Student No. ....

Barker College

2006
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics

AM FRIDAY 4 AUGUST

Staff Involved:

- VAB*
- RMH*
- AJD
- LMD
- GDH
- JML
- EAS
- GIC
- LJP
- CFR

155 copies

General Instructions

- Write using blue or black pen. Use pencil for diagrams.
- Write your Barker Student Number on every answer page.
- Start each question on a NEW page
- Write on one side of the page only
- All necessary working must be shown in every question.
- Marks may be deducted for careless or badly arranged working.
- Board-approved calculators may be used.
- Diagrams are not drawn to scale.
- A table of standard integrals is provided on the last page which may be detached for your use.

Total marks – 120

- Attempt Questions 1 – 10
- All questions are of equal value
- Reading time – 5 minutes
- Working time – 3 hours

Marks

Question 1 (12 marks)

(a) Simplify, expressing in scientific notation $\dfrac{1.26 \times 10^{25}}{7 \times 10^{7}}$ **1**

(b) Factorise fully $2 - 16x^3$ **2**

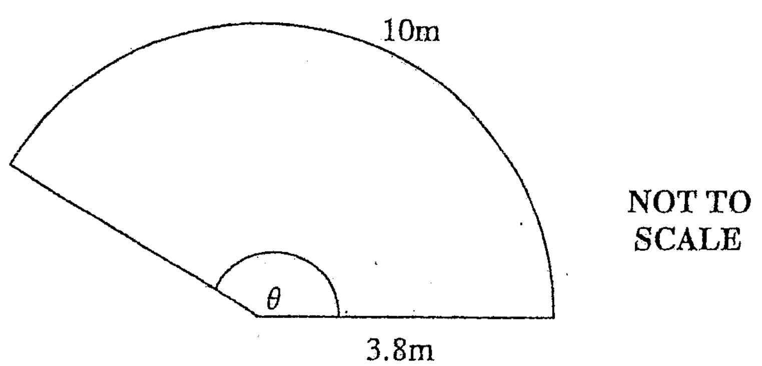
(c) Solve $5 - \dfrac{x}{7} < 3$ **2**

(d) Expand and simplify $\left(4 - \sqrt{5}\right)^2$ **2**

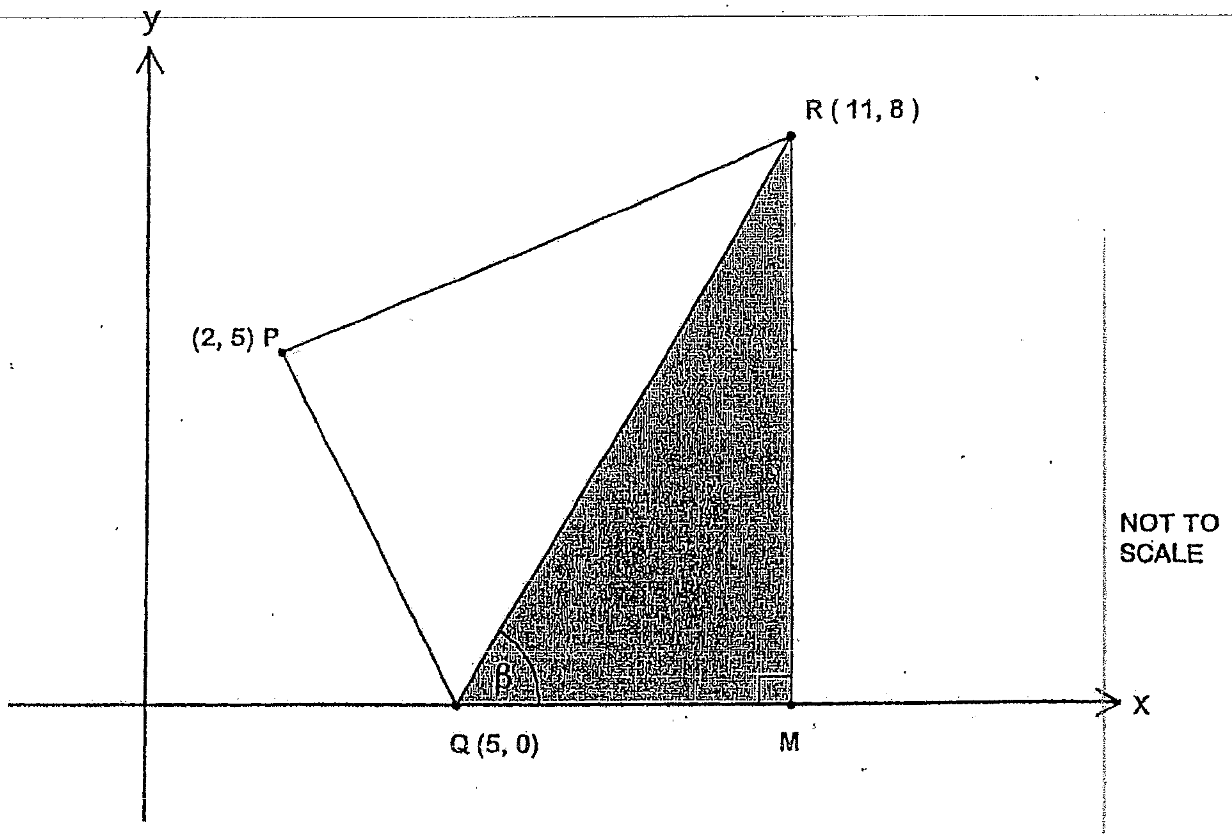
(e) Evaluate exactly $\cos \dfrac{\pi}{6}$ **1**

