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BARKER COLLEGE

FORM VI

MATHEMATICS

2/3 UNIT

AM THURSDAY 17 AUGUST
H.S.C. TRIAL 1989.
TIME: 3 HOURS

200 copies

INSTRUCTIONS:

- *ALL questions may be attempted, and are of equal value.*
- *All necessary working should be shown in every question. Marks may not be awarded for careless or badly arranged work.*
- *Standard integrals are printed at the end of the paper.*
- *Approved calculators may be used.*
- *Each question attempted is to be returned in a separate Booklet marked Question 1, Question 2, etc on the cover. Each Booklet must show your Candidate's Number.*
- *If you do not attempt a question, you must still hand in a Booklet for that question with NOT ATTEMPTED written clearly on the front.*

* * * * * * * * * * *

Question 1:

(a) Evaluate: $\dfrac{\sqrt{16\cdot 04}}{1\cdot 2 \times 3\cdot 56}$ correct to 2 decimal places.

(b) Solve the following equations:

(i) $y^2 - y - 12 = 0$

(ii) $\dfrac{x+1}{2} - \dfrac{x-1}{3} = 10$

(iii) $9^x = \dfrac{1}{27}$

(c) Factorise and simplify: $\dfrac{4m^2 - 4}{m - 1}$

(d) Rationalise the denominator and simplify: $\dfrac{\sqrt{2}}{\sqrt{2} - 1}$

(e) Given that $V = \frac{4}{3}\pi R^3$, find the value of R if V = 1000. Give your answer correct to one decimal place.

Question 2:

(a) The points A and B have coordinates (-3,6) and (5,2) respectively. Plot these points on a number plane.

(i) Find the equation of the line ℓ joining the points A and B.

(ii) The line k is drawn through B with a gradient of 2. Show that the equation of k is:

$$2x - y - 8 = 0$$

(iii) Show that the line k passes through P(2,-4).

(iv) Prove that the triangle ABP is right-angled.

(v) Find the area of triangle ABP.

(b) Sketch the graph of $y = \sqrt{9 - x^2}$. State its domain and range.

Question 3:

(a)

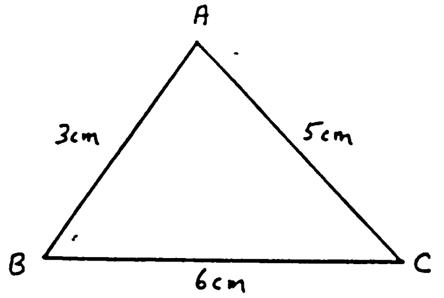


figure not to scale

(i) Use the Cosine rule to calculate $\angle BAC$ to the nearest degree.

(ii) Calculate the area of the triangle ABC. Give your answer correct to two significant figures.

(b) An aircraft flies 500 km from its base on a bearing of 050°T. It then flies 240 km due North. Calculate its distance (nearest km) from its base and the bearing (nearest degree) in which it must fly to return to its base in a direct line.

(c) Prove that: $\tan A \sin A + \cos A = \sec A$

(d) Sketch (not on graph paper): $y = 3 \sin 2x$ for $0 \le x \le 2\pi$

Question 4:

(a)

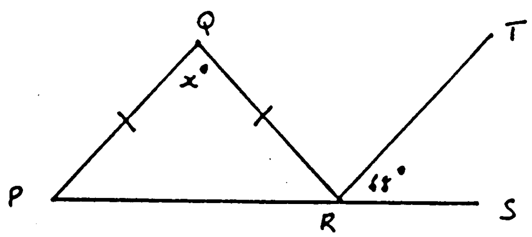
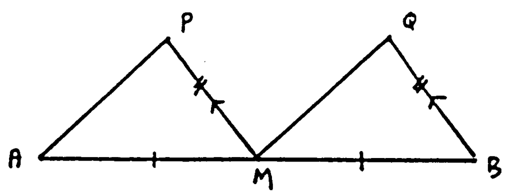


figure not to scale

Given that RT is parallel to PQ and that triangle PQR is isosceles, find the value of x, giving reasons.

(b)



M is the midpoint of AB, MP = BQ and MP||BQ

(i) draw a neat copy in your exam book.

(ii) Prove that $\Delta PAM \equiv \Delta QMB$

(iii) Prove: AP||MQ

/Q4.Cont.........

