BAULKHAM HILLS HS PERSEVERE

2020

TRIAL – YEAR 12
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2-5)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6-34)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section

Section I - 10 marks

Allow 15 minutes for this section

1. Which expression is equal to $\int \tan^2 x \, dx$?

(A) $\dfrac{\tan^3 x}{3} + C$

(B) $\tan x - x + C$

(C) $\tan x + x + C$

(D) $\sec^2 x + C$

2. $\dfrac{d}{dx} \log_e \dfrac{4x^2 - 9}{2x - 3}$ is equal to which of the following?

(A) $\dfrac{6}{2x - 3}$

(B) $\dfrac{2}{2x + 3}$

(C) $\dfrac{6(2x + 3)}{(2x - 3)^2}$

(D) $\dfrac{6(4x + 1)}{(2x - 3)^2}$

3. Which of the following could be a primitive for $f'(x) = \dfrac{x}{e^{x^2 - 8}}$?

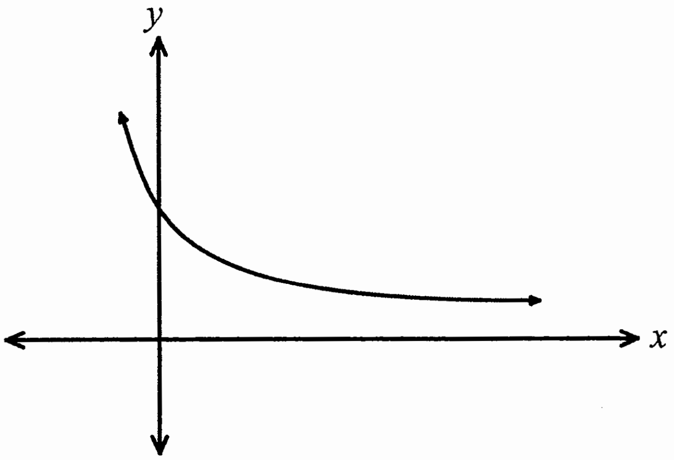
(A) $-\frac{1}{2}(e^{x^2 - 8}) + 8$

(B) $\frac{1}{2}\ln(e^{x^2 - 8}) + 8$

(C) $\ln(e^{8 - x^2}) - 8$

(D) $-\frac{1}{2}(e^{8 - x^2}) - 8$

4. For the curve shown, which inequalities are correct?



(A) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

(B) $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

(C) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

(D) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

5. Results for a test are given as z-scores. In this test Angela gained a z- score of 3.The test has a mean of 55 and standard deviation of 6. What was Angela's actual mark in this test?

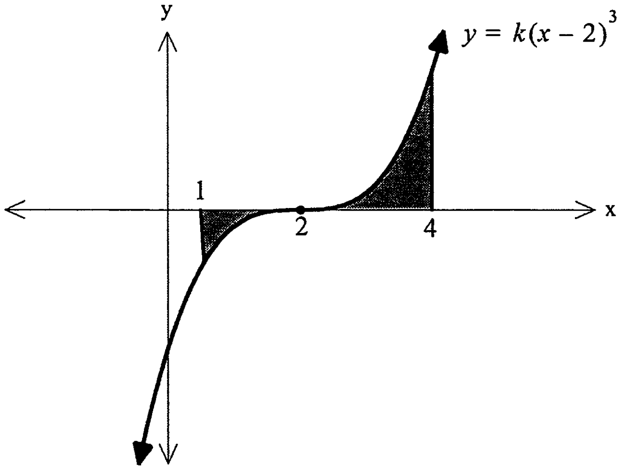
(A) 58

(B) 73

(C) 64

(D) 67

6. The graph with the equation $y = k(x - 2)^3$ is shown below, for some positive constant k.



If the area of the shaded region is 34, what is the value of k?

(A) $\frac{136}{15}$

(B) 8

(C) 4

(D) $\frac{34}{9}$

7. The time, T, in seconds that divers can hold their breath is normally distributed with $\mu = 120$ and $Var(T) = 400$. In what range of time length would you expect to find the middle 95%?

(A) $100 \leq x \leq 140$

(B) $80 \leq x \leq 160$

(C) $60 \leq x \leq 180$

(D) $40 \leq x \leq 200$

8. The exact value of $I = \int_1^2 \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln 2)^2$. The approximation of I using the Trapezoidal Rule with 2 function values is

(A) smaller by 28%

(B) larger by 28%

(C) smaller by 72%

(D) larger by 72%

9. Given a function $f(x) = \frac{x}{x^2 - 5}$

Which of the following statements is true?

(A) $f(x)$ is even and one-to-one.

(B) $f(x)$ is even and many-to-one.