(f) Solve $|4x - 5| = 15$ **2**

(g) The sector shown below has radius 3.8 metres and arc length 10 metres.
Find angle θ correct to the nearest degree. **2**



Question 2 (12 marks) **[START A NEW PAGE]** **Marks**



Answer by referring to the above diagram.

(i) Find distance RQ. **1**

(ii) Find the gradient of RQ. **1**

(iii) Find the size of angle β correct to the nearest degree. **1**

(iv) Show the equation of the line RQ is $4x - 3y - 20 = 0$ **1**

(v) Find the perpendicular distance of point P from the line RQ. **2**

(vi) Find the area of triangle PQR (which is **not** shaded). **2**

(vii) Point P is the midpoint of the interval RT, where T is a point not shown on the diagram. Find the coordinates of the point T. **2**

(viii) The shaded region which is triangle QMR can be described by three inequalities, one of which is $y \geq 0$. State the other two inequalities. **2**

Marks

Question 3 (12 marks) **[START A NEW PAGE]**

(a) Find $\frac{dy}{dx}$ given that

(i) $y = \log_e\left(5 + 7x^2\right)$ **1**

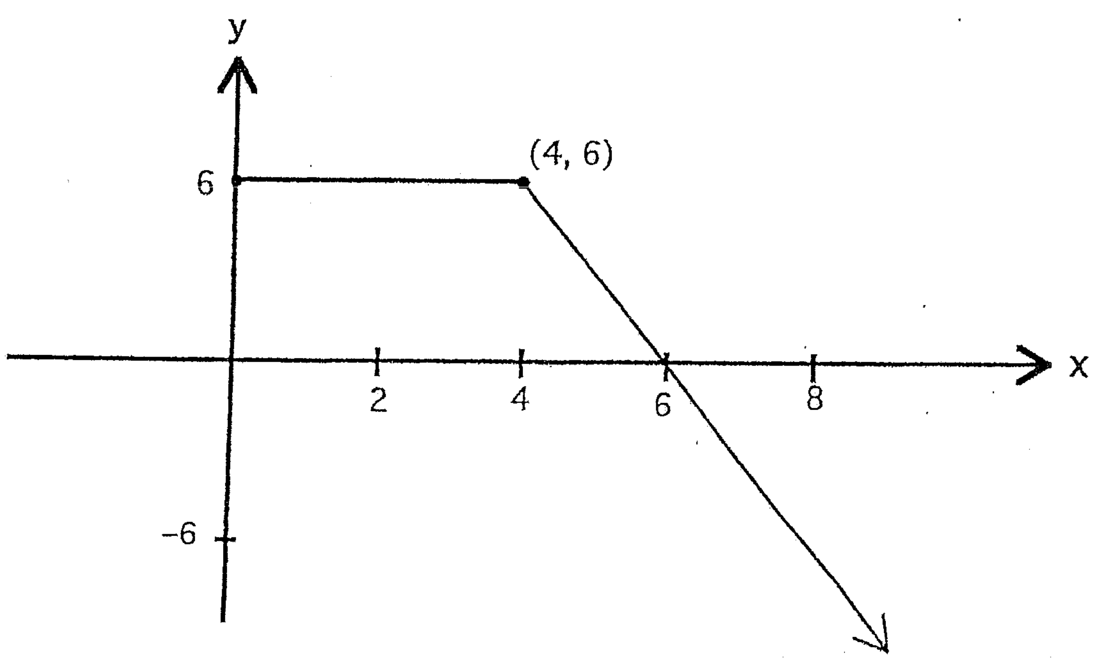
(ii) $y = \frac{\sin 3x}{x}$ **2**

(b) Given $f(x) = x\sqrt{x+1}$ find $f'(x)$
expressing your answer as a single simplified fraction. **3**

(c) Find $\int xe^{x^2}dx$ **2**

(d) Find the gradient of the tangent to the curve $y = \frac{12}{x}$ at the point $(2, 6)$ **2**