QUESTION 4 C'tinued:

(c)

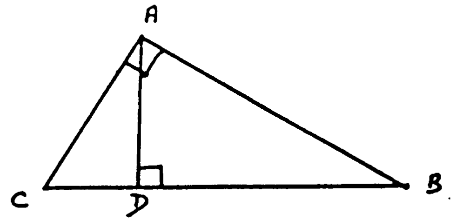


figure not to scale

ABC is a triangle, right-angled at A and AD is drawn perpendicular to BC.

AB = 15 cm and AD = 12 cm

(i) draw a neat sketch and mark on it all the given information

(ii) calculate the length of BD

(iii) *PROVE* ΔABC is similar to ΔDBA

(iv) find the length of AC

Question 5:

(a) Differentiate with respect to x:

(i) $y = 4x^3 - 2x + \frac{1}{x^2} - 1$

(ii) $y = x^2e^x$

(iii) $y = \frac{x + 1}{x - 1}$

(b) Evaluate the following integrals:

(i) $\int_1^4 (x - \frac{1}{x^2})\ d$

(ii) $\int_0^1 (2x - 1)^4\ d$

(iii) $\int_0^{\frac{\pi}{4}} \cos 2x\ d$

Question 6:

(a) Without solving the equation $2x^2 - 3x + 6 = 0$ which has the roots α and β, find the value of:

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $\alpha^2 + \beta^2$

(b) For what values of k does the equation $x^2 - (k + 1)x + 1 = 0$ have:

(i) equal roots

(ii) no real solutions

(c) Given that the parabola has the focus at (2,1) and vertex at (2,3) find:

(i) focal length

(ii) directrix

(iii) axis of symmetry

(iv) equation of the parabola

(v) sketch the parabola

Question 7:

(a) Which term of the sequence 2, 6, 18 .... is 486?

(b) A contractor undertakes to bore a well for \$9 for the first 10 metres, \$10 for the next 10 metres, \$11 for the next 10 metres and so on.
What is the cost of a well 500 metres deep?

(c) Given: $f(x) = x^3 - 12x$

(i) find the coordinates of the maximum and minimum turning points

(ii) find the coordinates of any points of inflexion (if they exist)

(iii) DRAW a neat sketch, indicating all essential features.

Question 8:

(a) Evaluate the $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 5}{2x^2 + 3x - 7}$

(b) Find the equation of the curve $y = f(x)$ given that $\dfrac{dy}{dx} = 1 - 5x^4$ and that the curve passes through (1,3)

(c) Sketch the curves $y = x^2 - 1$ and $y = 1 - x^2$ on the SAME number plane.
Hence find the area enclosed between the two curves.

(d) Calculate the area enclosed between the curve $y = \dfrac{1}{4x + 1}$, the x axis, $x = 1$ and $x = 5$

(i) by Simpson's rule using 3 function values

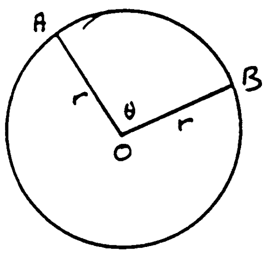
(ii) by integration (the exact value).

Question 9:

(a) Solve the equation: $2 \cos x + 1 = 0$ for $0 \le x \le 2\pi$

(b) Find the equation of the tangent to the curve $y = \sin 2x$ when $x = \dfrac{\pi}{2}$

(c)



In a circle radius r, the arc AB subtends an angle θ at the centre.

(i) Find an expression for the length of arc AB and hence write down an expression for the perimeter of the sector AOB

(ii) If the perimeter of the sector AOB is 8 cm, write an expression for θ in terms of r

(iii) Find the *maximum* area of the sector AOB when the perimeter of the sector AOB is 8 cm.

Question 10:

(a) Find the volume of the solid of revolution formed by rotating the curve $y = e^x + e^{-x}$ about the x axis between $x = -1$ and $x = 1$

(b) On the birth of their daughter Karen, the proud parents Richard and Helen decided to start an investment fund for her. They agreed to deposit \$500 on the 1st August 1989 (the date of birth) and \$500 on *every* subsequent birthday up to and *including* her 21st birthday. If we assume the interest is being paid at the rate of 15% p.a. (compound interest), find how much Karen will receive on her 21st birthday?

(c) The temperature of a cup of black coffee is given by $T = 100e^{-t/5}$ where t is the time in minutes.
If it is too hot to drink above 55°C and too cold below 25°C, calculate the length of time during which the coffee is drinkable (to the nearest second).

