BAULKHAM HILLS HIGH SCHOOL

TRIAL 2014

YEAR 12 TASK 4

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-16
- Marks may be deducted for careless or badly arranged work

Total marks – 100
Exam consists of 11 pages.

This paper consists of TWO sections.

Section 1 – Page 2-4 (10 marks)
Questions 1-10

- Attempt Question 1-10
- Allow about **15** minutes for this section

Section II – Pages 5-10 (90 marks)

- Attempt questions 11-16
- Allow about **2 hours and 45** minutes for this section

Table of Standard Integrals is on page 11

Section I - 10 marks
Use the multiple choice answer sheet for question 1-10

1. If $3\sqrt{5} + \sqrt{20} = \sqrt{a}$ then a is

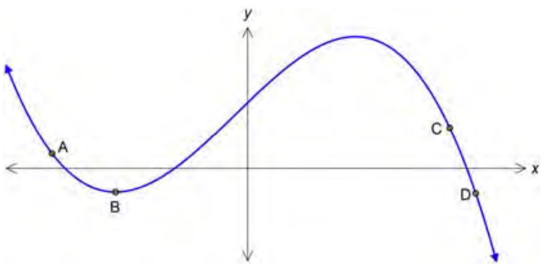
(A) 5 (B) 25 (C) 125 (D) 15

2. What is the exact value of $\cos\frac{7\pi}{6}$

(A) $\frac{\sqrt{3}}{2}$ (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

3. State which point on the diagram relates to the following

$y > 0,\ \frac{dy}{dx} < 0,\ \frac{d^2y}{dx^2} > 0.$



(A) A (B) B (C) C (D) D

4. If $\int_1^4 f(x)dx = 2$ then $\int_1^4 (2f(x) + 3)\,dx$ is equal to

(A) 2 (B) 13 (C) 7 (D) 10

5. The $\lim\limits_{\theta \to 0} \frac{sin\frac{\theta}{4}}{\theta} =$

(A) 1 (B) 4 (C) $\frac{1}{4}$ (D) 0

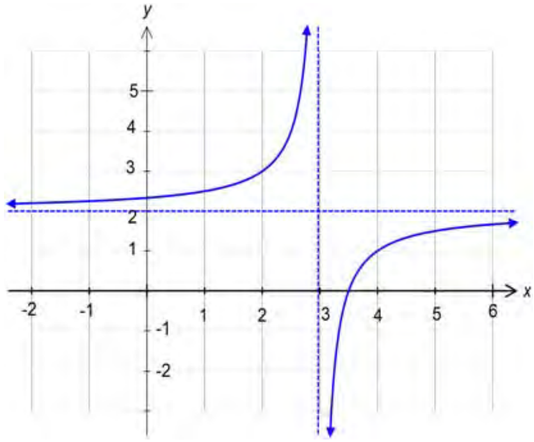
6. The function $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has a period of

(A) 3 (B) 10 (C) $\frac{\pi}{5}$ (D) $\frac{\pi}{10}$

7. The sum of the series $\frac{4}{9} + \frac{2}{9} + \frac{1}{9} + \frac{1}{18} + \frac{1}{36} + \frac{1}{72} \ldots \frac{1}{144}$ is

(A) $\frac{127}{36}$ (B) $\frac{127}{81}$ (C) $\frac{127}{144}$ (D) $\frac{81}{127}$

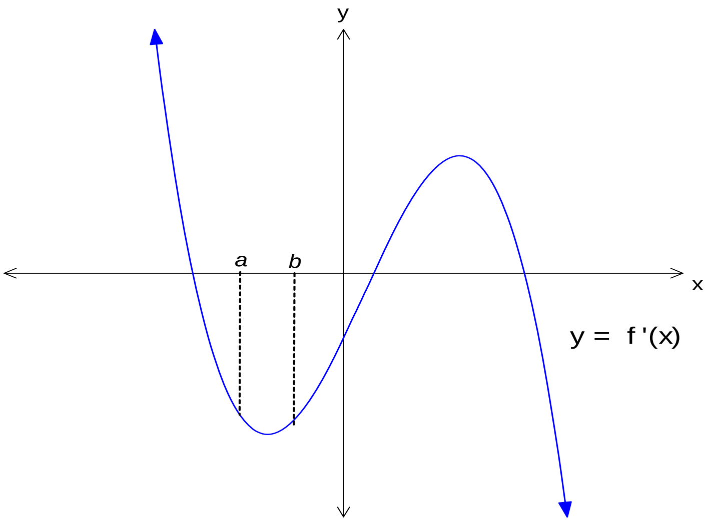
8. Part of the function $y = \frac{a}{x+b} + c$ is shown



