= \frac{1}{2}\ln 2^3$

$= \frac{3}{2}\ln 2$

ii) a) $\left.\begin{array}{l} y = 4x - x^2 - 3 \\ y = x^2 - 2x + 1 \end{array}\right\}$

$4x - x^2 - 3 = x^2 - 2x + 1$

$0 = 2x^2 - 6x + 4$

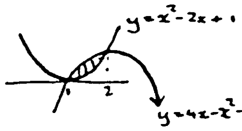
$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 1$ or 2

$\therefore$ pts of intersection are $(1,0)$ and $(2,1)$

b)



$A = \int_1^2 4x - x^2 - 3 - (x^2 - 2x + 1)\,dx$

$= \int_1^2 6x - 2x^2 - 4\,dx$

$= \left[3x^2 - \frac{2x^3}{3} - 4x\right]_1^2$

$= (12 - \frac{16}{3} - 8) - (3 - \frac{2}{3} - 4)$

$= 4 - 5\frac{1}{3} + 1 + \frac{2}{3}$

$= \frac{1}{3}\ u^2$

iii) Let m be the amount of each investment:

1st becomes: $m(1 + \frac{14}{400})^{100} = m(1.035)^{100}$

2nd " $m(1 + \frac{14}{400})^{99} = m(1.035)^{99}$

⋮

100th " $m(1 + \frac{14}{400})^1 = m(1.035)^1$

$\therefore$ Total is $m(1.035) + m(1.035)^2 + \dots + m(1.035)^{100}$

which is a GP $a = m(1.035)$

$r = 1.035$

$n = 100$

$\therefore$ Total is $\frac{m(1.035)(1.035^{100} - 1)}{0.035} = 500000$

$\therefore m = \frac{500000 \times 0.035}{1.035(1.035^{100} - 1)} = \$560$

⑨ a) $(x + \frac{1}{x})^2 = x^2 + 2.x.\frac{1}{x} + (\frac{1}{x})^2$

$= x^2 + \frac{1}{x^2} + 2$

b) $V = \pi\int_1^3 (x + \frac{1}{x})^2\,dx$

$= \pi\int_1^3 x^2 + x^{-2} + 2\,dx$

$= \pi\left[\frac{x^3}{3} - \frac{1}{x} + 2x\right]_1^3$

$= \pi\left[(9 - \frac{1}{3} + 6) - (\frac{1}{3} - 1 + 2)\right]$

$= \frac{40\pi}{3} = 13\frac{1}{3}\pi$

c) $\int_1^3 (x + \frac{1}{x})^2 \doteqdot \frac{1}{2}\left\{(1+1)^2 + (3+\frac{1}{3})^2 + 2\times(2+\frac{1}{2})^2\right\}$

$= \frac{1}{2}\left\{4 + \frac{100}{9} + \frac{25}{2}\right\}$

$= 13\frac{29}{36}$

$\therefore V \doteqdot 13\frac{29}{36}\pi$

$\therefore$ error is $(\frac{29}{36} - \frac{1}{3})\pi = \frac{17}{36}\pi$

ii)

$2\pi r + 2\pi r + 4l = 4$

$4(\pi r + l) = 4$

$\therefore l = 1 - \pi r$

$\therefore$ Area $= \pi r^2 + \pi r^2 + (1 - \pi r)^2$

ie $A = 2\pi r^2 + (1 - \pi r)^2$

$A' = 4\pi r + 2(1 - \pi r)(-\pi)$

$= 4\pi r - 2\pi(1 - \pi r)$

$= 0$ when $4\pi r = 2\pi(1 - \pi r)$

$2r = (1 - \pi r)$

$r(\pi + 2) = 1$

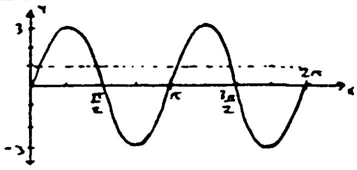
$r = \frac{1}{\pi + 2}$

$A'' = 4\pi + 2\pi^2 > 0$

$\therefore r = \frac{1}{\pi+2}$ gives a minimum area

& $l = 1 - \pi r = 1 - \frac{\pi}{\pi+2} = \frac{2}{\pi+2}$

⑩ i)



b) $\sin 2x = \frac{1}{3}$

$\Leftrightarrow 3\sin 2x = 1$

$\therefore$ solve graphically

sol^{ns} are:

$0.17, \frac{\pi}{2} - 0.17, \pi + 0.17$

and $\frac{3\pi}{2} - 0.17$

ie $0.17, 1.40, 3.31, 4.54$

ii) $y = xe^x$

$y' = xe^x + e^x$

$= e^x(x+1)$

$= 0$ when $x = -1$

test	x	-2	-1	0
	y'	$-$	0	$+$
		╲	—	╱

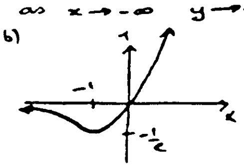
$\therefore (-1, -\frac{1}{e})$ is a min t.p.

when $x = 0$ $y = 0$

$\therefore$ curve passes through origin

as $x \to -\infty$ $y \to 0$ $\therefore$ x axis is asymptote on left

b)



c) by considering points of intersection with $y = k$, we can see that

$xe^x = k$ has

a) 2 sol^{ns} if $-\frac{1}{e} < k < 0$

b) 1 sol^n if $k \geq 0$ or $k = -\frac{1}{e}$