E N D O F P A P E R

Unit Solutions (Trials)

1. a) 0·94   2 d.p. (1)

b) i) $(y-4)(y+3)=0$

$y=4,-3$ (1)

ii) $3(x+1)-2(x-1)=60$ (1)

$x+5=60$

$x=55$ (1)

iii) $3^{2x}=3^{-3}$ (1) $\Rightarrow x=-\frac{3}{2}$ (1)

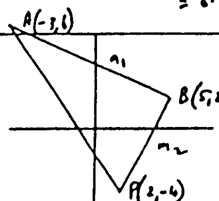
c) $\frac{4(m^2-1)}{m-1}=\frac{4(m-1)(m+1)}{(m-1)}$ (1) $=4(m+1)$ (1)

d) $\frac{\sqrt{2}.(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$ (1) $=\frac{2+\sqrt{2}}{2-1}=2+\sqrt{2}$ (1)

e) $R=\sqrt[3]{\frac{3\times 1000}{4\pi}}$ (1) $=6\cdot2035$

$=6\cdot2$ (1) 1 d.p.

2. a)



(1)

i) eqn AB $m_1=\frac{6-2}{-3-5}=-\frac{4}{8}=-\frac{1}{2}$ (1)

$y-6=-\frac{1}{2}(x+3)$

$2y-12=-x-3$

$x+2y-9=0$ (1)

ii) $m=2$, $(5,2)$ $y-2=2(x-5)$

$2x-y-8=0$ (1)

iii) if $(2,-4)$ lies on line ∴ must satisfy the eqn. $4+4-8=0$ YES (1)

iv) since $m_1=-\frac{1}{2}$ & that $m_2=2$ the gradient of line k & that $m_1m_2=-1$ (1)

∴ ΔABP is rt.∠'d

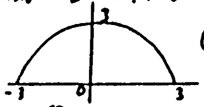
v) area $=\frac{1}{2}$BP.AB

$BP=\sqrt{3^2+6^2}=\sqrt{45}=3\sqrt{5}$ (1)

$AB=\sqrt{(8)^2+4^2}=\sqrt{80}=4\sqrt{5}$ (1)

∴ area $=\frac{1}{2}3\sqrt{5}.4\sqrt{5}=30$ unit^2 (1)

b)



(1)

$0\le y\le 3$, (1) $-3\le x\le 3$ (1)

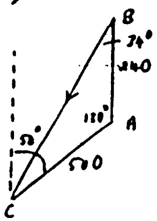
Q3. a) i) $\cos A=\frac{b^2+c^2-a^2}{2bc}$ (1) $=\frac{5^2+3^2-6^2}{2.15}=-\frac{1}{15}$ (1)

$\angle BAC=94°$ nearest deg. (1)

ii) area ΔBAC $=\frac{1}{2}\times3\times5\sin94$

$=7\cdot5cm^2$ 2 sig.fig. (1)

b)



Cos. Rule

$a^2=b^2+c^2-2bc\cos A$ (1)

$=500^2+240^2-2(500)(240)\cos120$

$a=674\cdot6$

$=680$ km (1)

Sine Rule $\frac{a}{\sin A}=\frac{b}{\sin B}$ (1)

$\frac{680}{\sin120}=\frac{500}{\sin B}$ now $\sin120=\sin50$

∴ $\sin B=\frac{500\sin50}{680}$

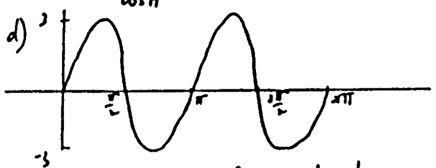
$\angle B=34°$ (1)

∴ bearing is $180+34=214°T$

c) $\frac{\sin A}{\cos A}.\sin A+\cos A$ (1)

$=\frac{\sin^2A+\cos^2A}{\cos A}=\frac{1}{\cos A}=\sec A$ (1)

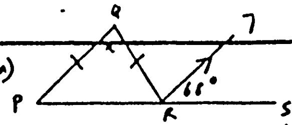
d)



(1) for period & shape

(1) for amplitude

Q4. a)

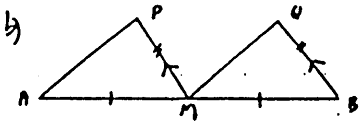


$\angle QPR=68°$ corresp. angles PQ‖RT

$\angle QPR=\angle QRP=68°$ base angle, isos.Δ

∴ $x=180-(2\times68)=44°$ (3)

b)



$\angle PMA=\angle QBM$ corresp ∠'s PM‖QB

PM = QM given

AM = MB given (3)

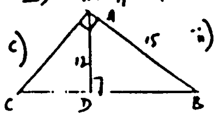
∴ ΔAPM ≡ ΔMQB (S.A.S.)