The value of a, b and c respectively are

(A) 2, 3, 2 (B) $2, -3, 2$ (C) $-2, -3, 2$ (D) $-2, 3, 2$

9. The function y= $f'(x)$ shown below



Between $x = a$ and $x = b$, the function $y = f(x)$ will have a

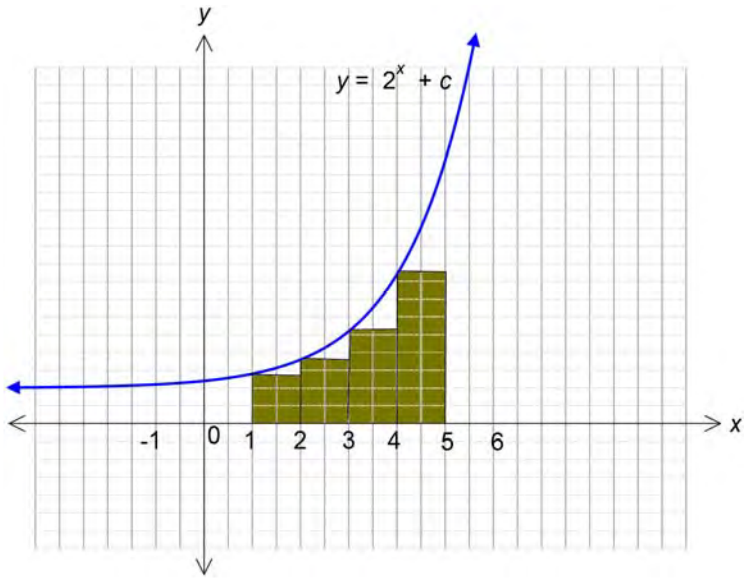
(A) negative gradient

(B) positive gradient

(C) local minimum value

(D) local maximum value

10. Consider the graph of $y = 2^x + c$, where c is a real number. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the $x-$axis and the line $x = 1$ and $x = 5$



If the total area of the shaded rectangles is 44, then the value of c is

(A) 14

(B) -4

(C) $\frac{14}{5}$

(D) $\frac{7}{2}$

End of Section 1

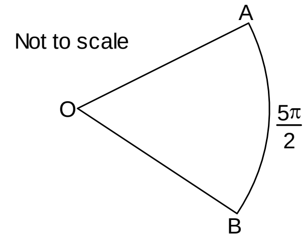
Section II – Extended Response
Attempt questions 11-16.
All necessary working should be shown in every question.

Question 11 (15 marks) (Answer on the appropriate page in your booklet)		
a)	Solve $\frac{1}{x} = x - 1$ leaving your answer in the exact form.	**2**
b)	Solve $\lvert 3x - 4 \rvert \leq 8$	**2**
c)	If $\tan\theta = \frac{7}{8}$ and $\cos\theta < 0$, find the exact value of $\sin\theta$.	**2**
d)	Solve for x: $9^{2x-3} = 27$	**2**
e)	Differentiate with respect to x	
	(i) $y = \cos(x^2)$	**2**
	(ii) $y = \frac{(1-e^{2x})}{x^3}$	**2**
f)	Find the equation of the normal to $y = 2x^3 - 3x^2$ at a point $x = -1$	**3**

End of Question 11

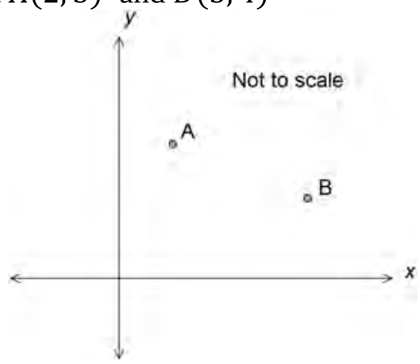
Question 12 (15 marks) (Answer on the appropriate page in your booklet)

a)



