BARKER 2000

QUESTION 1. [12 marks]

(a) If the surface area of a sphere is given by $A = 4\pi r^2$, find the radius (correct to 2 decimal places) if the surface area is $500\ \text{cm}^2$. [2m]

(b) Find the value of e^2 correct to 3 significant figures. [2m]

(c) Rationalise the denominator of $\dfrac{2}{\sqrt{2} - 1}$ [2m]

(d) Factorise fully $3 - 12x^2$ [2m]

(e) Solve $x(x - 4) = 5$ [2m]

(f) If $\sin\theta = \dfrac{2}{\sqrt{5}}$, evaluate $\tan\theta$ for $90° < \theta < 180°$ [2m]

166

QUESTION 2. [START A NEW PAGE] [12 marks]

(a)

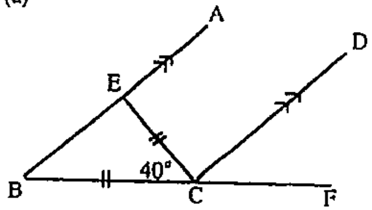


DIAGRAM NOT DRAWN TO SCALE

Given $AB \parallel CD$, $\angle BCE = 40°$ and $BC = EC$, find the size of $\angle ECD$, giving reasons. [3m]

(b) Draw a neat sketch showing the points $A(3, -1)$, $B(7, 2)$ and $C(1, 10)$ on a number plane.

(i) Find the gradients of BC and of AB. [2m]

(ii) State why ABC is a right-angled triangle. [1m]

(iii) Find the area of ΔABC. [2m]

(iv) Find M, the midpoint of AC. Show that the equation of the line passing through M and parallel to the line AB is $3x - 4y + 12 = 0$. [2m]

(v) Find the equation of the circle which passes through the points A, B and C with centre M. [2m]

QUESTION 3. [START A NEW PAGE] [12 marks]

(a) Differentiate the following functions:

(i) $y = (4x + 3)^2$ [2m]

(ii) $y = xe^{4x}$ [2m]

(b) Find the exact value of $\int_0^2 e^{3x}\,dx$ [2m]

(c) (i) Sketch the graph of $y = |x - 2|$, showing all the main features. [1m]

(ii) State the domain and range of $y = |x - 2|$ [1m]

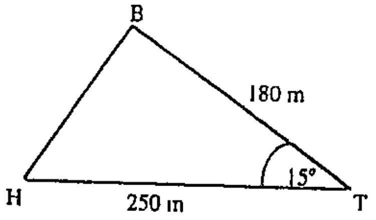
(d) If $f(x) = \begin{cases} x^2 - 1 & \text{for } x > 1 \\ x & \text{for } x \le 1 \end{cases}$

find the value of $f(-2) + f(1) + f(3)$ [2m]

(e) Evaluate $\lim_{x \to 0} \dfrac{2x}{x^2 + x}$ [2m]

QUESTION 4. [START A NEW PAGE] [12 marks]

(a)



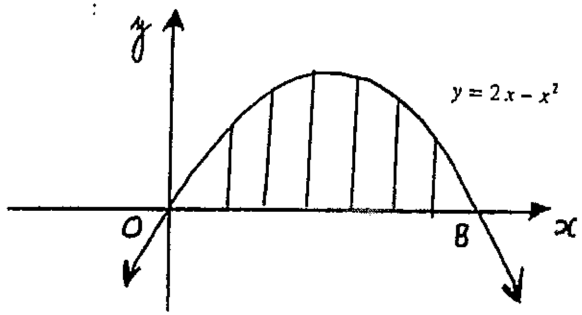
On a golf course, the distance from a tee, T, to the hole H is 250 metres.
A golfer's ball comes to rest at point B, 180 metres from T.
Angle HTB is 15°, as shown in the diagram.
How far is B from H? [3m]

(b) A supermarket displays cans of fruit in the form of a "pyramid".
There are 3 cans in the top row, 5 in the next row, 7 in the next row and so on.
If there are 20 rows on display, find:

(i) the number of cans in the bottom row;

(ii) the total number of cans in the display. [4m]

(c)



The diagram shows the graph of the function $y = 2x - x^2$.