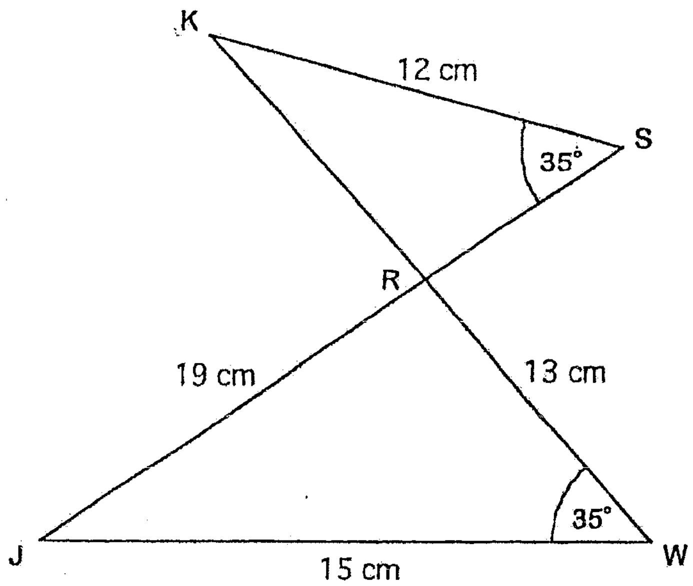
(e) The graph below shows the function $y = f(x)$ whose domain is $x \geq 0$
Trace or copy this graph on to your writing paper.
On the same axes sketch the graph of the function $y = f'(x)$ **2**



Marks

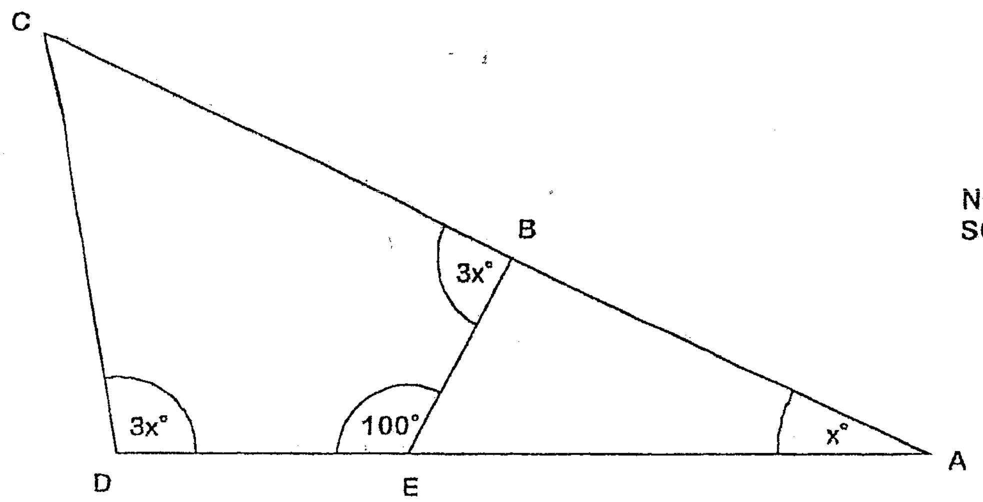
Question 4 (12 marks) [START A NEW PAGE]

(a) The triangles KSR and JWR shown below are similar.
KS = 12 cm, JW = 15 cm, JR = 19 cm, RW = 13 cm.
Find the length of the side SR. 2



NOT TO SCALE

(b) Find the value of x in the diagram below. 3
Show working and give reasons.

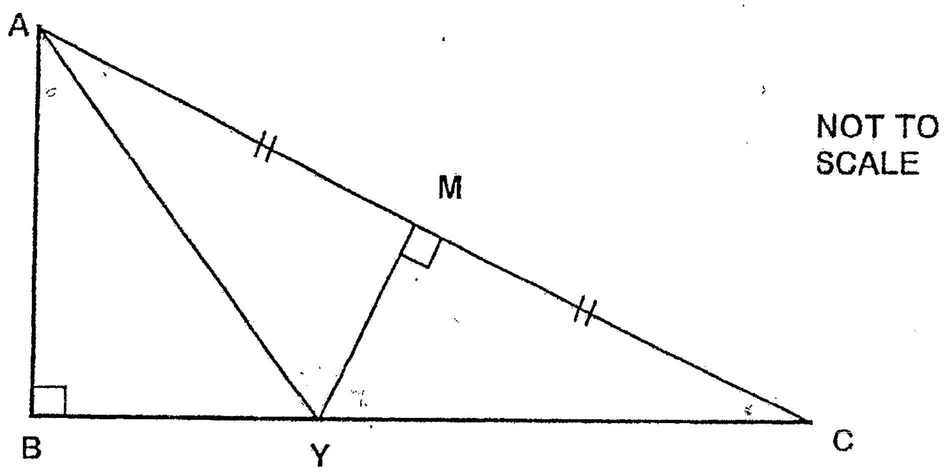


NOT TO SCALE

Question 4 continues on the next page.