In the diagram AB is an arc of a circle with the centre O. The length of the arc AB is $\frac{5\pi}{2}$ cm. The area of the sector AOB is $4\pi\ cm^2$. Find the radius of the sector. **2**

b) The diagram shows the point $A(2, 5)$ and $B(5, 4)$



(i) Show that equation of line AB is $x + 3y - 17 = 0$ **2**

(ii) Find the coordinates of M the midpoint of AB. **1**

(iii) Show that the equation of the perpendicular bisector of AB is **2**

$3x - y - 6 = 0$

(iv) The perpendicular bisector of AB cuts the x-axis at C. Find the coordinates of C. **1**

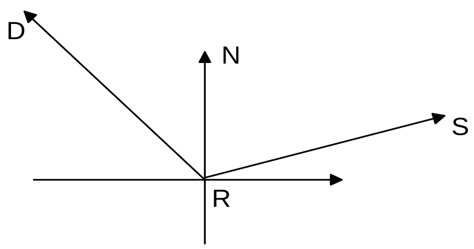
(v) Find the area of ΔABC **2**

c) Give the exact value of $\log_3\left(\frac{1}{\sqrt{3}}\right)$ **1**

d) Find $\int \sin\frac{x}{2}\,dx$. **2**

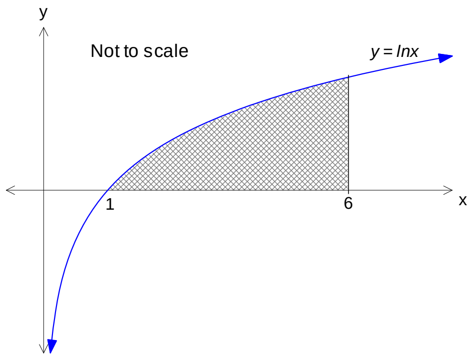
e) If the limiting sum of the series $1 - 2p + 4p^2 \ldots..$ is $\frac{4}{7}$. Find the value of p. **2**

End of Question 12

Question 13 (15 marks) (Answer on the appropriate page in your booklet)		
a)	Consider the function given by $f(x) = 2x^3 - 3x^2 - 36x + 26$. (i) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. (ii) Find the coordinates of any point of inflection. (iii) Hence sketch the graph of $f(x) = 2x^3 - 3x^2 - 36x + 26$ by showing the above information. (iv) For what values of x is the curve concave down and decreasing?	 **3** **2** **2** **2**
b)	Danny and Stella are pilots of two light planes which leave Rodeo Station at the same time. Danny flies on a bearing of 330°T at a speed of 180km/h and Stella flies on a bearing of 080°T at a speed of 240km/h. Copy the diagram below onto your answer page and mark the information on the diagram.  (i) How far apart are Danny and Stella after 2 hours? (ii) What is the bearing of Stella from Danny after 2 hours? (to the nearest degree)	 **2** **2**
c)	There is one red and three green jelly beans in a jar. One jellybean is selected at random, eaten, and then a second jellybean is selected at random and is also eaten. Find the probability: (i) The two jellybeans eaten are both green. (ii) The red jellybean is the second one eaten.	 **1** **1**

End of Question 13

Question 14 (15 marks) (Answer on the appropriate page in your booklet)

a)	The acceleration of a moving body is given by $a = \sqrt{2t+1}\ ms^{-2}$. If the body starts from rest, find its velocity after 4 seconds.	**2**
b)	Find the area of the region in the diagram bounded by curve $y = \log x$, x axis and line x=6. 	**2**
c)	The velocity of an object is given by the equation $v = 6t - 5 - t^2$ where t is in seconds and velocity v in $\mathrm{m/s}$. It begins its motion at $\mathrm{x} = 5$ metres.	
	(i) Find an equation for the displacement of the object.	**2**
	(ii) At what two times is the object stationary?	**1**
	(iii) Find the distance travelled by the object in first 3 seconds.	**2**
d)	Consider a parabola $2y = x^2 - 4x$	
	(i) Find the coordinates of the focus.	**2**
	(ii) Find the equation of the directrix.	**1**
e)	Let A be the point $(-2, 0)$ and $B(6, 0)$. The point $P(x, y)$ is such that $AP \perp PB$.	
	(i) Find the gradient of PA.	**1**
	(ii) Hence find the equation for the locus of P.	**2**