(i) Find the x coordinate of the point B where the curve crosses the positive x-axis.

(ii) Find the area of the shaded region contained by the curve $y = 2x - x^2$ and the x-axis.

(iii) Write down a pair of inequalities that specify the shaded region. [5m]

QUESTION 5. [START A NEW PAGE] [12 marks]

(a) Consider the curve given by $f(x) = x^3 - 12x$

(i) Find $f'(x)$

(ii) Find the coordinates of the stationary points and determine their nature.

(iii) Find any points of inflexion.

(iv) Draw a neat sketch of the curve.

(v) Show that $f(x)$ is odd.

[7m]

(b) Given $\log_a 6 = 1.792$ and $\log_a 2 = 0.693$, evaluate $\log_a 24$

[2m]

(c) Katrina's chance of obtaining a ticket to the Olympic Swimming Finals is 28%, but she has a 52% chance of obtaining a ticket to the Olympic Badminton Finals.

Using a tree diagram (or otherwise), find the probability that she obtains:

(i) both Final tickets;

(ii) only one of the Finals Tickets.

[3m]

QUESTION 6. [START A NEW PAGE] [12 marks]

(a) For the parabola $x^2 = 8(3 - y)$:

(i) Find the coordinates of the vertex.

(ii) Sketch this parabola on a number plane showing the vertex, focus and the equation of the directrix.

(iii) For what values of x is this parabola always positive?

[6n

(b) Find the value of k for which the equation $3x^2 + 10x + k = 0$ has:

(i) one root which is the reciprocal of the other;

(ii) equal roots.

[3m]

(c)

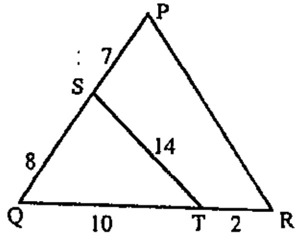


DIAGRAM IS NOT TO SCALE

Given the diagram:

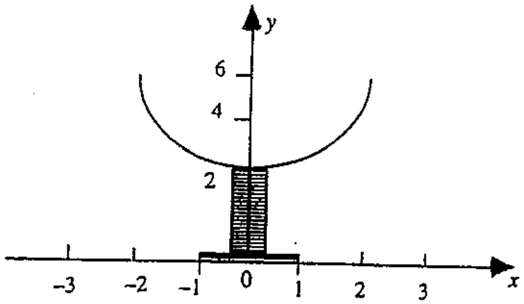
(i) Prove that ΔQST is similar to ΔQRP.

(ii) Hence find the length of PR.

[3m]

QUESTION 7. [START A NEW PAGE] [12 marks]

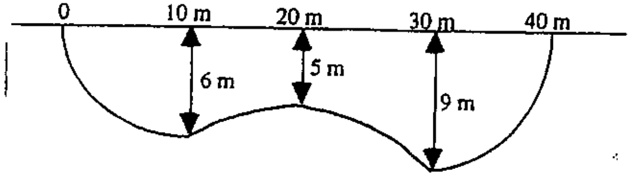
(a)



The bowl of a wine glass is formed by rotating the arc of the curve $y = x^2 + 2$ between (0, 2) and (2, 6) about the y-axis.

Find the volume of the bowl of the glass so formed. [3m]

(b)



The diagram above represents a cross-section through a river, with the depth of the river marked every 10 metres as shown.

(i) **COPY** and **COMPLETE** this table in your paper.

x	0	10	20	30	40
$f(x)$					

(ii) Use Simpson's Rule with 5 function values to estimate the area of the cross-section.