Marks

(c)



The diagram above shows a right-angled triangle ABC with $\angle ABC = 90°$.
The point M is the midpoint of AC, and Y is the point where the perpendicular to AC at M meets BC.

(i) Show that Δ AYM is congruent to Δ CYM, giving reasons. **2**

(ii) Suppose that it is also given that YA bisects $\angle BAC$.
Find the size of $\angle YCM$ and hence find the exact ratio MY: AC. **3**

(d) Draw a possible sketch of a function $y = f(x)$ which satisfies the following conditions: **2**

- The function has domain $0 \le x < 12$
- $\lim_{x \to 12} f(x) = \infty$
- The function is monotonically increasing.
- The curve has exactly one point of inflexion. Label this point I.

Marks

Question 5 (12 marks) **[START A NEW PAGE]**

(a) Consider the curve $y = x^3 - 6x^2 + 12x + 2$

(i) **Show** the curve has only one stationary point, find its **coordinates** and determine its **nature**. 3

(ii) State the values of x for which the curve is concave up. 1

(iii) State the values of x for which the curve is increasing. 1

(b) Use Simpson's rule with five function values to estimate $\int_0^2 \frac{1}{1+x^2}\,dx$ giving your answer correct to two decimal places. 4

(c) Solve for x:

$\log_4 6 + \log_4 x - 3\log_4 2 = 2$ 3

Marks

Question 6 (12 marks) **[START A NEW PAGE]**

(a) Solve $16 - x^2 > 0$ 1

(b) Find the values of k for which the equation $x^2 - (k-2)x + (k+1) = 0$ has real roots. 3

(c) The roots of the equation $2x^2 + 4x - 1 = 0$ are α and β

(i) State the values of $(\alpha + \beta)$ and $\alpha\beta$ 1

(ii) Evaluate $\alpha^2\beta^2$ 1

(iii) Show the value of $(\alpha^2 + \beta^2)$ is 5 . 1

(iv) Hence write down a quadratic equation whose roots are α^2 and β^2 1

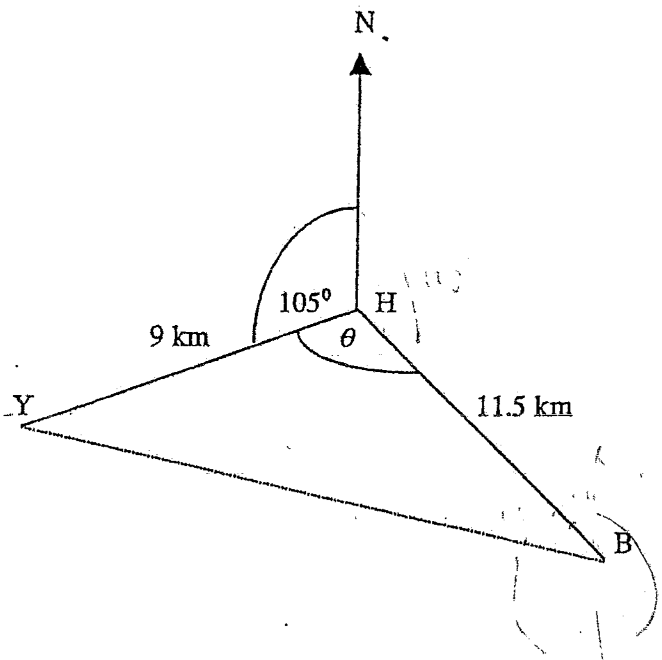
(d) A parabola has equation $x^2 - 12x = 8y - 52$

(i) By completing the square, express the equation in the form $(x-h)^2 = 8(y-k)$ 1

(ii) Hence find the coordinates of the **vertex** and the **focus** and the equation of the **directrix** for this parabola. 3

Question 7 [12 marks] **[START A NEW PAGE]**

(a) This diagram shows a harbour (H), a yacht (Y) and a boat (B).
The boat bears 110^0 from the harbour and $\angle YHN$ is $105°$ as shown.
The yacht is 9 km from the harbour and the boat is 11.5 km from harbour.