End of Question 14

Question 15 (15 marks) (Answer on the appropriate page in your booklet)

a)

(i) Show that $\sqrt{\dfrac{\operatorname{cosec}^2 x - \cot^2 x - \cos^2 x}{\cos^2 x}} = |\tan x|$ **2**

(ii) Find the value of x if $\sqrt{\dfrac{\operatorname{cosec}^2 x - \cot^2 x - \cos^2 x}{\cos^2 x}} = 1$ for $0 < x < \pi$ **1**

b) The population of a species of bacteria P at time t (minutes) grows such that

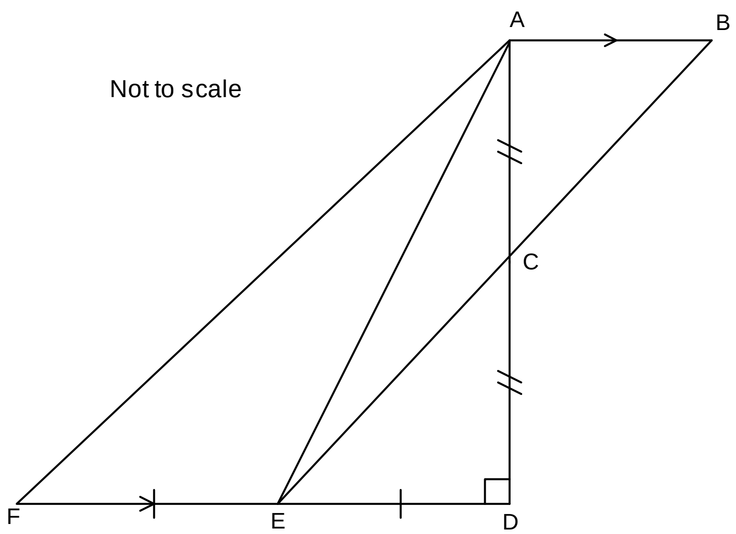
$$P = 2000e^{kt} \text{ where } k \text{ is the positive constant.}$$

(i) Show that the rate of increase of the population is proportional to the size of the population at that time. **1**

(ii) Given that the initial population doubles after 4 minutes, calculate the value of k, correct to 3 significant figures. **2**

(iii) Find the population after 6 minutes.(correct to nearest whole number) **2**

c)



In the diagram $AB \parallel FD$. ADF is a right angled triangle, C is the midpoint of AD and E is the midpoint of FD.

(i) Explain why $\angle CED = \angle ABC$. **1**

(ii) Show that $\Delta CDE \equiv \Delta CAB$. **3**

(iii) Show that $AF = 2BC$. **3**

End of Question 15

Question 16 (15 marks) (Answer on the appropriate page in your booklet)

a) Given that $2log\ (x^2y) = 3 + logx - logy$. Express y in terms of x. **3**

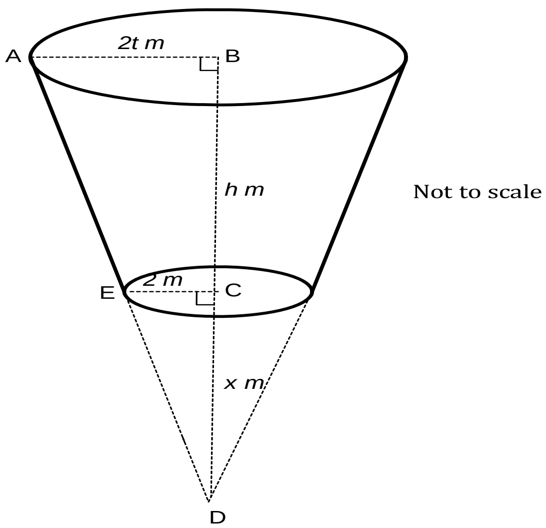
b) Sonia and Jenny want to save a deposit of $50 000 to buy a house. They devise a savings plan to allow them to achieve this goal. Beginning on 1^{st} January 2014, they deposit $1000 on the first day of each month into an account which pays 9% p.a compounded monthly.

(i) Find the amount they have in the account on 31^{st} January 2014. (i.e. at the end of the first month) **1**

(ii) Show that the amount they have in the account at the end of n months is given by $A = \frac{1007.5(1.0075^n - 1)}{0.0075}$ **2**

(iii) Hence find the least number of months they need to save their deposit. **2**

c) A truncated cone is to be used as a part of a hopper for a grain harvester. It has a height of h metres. The radius AB is to be t times greater than the bottom radius EC which is 2 metres.