(iii) If the river is flowing at the rate of 2 m per second, what volume of water passes through this cross-section each minute? [5m]

(c) Find the equation of the curve $y = f(x)$, given that $\frac{d^2y}{dx^2} = 2x + 1$ and that there is a stationary point at $(1, -2)$. [4m]

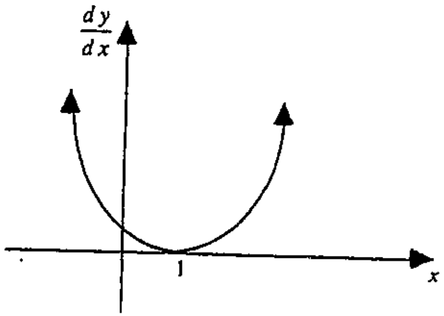
QUESTION 8. [START A NEW PAGE] [12 marks]

(a) A plant is observed over a period of time. Its initial height is 30 cm. It grows 5 cm during the first week of observation. In each succeeding week the growth, in height, is 80% of the previous week's growth.

Assuming this pattern continues, calculate the plant's ultimate height. [3m]

(b) Solve $4^x - 2(2^x) - 8 = 0$ for x [4m]

(c) Consider the graph of the derivative $\frac{dy}{dx}$ given below.



(i) Comment on the sign of $\frac{dy}{dx}$ for all x except $x = 1$.

What does this imply about the curve $y = f(x)$ for all x, except $x = 1$?

(ii) What can you conclude about $y = f(x)$ when $x = 1$?

(iii) Sketch $\frac{d^2y}{dx^2}$

(iv) Sketch a possible graph of $y = f(x)$. [5m]

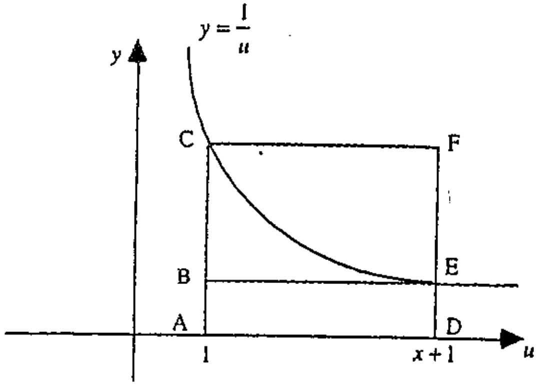
QUESTION 9. [START A NEW PAGE] [12 marks]

(a) (i) Find the derivative of $y = e^{x^2}$

(ii) Hence, evaluate $\int_0^2 x e^{x^2} dx$ **[2m]**

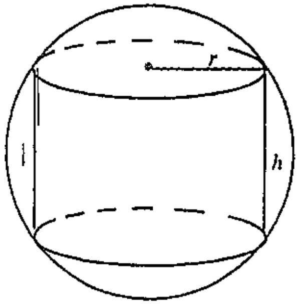
(b) $\int \frac{2x \, dx}{2 - 5x^2}$ **[2m]**

(c)



Given this diagram and using the fact that area ABED < area ACED < area ACFD show that $\frac{x}{x+1} < \ell n\,(x+1) < x$ **[3m]**

(d)



A cylinder of radius (r) and height (h) fits exactly inside a sphere of radius (R).

If the sphere has a radius of 12 cm:

(i) show that the volume of the cylinder can be written as

$$V = \pi\left(144h - \frac{h^3}{4}\right)$$

(ii) determine the exact value of the height (h) of the cylinder for it to be a maximum volume. **[5m]**

170

QUESTION 10. [START A NEW PAGE] [12 marks]

(a) Mr and Mrs Lee wish to buy a home unit in Hornsby. They obtained a loan of \$120 000 from a bank which they agreed to repay by equal monthly repayments. We assume that compound interest is calculated at a fixed rate of 7.2% p.a.

(i) If the **monthly** repayment is \$M, then show $A_1 = \$(120000 \times 1.006 - M)$ is the amount owing at the end of one month and hence write an expression for A_2, the amount owing at the end of two months.

(ii) Write an expression for A_n, the amount owing after n months.

(iii) If the \$120 000 loan (including interest charges) is exactly repaid at the end of 25 years, write an expression for the monthly repayment, \$M.

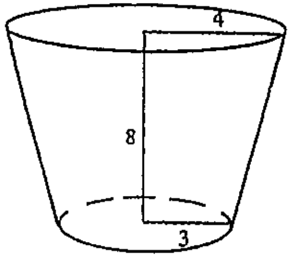
(iv) Calculate the monthly repayment. **[4m]**

(b) The curve $y = \frac{1}{2}\left(e^x + e^{-x}\right)$ is called a catenary.