(i) Find θ and, hence, find the distance YB (1 decimal place) 2

(ii) Find $\angle$ HBY to the nearest degree. 2

(iii) Hence, find the bearing of the yacht from the boat. 1

(b) Simplify fully $\cos^2\theta\left(\sec\theta - 1\right)\left(\sec\theta + 1\right)$ 3

(c) For the series $\cos^4\theta + \cos^4\theta\sin^2\theta + \cos^4\theta\sin^4\theta + \ldots\ldots$

(i) Find the simplest expression for the limiting sum of the series, assuming it exists. 2

(ii) For what values of θ in the interval $0 \le \theta \le 360°$ does the limiting sum exist? 2

Marks

Question 8 (12 marks) **[START A NEW PAGE]**

(a) A function $f(x)$ is defined as follows: $f(x) = \begin{cases} -5 & \text{if } x \leq -1 \\ 3x-4 & \text{if } x > -1 \end{cases}$

Evaluate (i) $f(-1) + f(-3)$ **1**

(ii) $f(a^2)$ **1**

(iii) $f(f(0))$ **1**

(b) A heavy object is dropped from a plane. During the 1st second it falls 4.9 metres. During the 2nd second it falls 14.7 metres. During the 3rd second it falls 24.5 metres. These distances continue in arithmetic progression.

(i) Find the distance the object falls **during the 15th second.** **2**

(ii) Find the **total distance** the object has fallen after 15 seconds. **2**

(c) David invested \$1200 on 1st January every year.
He was paid 6.5% per annum interest compounded annually.

(i) How much was the investment worth at the end of 15 years? **3**

(ii) How many years in total would it take until the accumulated value of the investment was \$64 000? **2**

Marks

Question 9 (12 marks) **[START A NEW PAGE]**

(a) Evaluate exactly $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} 4\sec^2 2x \, dx$ **3**

(b) (i) Sketch the curve $y = 3\sin 2x$ in the domain $0 \le x \le 2\pi$ **2**

(ii) On the same axes, sketch the line $y = x - 3$ **1**

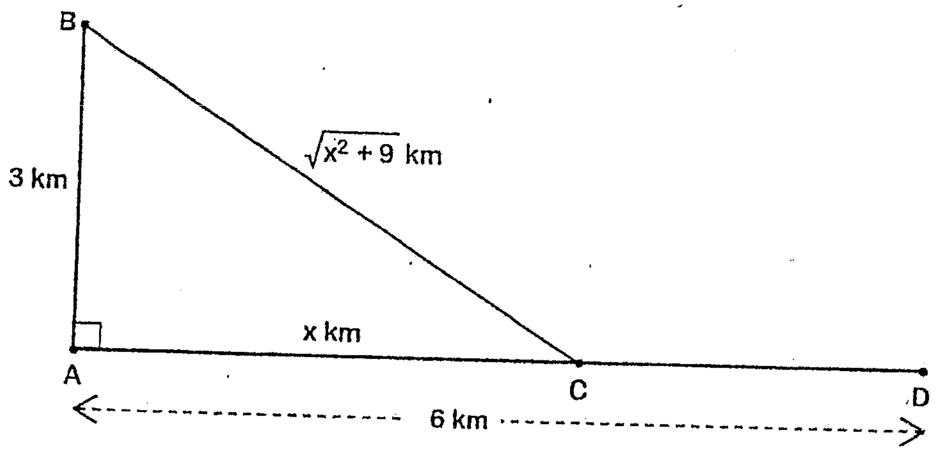
(iii) By referring to your sketch, state **how many** solutions there are to the equation $3\sin 2x - x + 3 = 0$ **1**
(You do **not** need to find the solutions.)

(c) (i) Sketch the curve $y = e^x + 1$ and shade the region bounded by the curve, the x–axis and the lines $x = 0$ and $x = \log_e 3$ **1**

(ii) The region in part (i) is rotated about the x–axis. **4**
Find the volume of the resulting solid of revolution.
Give your answer in simplest exact form.

Question 10 (12 marks) **[START A NEW PAGE]** **Marks**

(a)



A man is in a boat at point B on a lake and AD is a straight stretch of the lake's edge. B is 3 kilometres from a point A on the river bank. The man wishes to travel from point B to point D. He intends to row in a straight line to point C and then walk to D. He can row at 4 km /h and walk at 5 km /h.

Let the distance AC be x kilometres and let the total time for the trip be T hours.

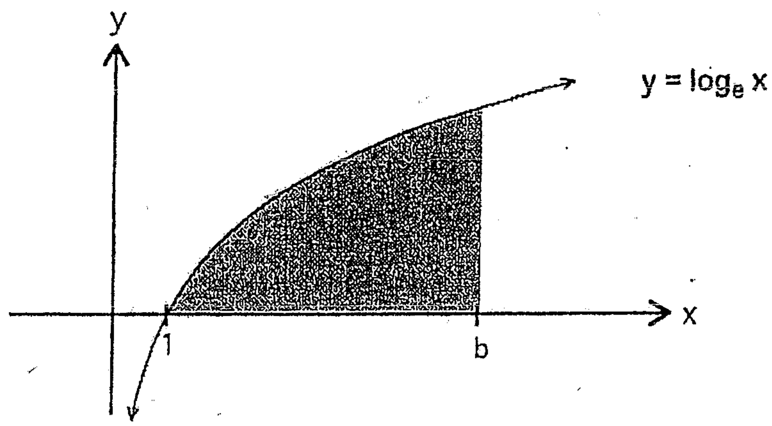
(i) Explain why $T = \dfrac{\sqrt{x^2+9}}{4} + \dfrac{6-x}{5}$ **1**

(ii) Find the value of x which will enable him to complete the trip in the minimum time. **4**

Question 10 continues on the next page.