(i) If x is the height of the removed section of the original cone, show that **2**

$x = \frac{h}{t-1}$

(ii) Show that the volume of the truncated cone is given by **2**

$V = \frac{4\pi h}{3}(t^2 + t + 1)$

(iii) If the upper radius plus the lower radius plus the height of the truncated cone must total 12 metres, calculate the maximum volume of the hopper. **3**

END OF EXAM

Solution

MC

1. C
2. B
3. A
4. B
5. C
6. B
7. C
8. C
9. A
10. D

Q11 a $\frac{1}{x} = x - 1$

$x^2 - x = 1$ ✓

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$ ✓

b $3x - 4 \le 8$ | $-(3x-4) \le 8$

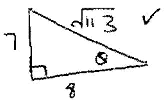
$3x \le 12$ | $3x - 4 \ge -8$ ✓

$x \le 4$ | $3x \ge -4$

$x \ge -4/3$

$-\frac{4}{3} \le x \le 4$ ✓

c



✓ $\sin\theta = \frac{7}{\sqrt{113}}$ ✓

d $(3^2)^{2x-3} = 3^3$ ✓

$4x - 6 = 3$

$4x = 9$

$x = 9/4$ ✓

e (i) $y' = -\sin x^2 \times 2x$ ✓ ✓

(ii) $y' = \frac{x^3 \cdot -2e^{2x} - 3x^2(1-e^{2x})}{x^6}$ ✓ ✓

$= \frac{-x^2(xe^{2x} - 3 + 3e^{2x})}{x^{\cancel{6}\,4}}$

f. $y = 2x^3 - 3x^2$

$y' = 6x^2 - 6x$

$m_T = 12$ ✓

$m_N = -1/12$

$x = 1, \quad y = -5$

$\therefore$ eq of normal is

$y + 5 = -\frac{1}{12}(x+1)$ ✓

$x + 12y + 61 = 0$

Q12 a $\frac{1}{2} r^2 \theta = 4\pi$

$r^2 \theta = 8\pi$ —(1) ✓

$\theta = 5\pi/2$ —(2)

Divide (1) by (2)

$r = \frac{8\pi}{5\pi/2} = \frac{16}{5}$ cm. ✓

b. (i) $m_{AB} = \frac{4-5}{5-2} = -1/3$ ✓

eq of AB is

$y - 5 = -\frac{1}{3}(x-2)$ ✓

$3y - 15 = -x + 2$

$x + 3y - 17 = 0$.

(OR if students doing sub., they have to sub. both the pts to get 2 marks)