∴ $\angle PAM=\angle QMB$ corresp.∠. in congruent Δ's

since corresp. ∠'s equal ie $\angle PAM$ & $\angle QMB$

⇒ AP ∥ MQ (1)

c)

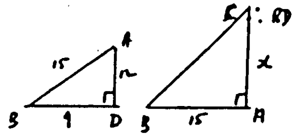


ii) in ΔADB Pythag.

$BD^2=AB^2-AD^2$

$=15^2-12^2$ (1)

∴ BD = 9



iii) $\angle BDA=\angle BAC=90°$ given (2)

$\angle ABD=\angle CBA$ common angle

∴ ΔABC is similar to ΔDBA

iv) let AC = x $\frac{9}{15}=\frac{12}{x}$ $x=\frac{12\times15}{9}$

$=20$ (2)

Q5. a) i) $y'=12x^2-2-2x^{-3}$ (2)

ii) $y'=2xe^x+x^2e^x$ (2)

$=xe^x(2+x)$

iii) $y'=\frac{(x-1)1-(x+1)1}{(x-1)^2}=\frac{-2}{(x-1)^2}$ (2)

b) i) $\int_1^4(x-x^{-2})dx=\left[\frac{1}{2}x^2+x^{-1}\right]_1^4$

$=(8+\frac{1}{4})-(\frac{1}{2}+1)$ (2)

$=6\frac{3}{4}$

ii) $\int_0^1(2x-1)^4dx=\left[\frac{(2x-1)^5}{2.5}\right]_0^1$ (2)

$=\frac{1}{10}-(\frac{-1}{10})=\frac{1}{5}$

iii) $\int_0^{\frac{\pi}{4}}\cos2x\,dx=\left[\frac{1}{2}\sin2x\right]_0^{\frac{\pi}{4}}$

$=\frac{1}{2}\sin\frac{\pi}{2}-\frac{1}{2}\sin0$ (2)

$=\frac{1}{2}$

Q6. a) $2x^2-3x+6=0$

i) $\alpha+\beta=-\frac{b}{a}=\frac{3}{2}$ (1)   ii) $\alpha\beta=\frac{c}{a}=3$ (1)

iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ (1)

$=(\frac{3}{2})^2-2.3=\frac{9}{4}-6$

$=-3\frac{3}{4}$ (1)

b) i) Δ = 0 for equal roots

$b^2-4ac=(k+1)^2-4(1)(1)=0$ (1)

$k^2+2k+1-4=0$

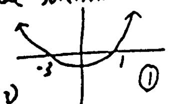
$k^2+2k-3=0$

$(k+3)(k-1)=0$ (1)

$k=1,-3$

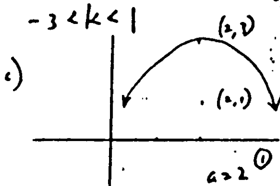
ii) Δ < 0 for no real solutions

ie. $(k+3)(k-1)<0$



$-3<k<1$ (1)

c)



(1) for graph

a=2 (1) $x^2=-4ay$

$(x-h)^2=-8(y-k)$

$(x-2)^2=-8(y-3)$ (1)

d: y=5 (1), axis x=2 (1)

Q7. a) $a=2, \quad r=3$

$T_n = ar^{n-1} = 486$ ①

$2.3^{n-1} = 486$

$3^{n-1} = 243 = 3^5$ ①

$n-1 = 5$

$n = 6$ ①

b) $n=50, \quad a=9, \quad d=1$ ①

$S_n = \frac{n}{2}[2a+(n-1)d]$ ①

$= 25(18+49) = \$1675$ ①

c) $y = x^3 - 12x$

① $y' = 3x^2 - 12 = 3(x^2-4)$

$3(x-2)(x+2) = 0$

$x = 2, -2 \quad (2,-16)\ (-2,16)$

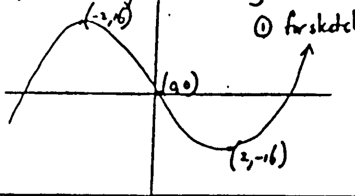
① $y'' = 6x$, when $x=2$, $y''>0$ $(2,-16)$ min

① $x=-2$, $y''<0$ $(-2,16)$ max

① at $x=0$, $y''=0$ $(0,0)$ pt. of infl.