(b) (i) Show the derivative of $(x\log_e x - x)$ is $\log_e x$ **1**

(ii) The diagram below shows the area bounded by the curve $y = \log_e x$, the x–axis and the line $x = \text{b}$, where b is some number greater than 1. Find the simplest expression for the area in terms of b. **2**



(iii) If the area in part (ii) has magnitude (2b + 1) units^2, find the exact value of b. **2**

(iv) Exactly one point on the curve $y = \log_e x$ has a tangent which passes through the origin. Find the coordinates of this point. **2**

End of Paper

1\. a) 1.8×10^{17}

b) $2(1-8x^3) = 2(1-2x)(1+2x+4x^2)$

c) $5 - \frac{x}{7} < 3$

$35 - x < 21$

$14 < x \quad \therefore x > 14$

d) $16 - 8\sqrt{5} + 5 = 21 - 8\sqrt{5}$

e) $\sqrt{3}/2$

f) $4x - 5 = 15 \qquad 4x - 5 = -15$

$4x = 20 \qquad 4x = -10$

$x = 5 \qquad x = -\frac{5}{2}$

g) $l = r\theta$

$10 = 3.8 \times \theta$

$\theta = \frac{10}{3.8}$ radians

$\theta = \frac{10}{3.8} \times \frac{180}{\pi}^{\circ}$

$\theta = 151^{\circ}$ (nearest degree)

2\. i) $RQ = \sqrt{(11-5)^2 + (8-0)^2}$

$= \sqrt{100}$

$= 10$ units

ii) grad. $RQ = \frac{8-0}{11-5} = \frac{8}{6} = \frac{4}{3}$

iii) $\tan\beta = 4/3$

$\beta = \tan^{-1} 4/3$

$\beta = 53^{\circ}$ (nearest degree)

iv) RQ: $Q(5,0) \quad m = 4/3$

$y - 0 = \frac{4}{3}(x-5)$

$3y = 4x - 20$

$4x - 3y - 20 = 0$

v) $P(2,5)$ RQ: $4x - 3y - 20 = 0$

⊥ dist $= \left|\frac{4\times 2 + (-3)\times 5 + -20}{\sqrt{4^2 + (-3)^2}}\right|$

$= \frac{27}{5}$ units

vi) $A = \frac{1}{2} \times 10 \times \frac{27}{5} = 27$ sq. units

vii) $\frac{x+11}{2} = 2, \quad \frac{y+8}{2} = 5$

$\therefore x = -7 \quad \therefore y = 2$

$T(-7, 2)$

viii) $x \leq 11$ and $4x - 3y - 20 \geq 0$

Q3. a) i) $y' = \frac{14x}{5+7x^2}$

ii) $y' = \frac{3x\cos 3x - \sin 3x}{x^2}$

b) $f'(x) = x \cdot \frac{1}{2}(x+1)^{-1/2} + \sqrt{x+1}$

$= \frac{x}{2\sqrt{x+1}} + \sqrt{x+1} \times \frac{2\sqrt{x+1}}{2\sqrt{x+1}}$

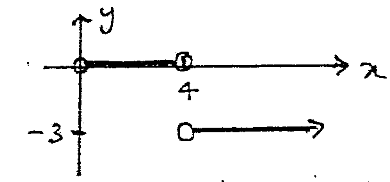
$= \frac{3x+2}{2\sqrt{x+1}}$

c) $I = \frac{1}{2}e^{x^2} + c$

d) $y' = \frac{-12}{x^2}$ when $x = 2$, $y' = -3$

$\therefore$ grad $= -3$.

e)



Q4.

a) $\frac{SR}{13} = \frac{12}{15}$

$SR = 10.4$ cm

b) $\angle BEA = 80^{\circ}$ (st. ∠)

$80 + x = 3x$ (ext ∠.