(ii) $M_{AB} = \left(\frac{7}{2}, \frac{9}{2}\right)$ ✓

(iii) grad. of bisector $= 3$ ✓

$\therefore$ eq is

$y - \frac{9}{2} = 3(x - 7/2)$ ✓

$2y - 9 = 3(x - 7)$

$6x - 2y - 12 = 0$

$3x - y - 6 = 0$

(iv) $C(2, 0)$ ✓

(v) $A = \frac{1}{2} \times 3 \times 5$ ✓✓

$= 7.5$ unit^2

c $\log_3 (3)^{-1/2} = -1/2$ ✓

d $\frac{-\cos\frac{x}{2}}{1/2} + c = -2\cos\frac{x}{2} + c$ ✓ ✓

e $1 - 2p + 4p^2 \cdots = \frac{4}{7}$

$\frac{4}{7} = \frac{1}{1+2p}$ ✓

$4 + 8p = 7$

$p = 3/8$ ✓

Q13 a $f(x) = 2x^3 - 3x^2 - 36x + 26$

$f'(x) = 6x^2 - 6x - 36$

For stat pts $6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$x = 3, -2$

$\therefore$ coordinates of stat pts are $(3, -55)$ + $(-2, 70)$

$\frac{d^2y}{dx^2} = f''(x) = 12x - 6$

at $x = 3$ $\frac{d^2y}{dx^2} = 36 - 6 = 30 > 0$ ✓

$\therefore$ min at $(3, -55)$

at $x = -2$

$\frac{d^2y}{dx^2} = -24 - 6 = -30 < 0$ $\therefore$ max at $(-2, 70)$ ✓

(ii) For possible P.O.I $\frac{d^2y}{dx^2} = 0$

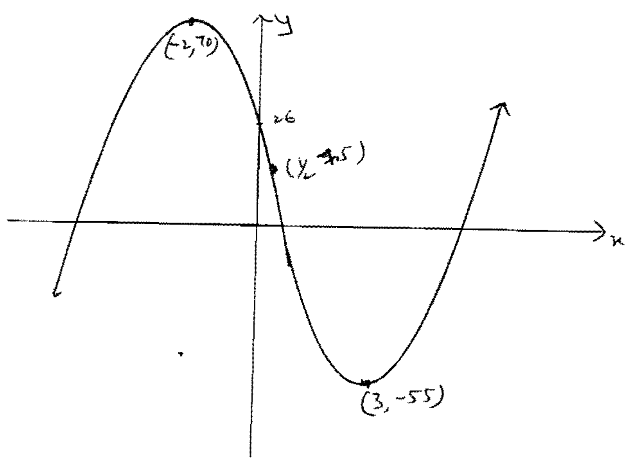
$12x - 6 = 0 \Rightarrow x = 1/2$ ✓

x	-1	$1/2$	1
$\frac{d^2y}{dx^2}$	-18	0	6

$\therefore$ There is change in concavity at $x = 1/2$

$\therefore \left(\frac{1}{2}, 7.5\right)$ is P.O.I ✓

(must show change in concavity)



(iv) For decreasing fn then $f'(x) < 0$

$6x^2 - 6x - 36 < 0$

$(x-3)(x+2) < 0$

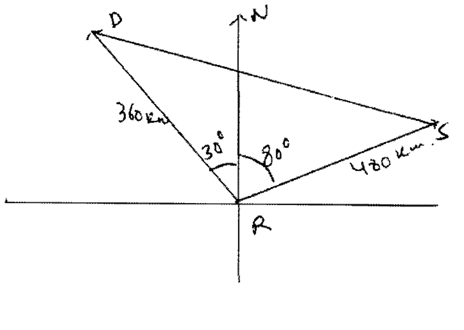
$-2 < x < 3$ —(1)

For concave down $12x - 6 < 0$

$x < 1/2$

$\therefore -2 < x < 1/2$ —(1)

b)



) $\angle DRS = 30° + 80 = 110°$ ✓

$DS^2 = DR^2 + RS^2 - 2 \times DR \times RS \cos\theta$

$= 360^2 + 480^2 - 2\ 360 \times 480 \times \cos 110°$ ✓

$= 478202 \cdot 16 \ldots$

$DS = 692$ to the nearest km.

(691·5)

ii) $\frac{\sin\theta}{480} = \frac{\sin 110°}{691 \cdot 52 \ldots}$ ✓

$\sin\theta = \frac{480 \times \sin 110°}{691 \cdot 52 \ldots}$

$= 0 \cdot 65226 \ldots$

$\theta = 40 \cdot 712 \ldots$

$= 41°$

Bearing $= 180 - 41 - 30$ ✓

$= 109° T$

c (i) $P(G,G) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$ ✓

(ii) $P(G,R) = \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ ✓

Q14 $a = \sqrt{2t+1}$

a $t=0$, V at $t=4$

$v=0$

$V = \frac{2(2t+1)^{3/2}}{3 \times 2} + c$

$= \frac{(2t+1)^{3/2}}{3} + c$

at $t=0, v=0$

$0 = \frac{1}{3} + c$

$c = -\frac{1}{3}$

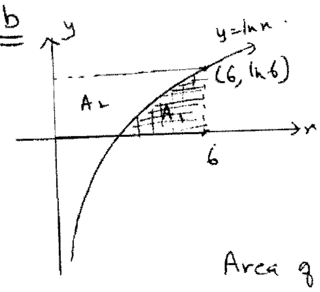
$V = \frac{(2t+1)^{3/2}}{3} - \frac{1}{3}$ ✓