(i) Draw a neat sketch of the catenary for $-2 \le x \le 2$.

(ii) Hence, find the volume of the solid of revolution by rotating around the x-axis the region under the curve in (i) and above the x-axis and between $x = -2$ and $x = 2$. **[4m]**

(c) A drinking glass has the shape of a truncated cone.



(i) If the internal radii of the base and the top are 3 cm and 4 cm respectively and the height is 8 cm, calculate the capacity of the glass.

(ii) If the glass is filled with water to a depth of 4 cm, find the volume of the water. **[4m]**

END OF EXAM

YEAR 12 2U TRIAL. 200

1. a) $A = 4\pi r^2$

$500 = 4\pi r^2$

$r = \sqrt{\frac{500}{4\pi}}$

$\doteqdot 6.31$ (2 d.p.)

b) $e^2 \doteqdot 7.39$

c) $\frac{2}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \frac{2\sqrt{2}+2}{2-1}$

$= 2\sqrt{2}+2.$

d) $3(1-4x^2)$

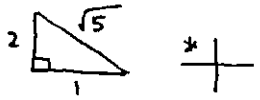
$= 3(1-2x)(1+2x)$

e) $x^2-4x-5=0$

$(x-5)(x+1)=0$

$x = 5, -1.$

f)



$\therefore \tan\theta = -\frac{2}{1} = -2.$

2.

i) ΔBEC is isos. (given)

∴ ∠CEB = 70° (∠ sum of isos. Δ.)

∴ ∠ECD = 70° (alt. ∠s // lines)

b) i) BC: $m_1 = \frac{2-10}{7-1} = -\frac{4}{3}$

AB: $m_2 = \frac{2-1}{7-3} = \frac{3}{4}.$

ii) $m_1 \times m_2 = -1.$

iii) AB: $d = \sqrt{(7-3)^2+(2-1)^2}$

$= 5$ units

BC: $d = \sqrt{(7-1)^2+(2-10)^2}$

$= 10$ units.

$\therefore A = \frac{1}{2} \times 10 \times 5$

Q3. a)

i) $y' = 2(4x+3) \times 4$

$= 8(4x+3).$

ii) $y' = x \cdot 4e^{4x} + e^{4x} \cdot 1$

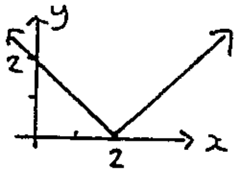
$= e^{4x}(4x+1).$

b) $I = \frac{1}{3}\left[e^{3x}\right]_0^2$

$= \frac{1}{3}(e^6 - e^0)$

$= \frac{1}{3}(e^6-1).$

c) i)



ii) $x \in \mathbb{R}$

$y \geq 0$

d) $-2+1+(3^2-1)$

$= 7.$

e) $\text{Limit} = \lim_{x\to 0} \frac{2x}{x(x+1)}$

$= \lim_{x\to 0} \frac{2}{x+1}$

$= \frac{2}{0+1}$

$= 2.$

Q4. a)

$a^2 = b^2+c^2-2bc\cos A$

$BH^2 = 180^2+250^2-2 \times 180 \times 250 \times \cos 15°$

$BH^2 = 7966.675625$

$\therefore BH = 89.25623578$ m

$\doteqdot 89.3$ (1 d.p.)

b) 3, 5, 7, ...

AP. $a=3$; $d=2$

i) $T_n = a+(n-1)d$

$T_{20} = 3+19 \times 2$

$= 41.$

∴ 41 cans in bottom row.