$80 = 2x$

$\therefore x = 40^{\circ}$

c) i) In △AYM, △CYM

YM is common

AM = CM (given)

∠YMC = ∠AMY (st. ∠)

∴ △AYM ≡ △CYM (SAS)

ii) let $\angle YAM = x \quad \therefore \angle BAY = x$ (data)

$\angle YCM = x$ (corresp. ∠ in cong △)

$x + x + 90 + x = 180^{\circ}$ (∠ sum △)

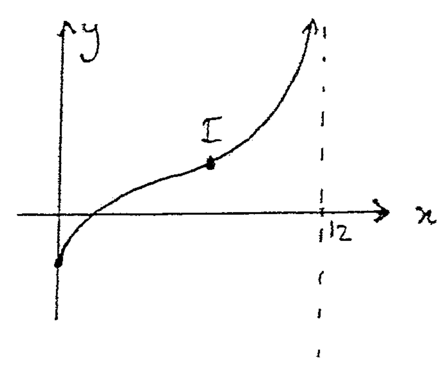
$3x + 90 = 180$

$x = 30^{\circ} \quad \therefore \angle YCM = 30^{\circ}$

$\frac{MY}{AM} = \tan 30^{\circ} = \frac{1}{\sqrt{3}}$

$\therefore \frac{MY}{AC} = \frac{1}{2\sqrt{3}}$ since $AM = \frac{1}{2}AC$.

d)



$y' = 3x^2 - 12x + 12$

let $y' = 0$

$3x^2 - 12x + 12 = 0$

~~$3(x^2 - 4x + 4) = 0$~~

$3(x-2)^2 = 0$

$\therefore x = 2$ One st. pt at $(2,10)$

$y'' = 6x - 12$

when $x = 2$, $y'' = 0$

x	1.9	2	2.1
y''	-.6	0	0.6

y'' changes sign

$\therefore$ horizontal point of inflect. at $(2,10)$

ii) $y'' > 0$

$6x - 12 > 0 \quad \therefore x > 2$

iii) $y' > 0 \quad \therefore$ all x except $x = 2$

x	0	0.5	1	1.5	2
$\frac{1}{1+x^2}$	1	0.8	0.5	0.308	0.2

$\doteqdot \frac{0.5}{3}\left[1 + 0.2 + 4(0.8 + 0.308) + 2(0.5)\right]$

$= 1.11$ (2 d.p)

) $\log_4\left(\frac{6x}{2^3}\right) = 2 \quad \therefore 4^2 = \frac{6x}{8}$

$\frac{4^2 \times 8}{6} = x$

$\therefore x = 21.\dot{3}$

. a) $(4-x)(4+x) > 0$

$\therefore -4 < x < 4$

b) $\Delta \geq 0$

$b^2 - 4ac \geq 0$

$[-(k-2)]^2 - 4 \times 1 \times (k+1) \geq 0$

$k^2 - 4k + 4 - 4k - 4 \geq 0$

$k^2 - 8k \geq 0$

$k(k-8) \geq 0$

$\therefore k \leq 0, \ k \geq 8$

:) i) $\alpha + \beta = -\frac{4}{2} = -2, \quad \alpha\beta = -\frac{1}{2}$

ii) $\alpha^2\beta^2 = \frac{1}{4}$

iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 4 - 2 \times -\frac{1}{2} = 5.$

iv) $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

$\therefore x^2 - 5x + \frac{1}{4} = 0.$

) i) $x^2 - 12x + (-6)^2 = 8y - 52 + (-6)^2$

$(x-6)^2 = 8y - 16$

$(x-6)^2 = 8(y-2)$

ii) vertex $(6,2)$; focus $(6,4)$

directrix : $y = 0$

Q7. a) i) $\theta = 360° - 105 - 110 = 145°$

$YB^2 = 9^2 + 11.5^2 - 2 \times 9 \times 11.5 \times \cos 145°$

$= 382.8144732$

$\therefore YB = 19.6$ km (1 d.p.)

ii) $\frac{\sin \angle HBY}{9} = \frac{\sin 145°}{19.6}$

$\sin \angle HBY = \frac{9 \sin 145}{19.6}$

$\therefore \angle HBY = 15°$

iii) $360° - 70° - 15° = 275°$

b) $\cos^2\theta(\sec^2\theta - 1) = \cos^2\theta \times \tan^2\theta$

$= \cos^2\theta \times \frac{\sin^2\theta}{\cos^2\theta}$

$= \sin^2\theta.$

c) i) $S = \frac{a}{1-r} = \frac{\cos^4\theta}{1 - \sin^2\theta}$

$= \frac{\cos^4\theta}{\cos^2\theta}$

$= \cos^2\theta$

ii) $1 - r \neq 0$

$1 - \sin^2\theta \neq 0 \quad \therefore \sin\theta \neq \pm 1$

$\theta \neq 90°, 270°$

$\therefore$ all θ except $90°, 270°$.