at $t=4$

$v = \frac{(9)^{3/2}}{3} - \frac{1}{3}$

$= 9 - \frac{1}{3} = \frac{26}{3}$ m/sec to the right ✓

b



$y = \log x$

$e^y = x$ ✓

$A_2 = \int_0^{\ln 6} e^y\,dy$

$= e^y\Big]_0^{\ln 6} = 6 - 1 = 5$

Area of rectangle $= \ln 6 \times 6 = 10.7505\ldots$ ✓

$A_1 =$ Area $- A_2$

$= 10.7505 - 5$

$= 5.75$ unit^2

c $V = 6t - 5 - t^2$

(i) $x = \frac{6t^2}{2} - 5t - \frac{t^3}{3} + c$ ✓

at $t=0$ $x=5$

$5 = c$ ✓

$x = 3t^2 - \frac{t^3}{3} - 5t + 5$

(ii) Stationary when $v=0$

$6t - 5 - t^2 = 0$

$(t-5)(t-1) = 0$

$t = 1, 5$ sec. ✓

(iii) dist $= \left|\int_0^1 6t - 5 - t^2\,dt\right| + \int_2^3 (6t - 5 - t^2)\,dt$ ✓

$= \left|\left[3t^2 - 5t - \frac{t^3}{3}\right]_0^1\right| + \left[3t^2 - 5t - \frac{t^3}{3}\right]_2^3$

$= \left|(3 - 5 - 1 - 0)\right| + (27 - 15 - 9) - (12 - 10 - \frac{8}{3})$

$= 3 + (3 - -\frac{2}{3}) = 6\frac{2}{3}$ m. ✓

d $2y = x^2 - 4x$

$x^2 - 4x + 4 = 2y + 4$

$(x-2)^2 = 2(y+2)$ ✓

vertex $= (2, -2)$

$4a = 2$

$a = \frac{1}{2}$

(i) Focus $(2, -1\frac{1}{2})$ ✓

(ii) directrix is $y = -2\frac{1}{2}$ ✓

e) (i) $m_{PA} = \frac{y-0}{x+2} = \frac{y}{x+2}$ ✓

(ii) $m_{PB} = \frac{y-0}{x-6} = \frac{y}{x-6}$

Since $AP \perp PB$

$\therefore m_{PA} \times m_{PB} = -1$ ✓

$\frac{y}{x+2} \times \frac{y}{x-6} = -1$

$y^2 = -(x+2)(x-6)$

$x^2 - 4x + y^2 = 12$ ✓ or $(x-2)^2 + y^2 = 16$

Q15 a(i) LHS $\sqrt{\frac{\operatorname{cosec}^2 x - \cot^2 x - \cos^2 x}{\cos^2 x}}$

$= \sqrt{\frac{1 + \cot^2 x - \cot^2 x - \cos^2 x}{\cos^2 x}}$

$= \sqrt{\frac{1 - \cos^2 x}{\cos^2 x}} = \sqrt{\frac{\sin^2 x}{\cos^2 x}} = \frac{\sin x}{\cos x} = \tan x =$ RHS.

(ii) $\tan x = 1$ $\therefore x = \pi/4$ ✓

b $P = 2000e^{kt}$

(i) $\frac{dP}{dt} = k \times 2000e^{kt}$

$\frac{dP}{dt} = kP$ $\therefore$ proportional ✓

(ii) $t=0$ $P = 2000e^0 = 2000$

$t=4$ $P = 4000$ ✓

$\therefore 4000 = 2000e^{4k}$

$2 = e^{4k}$

$\ln 2 = 4k \ln e$

$k = \frac{\ln 2}{4} = 0.17328\ldots$

$= 0.173$ ✓

iii) $t = 6$

$P = 2000e^{6k}$ ✓

$= 5626.62$

$= 5627$ ✓

c (i) $\angle CED = \angle ABC$

(alternate ∠s on $AB \parallel FD$) ✓

(ii) In $\triangle CDE$ & $\triangle ACB$

$\angle CED = \angle ABC$ proved above

$\angle CDE = \angle CAB$ (alternate ∠ on $AB \parallel FD$) — ①

$AC = CD$ (given) — ①

$\therefore \triangle CDE \equiv \triangle CAB$ (AAS) — ✓

(iii) $AB = ED$ (matching sides of ≡ ∆s)

$ED = EF$ (E is the mid pt of FD) ✓

$\therefore AB = EF$ & Also $AB \parallel EF$

$\therefore ABFE$ is a parallelogram ✓

$AF = BE$ (opposite sides of parallelogram)