Q4. b) ii) $S_{20} = \frac{20}{2}(3+41)$

$= 440.$

440 cans in display.

c) i) $y = 2x-x^2$

$y = x(2-x).$

let $y=0$ ∴ $x(2-x)=0$

$x = 0$ & $2.$

∴ $B(2,0).$

ii) $A = \int_0^2 (2x-x^2)\,dx$

$= \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= \left(4-\frac{8}{3}\right) - 0$

$= \frac{4}{3}$ units^2

iii) $y \geq 0$, $y \leq 2x-x^2.$

Q5. a) $f(x) = x^3-12x$

i) $f'(x) = 3x^2-12$

ii) let $f'(x)=0$

$\therefore 3x^2-12=0$

$3(x^2-4)=0$

$x = +2, -2.$

when $x=2$, $f(2) = -16$ ∴ (2, -1

when $x=-2$ $f(-2) = 16$ ∴ (- , 1

$f''(x) = 6x$

when $x=2$, $f''(2) = 12$ ∴ mi t.

when $x=-2$ $f''(-2) = -12$ ∴ maxi t.

min t.p. $(2,-16)$ max t.p. $(-2, 16)$.

iii) $f''(x) = 6x$

let $f''(x)=0$

$\therefore 6x=0$

$x=0, y=0$

x	0^-	0	0^+
$f''(x)$	$-$	0	$+$

$f''(x)$ chang sign

∴ point of inflexion at (0,0

Q2 iv) $M = (2, 4\frac{1}{2})$ $m_{AB} = \frac{3}{4}$

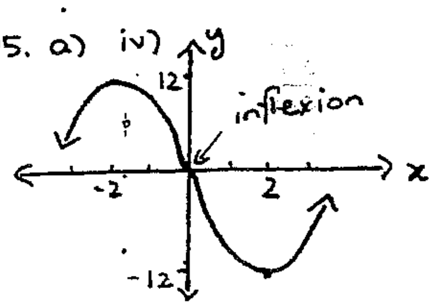
→ $4y-18 = 3x-6$

$\therefore 3x-4y+12=0$

→ diameter $\doteqdot \sqrt{125} = 5\sqrt{5}$

radius $= \frac{5}{2}\sqrt{5}$, centre

Q5. a) iv)



v) $f(-x) = (-x)^3 - 12(-x)$

$= -x^3 + 12x$

$= -(x^3 - 12x)$

$= -f(x).$

since $f(-x) = -f(x)$

$f(x)$ is odd.

b) $\log_a 24 = \log_a (6 \times 4)$

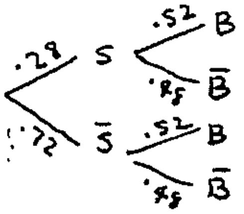
$= \log_a (6 \times 2^2)$

$= \log_a 6 + 2\log_a 2$

$= 1.792 + 2 \times .693$

$= 3.178$

c)



i) $P(SB) = 0.28 \times 0.52$

$= 0.1456$

ii) $P(S\bar{B}) + P(B\bar{S})$

$= .28 \times .48 + .52 \times .72$

$= .5088.$

6. a) i) $x^2 = 8(3-y)$

$x^2 = -8(y-3)$

$(x-h)^2 = -4a(y-k)$

i) vertex $(0,3)$

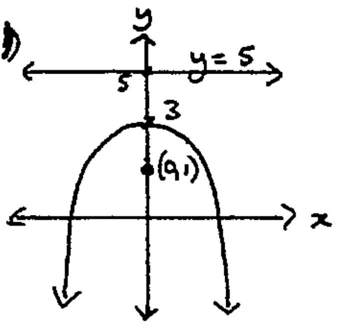
i) focal length $4a = 8$

$\therefore a = 2$

focus $(0,1)$

directrix $y = 5$.

ii)



iii) let $y = 0$

$\therefore x^2 = 24$

$x = \pm 2\sqrt{6}.$

Parabola is +ive

$-2\sqrt{6} < x < 2\sqrt{6}$.

b) i)

let roots be $\alpha, 1/\alpha$.

prod. of roots $= \frac{c}{a} = 1.$

for $3x^2 + 10x + k = 0$

$\alpha \cdot \frac{1}{\alpha} = \frac{k}{3}$

$1 = \frac{k}{3}$

$\therefore k = 3.$

ii) $\Delta = 0$

$b^2 - 4ac = 100 - 12k = 0$

$k = \frac{100}{12}$

$\therefore k = \frac{25}{3}$.

c) i) $\frac{QS}{QR} = \frac{QT}{QP}$

$\frac{8}{12} = \frac{10}{15}$

$\frac{2}{3} = \frac{2}{3}$.