① check - change in concavity

① for sketch



Q8. a) $\lim_{x\to\infty} \dfrac{1-\frac{5}{x}+\frac{5}{x^2}}{2+\frac{3}{x}-\frac{2}{x^2}} = \frac{1}{2}$ ①

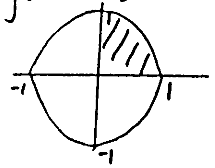
b) $y' = 1-5x^4$

$y = x - x^5 + c_1$ ①

$3 = 1 - 1 + c_1 \quad (c_1 = 3)$

$y = x - x^5 + 3$ ①

c)



① for top graph

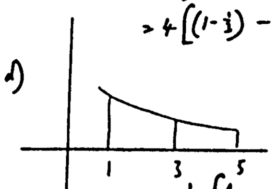
① for bottom graph

area enclosed $= 4\int_0^1 (1-x^2)\,dx$ ①

$= 4\left[x - \frac{1}{3}x^3\right]_0^1$

$= 4\left[(1-\frac{1}{3}) - 0\right] = \frac{8}{3}$ ①

d)



①

i) Simpson's Rule $A \doteqdot \frac{h}{3}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]$

$h = \frac{b-a}{n} = \frac{5-1}{2} = 2$ ①

$= \frac{2}{3}\left[\frac{1}{5}+\frac{1}{21}+4.\frac{1}{13}\right]$

$= 0.37$ ①

ii) $A = \int_a^b y\,dx = \int_1^5 \frac{1}{4x+1}\,dx = \frac{1}{4}\int_1^5 \frac{4}{4x+1}\,dx$ ①

$= \frac{1}{4}\left[\log(4x+1)\right]_1^5$

$= \frac{1}{4}\left[\log 21 - \log 5\right] = \frac{1}{4}\log\left(\frac{21}{5}\right)$ ①

Q9. a) $2\cos x + 1 = 0, \quad \cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$ ②

b) $y' = 2\cos 2x$ at $(\frac{\pi}{2}, 0)$

$m = 2\cos\pi = 2\times -1 = -2$ ①

$y - 0 = -2(x - \frac{\pi}{2})$

$2x + y - \pi = 0$ ①

c) arc $AB = r\theta$ ①

let $P = 2r + r\theta$

now $P = 8 = 2r + r\theta$

$r\theta = 8 - 2r, \quad \theta = \frac{8}{r} - 2$ ①

area sector $A = \frac{1}{2}r^2\theta$ ①

$= \frac{1}{2}r^2\left(\frac{8}{r}-2\right)$

$= 4r - r^2$ ①

$\frac{dA}{dr} = 4 - 2r = 0$ ①

$r = 2$

$\frac{d^2A}{dr^2} = -2 < 0 \Rightarrow$ MAX ①

$\therefore$ max area $= \frac{1}{2}r^2\theta$ when $r=2$ ①

$\theta = \frac{8}{2} - 2 = 2$

$= \frac{1}{2}.4.2$

$= 4\text{ cm}^2$ ①

Q10 a) $V = \int_a^b \pi y^2\,dx$ around x axis ①

$= \int_{-1}^{1} \pi(e^x+e^{-x})^2\,dx = 2\int_0^1 \pi(e^x+e^{-x})^2\,dx$

since even fn.

$= 2\pi\int_0^1 \left(e^{2x} + 2e^x.e^{-x} + e^{-2x}\right)dx$ ①

$= 2\pi\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_0^1$ ①

$= 2\pi\left[\left(\frac{1}{2}e^2+2-\frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^0+0-\frac{1}{2}e^0\right)\right]$

$= 2\pi\left(\frac{1}{2}e^2+2-\frac{1}{2}e^{-2}-\frac{1}{2}+\frac{1}{2}\right)$

$= 2\pi\left(\frac{1}{2}e^2+2-\frac{1}{2}e^{-2}\right)$ ①

b) $A = P\left(1+\frac{r}{100}\right)^n$ ①

$A = 500(1.15)^{21} + 500(1.15)^{20} + \dots + 500(1.15)$

21st birthday - no interest ①

$= 500\left[1 + 1.15 + 1.15^2 + \dots + 1.15^{21}\right]$

$= 500\,\frac{a(r^n-1)}{r-1} = 500\,\frac{1.(1.15^{22}-1)}{1.15-1}$ ①

$= \frac{500\,(21.644746-1)}{0.15}$

$= \$68815.82$ ①

c) $T = 100.e^{-t/5}$

$T = 55 = 100\,e^{-t/5}$

$0.55 = e^{-t/5}$

$\log(0.55) = -t/5$

$t = 2.989$ min ②

$T = 25 = 100.e^{-t/5}$

$0.25 = -t/5$

$t = 6.9314$

change in time $= 3.9$ min ②

$= 3$ min 56 s (nearest sec.)