Q8. a) i) $-5 + -5 = -10$

ii) $3a^2 - 4$

iii) $f(3 \times 0 - 4) = f(-4) = -5$.

b) 4.9, 14.7, 24.5, ...

i) $T_{15} = a + 14d$

$= 4.9 + 14 \times 9.8$

$= 142.1$ m

ii) $S_{15} = \frac{15}{2}[2 \times 4.9 + 14 \times 9.8]$

$= 1102.5$ m

c) i) $A = 1200[1.065 + 1.065^2 + .. + 1.065^{15}]$

$= 1200\left[\frac{1.065(1.065^{15} - 1)}{.065}\right]$

$= \$30904.81$

Q8. c) ii)

$$1200\left[\frac{1.065(1.065^n - 1)}{.065}\right] = 64\,000$$

$$1.065^n - 1 = \frac{64000 \times .065}{1200 \times 1.065}$$

$$1.065^n = 4.255086072$$

$$n = \log_{1.065}(4.255086072)$$

$$n = 22.995$$

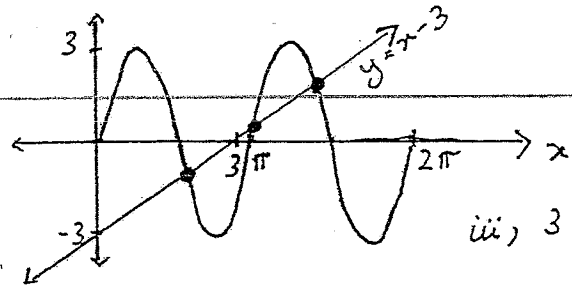
$\therefore n = 23$ years

Q9. a) $I = 2\left[\tan 2x\right]_{\pi/8}^{\pi/6}$

$$= 2\left[\tan \frac{\pi}{3} - \tan\frac{\pi}{4}\right]$$

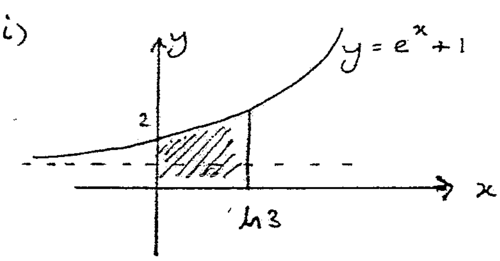
$$= 2(\sqrt{3} - 1)$$

b) i), ii)



iii) 3 solutions

c) i)



ii) $V = \pi \int_0^{\ln 3} (e^x + 1)^2 \, dx$

$$= \pi \int_0^{\ln 3} (e^{2x} + 2e^x + 1)\, dx$$

$$= \pi\left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^{\ln 3}$$

$$= \pi\left[\left(\frac{9}{2} + 6 + \ln 3\right) - \left(\frac{1}{2} + 2 + 0\right)\right]$$

$$= \pi\left[8 + \ln 3\right] \text{ cubic units}$$

Q10. a) i) BC: time $= \dfrac{\sqrt{x^2+9}}{4}$

CD: dist $= 6 - x$

$\therefore$ time $= \dfrac{6-x}{5}$

Total time $= \dfrac{\sqrt{x^2+9}}{4} + \dfrac{6-x}{5}$

Q10. ii)

$$\frac{dT}{dx} = \frac{\frac{1}{2}(x^2+9)^{-1/2} \times 2x}{4} + -\frac{1}{5}$$

$$= \frac{x}{4\sqrt{x^2+9}} - \frac{1}{5}$$

let $\frac{dT}{dx} = 0$ $\quad \therefore \dfrac{x}{4\sqrt{x^2+9}} = \dfrac{1}{5}$

$$5x = 4\sqrt{x^2+9}$$

$$25x^2 = 16(x^2+9)$$

$$9x^2 = 16 \times 9$$

$$x^2 = 16$$

$\therefore x = 4$ since $x > 0$

x	3.9	4	4.1
$\frac{dT}{dx}$	-1.8	0	.0017

$\therefore$ minimum when $x = 4$ km

b) i) $y' = x \times \frac{1}{x} + \log_e x \times 1 - 1$

$$= 1 + \log_e x - 1$$

$$= \log_e x$$

ii) $A \doteqdot \int_1^b \log_e x \, dx$

$$= \int_1^b \log_e x \, dx$$

$$= \left[x \log_e x - x\right]_1^b$$

$$= (b\log_e b - b) - (\log_e 1 - 1)$$

$$= b\log_e b - b + 1$$

iii) $2b + 1 = b\log_e b - b + 1$

$$3b = b \log_e b$$

$$3 = \log_e b$$

$\therefore b = e^3$

iv) $y' = \frac{1}{x}$

$$m = \frac{y-0}{x-0} = \frac{y}{x}$$

$\therefore \frac{1}{x} = \frac{y}{x} \Rightarrow y = 1$

$\log_e x = 1 \Rightarrow x = e$

$\therefore$ point is $(e, 1)$