$\angle SQT = \angle PQR$

$\therefore \Delta QST \mathbin{|||} \Delta QRP.$

2 sides in proportion and 1 angle (inc). equal.

ii) $\frac{ST}{PR} = \frac{2}{3}$

$\frac{14}{PR} = \frac{2}{3}$

Q7. a) $V = \pi \int_2^6 x^2 \cdot dy$

$= \pi \int_2^6 y - 2 \cdot dy$

$= \pi \left[\frac{y^2}{2} - 2y \right]_2^6$

$= \pi [18 - 12 - (2 - 4)]$

$= 8\pi$ units^3.

b) i)

x	0	10	20	30	40
$f(x)$	0	6	5	9	0
	y_1	y_2	y_3	y_4	y_5

ii) $A \doteqdot \frac{h}{3}\{0 + 0 + 4(6+9) + 2 \times 5$

$= 233\frac{1}{3}$ m^2 (h = 10)

iii) $V = A \times H$ (2m/se

$= 233\frac{1}{3} \times 120$ = 120m/

$= 28000$ m^3

c) $\frac{dy}{dx} = x^2 + x + c.$

when $x = 1$, $\frac{dy}{dx} = 0.$

$\therefore 1^2 + 1 + c = 0 \Rightarrow c = -2$

$\therefore \frac{dy}{dx} = x^2 + x - 2.$

$y = \frac{x^3}{3} + \frac{x^2}{2} - 2x + k$

when $x = 1$, $y = -2.$

$\therefore -2 = \frac{1}{3} + \frac{1}{2} - 2 + k$

$\therefore k = -5/6$

$\therefore y = \frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{5}{6}$.

Q8. a) $a = 5$, $r = 0.8$

$S_\infty = \frac{a}{1-r} = \frac{5}{1-0.8} = 25.$

$\therefore$ Ult. height $= 30 + 25 = 55$c

b) $4^x - 2(2^x) - 8 = 0.$

let $m = 2^x$

$m^2 - 2m - 8 = 0$

$(m-4)(m+2) = 0$

$\therefore m = 4, -2$

$\therefore 2^x = 4$ or $2^x = -2.$

$\frac{dy}{dx} > 0$ for all

x, $x \neq 1$. The

has positive

nt, except when $x=1$

is a stationary

on $y=f(x)$ as

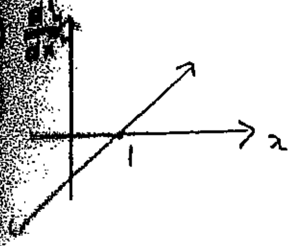
0 and if

$x<1$, $\frac{dy}{dx} > 0$

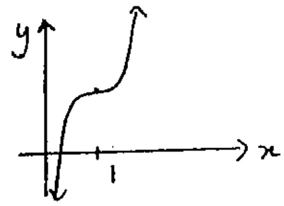
$x>1$, $\frac{dy}{dx} > 0$.

horizontal point

inflexion at $x=1$.



v)



9. a) i) $y' = 2xe^{x^2}$

$I = \frac{1}{2}\left[e^{x^2}\right]_0^2$

$= \frac{1}{2}(e^4 - e^0)$

$= \frac{1}{2}(e^4 - 1)$.

$I = \frac{-2}{10}\int \frac{-10x}{2-5x^2}\,.dx$

$= \frac{-1}{5}\ln(2-5x^2) + c$.

Area $ABED = [(x+1)-1] \times \frac{1}{x+1}$

$= \frac{x}{x+1}$.

Area $ACED = \int_1^{x+1} \frac{1}{u}\,.du$

$= [\ln u]_1^{x+1}$

$= \ln(x+1) - \ln 1$

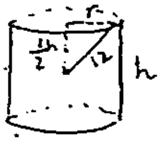
$= \ln(x+1)$.

Area $ACFD = [(x+1)-1] \times \frac{1}{1}$

$= x$.