But $BE = BC + CE$

$= 2BC$ (as $BC = CE$ given) ✓

$\therefore AF = 2BC$

Q16 a $2\ln(x^2y) = 3 + \ln x - \ln y$

$2\ln x^2 + 2\ln y = 3 + \ln x - \ln y$

$4\ln x + 2\ln y = 3 + \ln x - \ln y$

$3\ln x + 3\ln y = 3$

$\ln x + \ln y = 1$

$\ln xy = 1$

$xy = e$

$y = \frac{e}{x}$

3 marks correct answer for correct working.
2 marks: substantial log knowledge
1 mark some solid working.

b (i) $A_1 = 1000\left(1 + \frac{0.09}{12}\right)^1$

$= 1000(1.0075)$

$= \$1007.50$ ✓ (must calculate)

(ii) $A_2 = 1000(1.0075)^2$

$A_3 = 1000(1.0075)^3$

$\vdots$

$A_n = 1000(1.0075)^n$

$\therefore A = 1000(1.0075) + 1000(1.0075)^2 + \cdots (1.0075)^n$ ✓

$= 1000[\underbrace{1.0075 + 1.0075^2 + \cdots 1.0075^n}_{G.P.}]$

$= 1000\left[\frac{1.0075(1.0075^n - 1)}{1.0075 - 1}\right]$ ✓

$= 1007.50\left(\frac{1.0075^n - 1}{0.0075}\right)$

(iii) $5000 = 1007.5\frac{(1.0075^n - 1)}{0.0075}$

$0.3722 = 1.0075^n - 1$

$1.3722 = 1.0075^n$

$\log 1.3722 = n \log 1.0075$

$n = \frac{\log 1.3722}{\log 1.0075}$ ✓

$n = 42.347$

$\therefore$ least no. of months $= 43$ ✓

c (i) In $\triangle ABD$ & $\triangle ECD$

$\angle ECD = \angle ABD = 90°$ (given)

$\angle ABD$ is common

$\triangle ABD \parallel\parallel \triangle ECD$

✓ $\therefore \frac{x}{xt} = \frac{x}{x+h}$ (Ratio of all matching sides are =)

$h + x = tx$

✓ $h = xt - x$

$x = \frac{h}{t-1}$

- could use Trig but ratio had to be correct

(ii) $V_{cone} = \frac{1}{3}\pi r^2 H$.

$V_{truncated\ cone} = V_{cone} - V_{small\ cone}$

$= \frac{1}{3}\pi(2t)^2(h+x) - \frac{1}{3}\pi 4x$

$= \frac{4}{3}\pi(t^2h + t^2x - x)$

$= \frac{4}{3}\pi\left(t^2h + \frac{t^2h}{t-1} - \frac{h}{t-1}\right)$

$= \frac{4}{3}\pi h\left(t^2 + \frac{t^2}{t-1} - \frac{1}{t-1}\right)$ ✓

$= \frac{4}{3}\pi h\left(t^2 + \frac{t^2-1}{t-1}\right)$ ✓

$= \frac{4}{3}\pi h(t^2 + t + 1)$

(iii) $2t + 2 + h = 12$

$2t + h = 10$

$h = 10 - 2t$ ✓

$V = \frac{4}{3}\pi(10 - 2t)[t^2 + t + 1]$

$= \frac{8\pi}{3}[(5-t)(t^2+t+1)]$

$= \frac{8\pi}{3}[5t^2 + 5t + 5 - t^3 - t^2 - t]$

$= \frac{8\pi}{3}[-t^3 + 4t^2 + 4t + 5]$

$\frac{dV}{dt} = \frac{8\pi}{3}[-3t^2 + 8t + 4]$

For max vol. $\frac{dV}{dt} = 0$

$-3t^2 + 8t + 4 = 0$

$3t^2 - 8t - 4 = 0$

$t = \frac{8 \pm \sqrt{112}}{6}$

$= \frac{4 \pm 2\sqrt{7}}{3}$ ✓

But $t = \frac{4 + 2\sqrt{7}}{3} = 3.09$

as $t > 0$

$\frac{d^2V}{dt^2} = \frac{8\pi}{3}[-6t + 8]$

$= -88.29 < 0$ ✓ Volume + testing.

∴ max at $t = 3.09$

Hence $V = \underline{218.225}$ unit³