$\therefore \frac{x}{x+1} < \ln(x+1) < x$.

d)



$r^2 = 12^2 - \left(\frac{h}{2}\right)^2$

$r^2 - 144 - \frac{h^2}{4}$

i) $V = \pi r^2 h$

$= \pi\left(144 - \frac{h^2}{4}\right).h$

$= \pi\left(144h - \frac{h^3}{4}\right)$.

ii) $\frac{dV}{dh} = \pi\left(144 - \frac{3h^2}{4}\right)$

let $\frac{dV}{dh} = 0$.

$\therefore \frac{3h^2}{4} = 144$

$h^2 = 192$

$h = \pm\sqrt{192}$

but h is height

$\therefore h = +\sqrt{192}$

h	10	$\sqrt{192}$	20
$\frac{dV}{dh}$	+69	0	-156

+ 0 −

/ — \

∴ maximum volume when

$\therefore h = 8\sqrt{3}$ cm.

Q10. a) i) 0.6% month.

$A_1 = 120000 \times 1.006 - M$

$A_2 = A_1 \times 1.006 - M$

$= (120000 \times 1.006 - M) \times 1.006 - M$

$= 120000 \times 1.006^2 - 1.006M - M$

$= 120000 \times 1.006^2 - M(1 + 1.006)$

ii) $A_n = 120000 \times 1.006^n - M(1 + 1.006 + \cdots + 1.006^{n-1})$

iii)

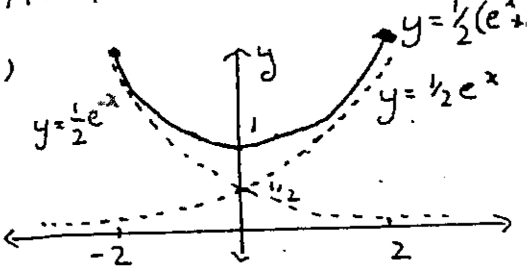
$\therefore 0 = 120000 \times 1.006^{200} - M(1 + 1.006 + \cdots + 1.0$

$0 = 120000 \times 1.006^{300} - M\left[\frac{1(1.006^{100} - 1)}{.006}\right]$

$\therefore M = 120000 \times 1.006^{300} \times \frac{.006}{1.006^{200} - 1}$

iv) M: \$863.51

b) i)



ii) $V = \pi\int_{-2}^{2}\left[\frac{1}{2}(e^x + e^{-x})\right]^2 .dx$

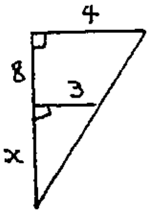
$= \frac{2\pi}{4}\int_0^2 (e^{2x} + 2 + e^{-2x})$.

$= \frac{\pi}{2}\left[\frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2}\right]_0^2$

$= \frac{\pi}{2}\left[\left(\frac{e^4}{2} + 4 - \frac{e^{-4}}{2}\right) - \left(\frac{e^0}{2} + 0 - \frac{e}{2}\right.\right.$

$= \frac{\pi}{4}(e^4 + 8 - e^{-4})$ units^3.

c) i)



similar Δs

$\therefore \frac{x}{x+8} = \frac{3}{4}$

$4x = 3x + 24$

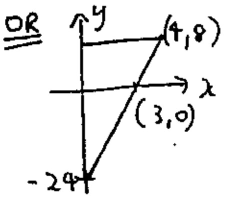
$\therefore x = 24$

Vol = vol. of big cone − vol. small c

$= \frac{1}{3}\pi.4^2.32 - \frac{1}{3}\pi.3^2.24$

$= 98\frac{2}{3}\pi$ cm^3.

OR



$m = \frac{8-0}{4-3} = 8$.

equation of line i

$y - 0 = 8(x-3)$

$y = 8x - 24$

$\therefore$ y intercept $= -24$.

if $y = 8x - 24 \Rightarrow x = \frac{y+24}{8}$

$V = \pi\int_0^8 x^2.dy = \pi\int_0^8 \left(\frac{y+24}{8}\right)^2.d$

$= 98\frac{2}{3}\pi$ un

ii) $V = \pi\int_0^4 \left(\frac{y+24}{8}\right)^2.dy$

$= \frac{127}{3}\pi$ cm^3.

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