(C) $f(x)$ is odd and one-to-one.

(D) $f(x)$ is odd and many-to-one.

10. The amount M of certain medicine present in the blood after t hours is given by $M = 9t^2 - t^3$ for $0 \leq t \leq 9$.

When is the amount of medicine in the blood increasing most rapidly?

(A) $t = 0$

(B) $t = 9$

(C) $t = 6$

(D) $t = 3$

END OF SECTION I

Section II- Extended Response

Attempt Questions 11-16.

Allow about 2 hours and 45 minutes for this section.

Question 11(15 Marks)

a) Differentiate the following

(i) $y = (4x - 5)(4x + 5)$ **1**

(ii) $y = \sin^2 x$ **2**

b) In an arithmetic series, the third term is 5 and the tenth term is 26. Find the sum of the first 14 terms. **2**

Question 11 continued on the next page

c) Evaluate **2**

$$\int_1^4 5(9x-4)^4\,dx$$

....

....

....

....

....

....

....

....

....

....

d) Solve the following equation for x. **2**

$$e^{2x} + 3e^{x} - 10 = 0.$$

....

....

....

....

....

....

....

....

....

....

....

....

....

Question 11 continued on the next page

e) (i) Show that $\dfrac{d}{dx}(\sec^2 x) = 2\tan x\ \sec^2 x$. **2**

(ii) Hence find $\displaystyle\int \tan x\ \sec^2 x\ dx$. **1**

Question 11 continued on the next page

f) Given a function $f(x) = \begin{cases} 6x - 6x^2 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$

(i) Show that $f(x)$ represents probability density function. **2**

(ii) Find the mode of the probability density function $f(x)$. **1**

End of Question 11

Question 12 (13 Marks)

a) Find the value(s) of b such that $y = 2x + b$ is a tangent to the parabola **2**

$$y = 2x^2 + 6x - 5.$$

Question 12 continued on the next page

b) Angela guesses three questions in her multiple choice test, which has four options per question. Find the probability that Angela gets:

(i) Only one correct answer. **1**

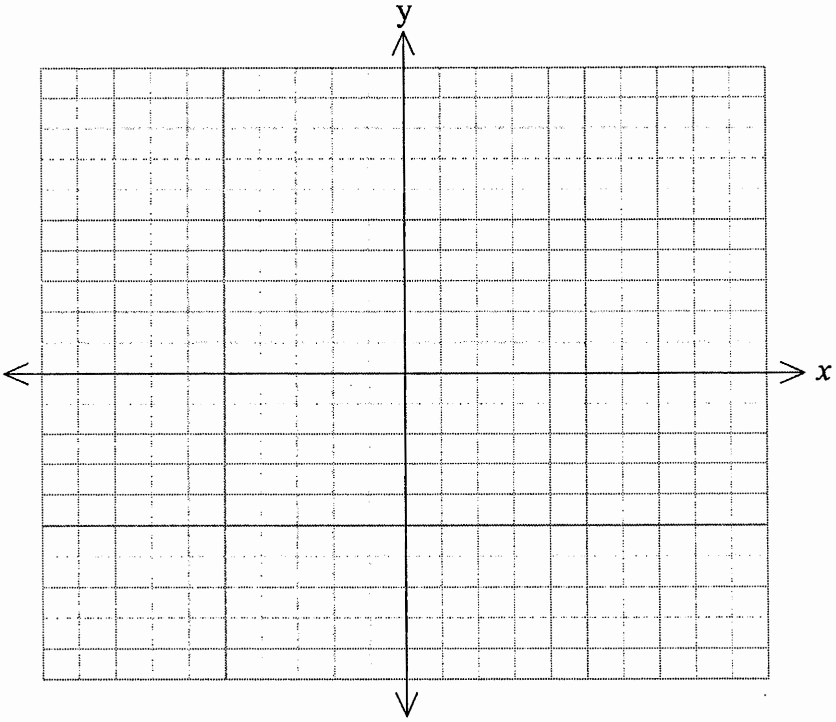
(ii) At least one correct answer. **1**

Question 12 continued on the next page

c)

(i) Sketch the hyperbola by shifting $y = \dfrac{1}{x-1}$ horizontally 3 units to the right **2**

and 1 unit down.



(ii) State the equation of the shifted hyperbola, then find all the intercepts **2**

of the shifted hyperbola with the axes and mark them on your graph in

part (i).

....

....

....

....

....

....

....

....

Question 12 continued on the next page

d) Consider the piece -wise defined function.

$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ 4 - x^2 & x > 1 \end{cases}$$

(i) Find $f(1)$ **1**

....

....

(ii) Find x if $f(x) = 0$ **2**

....

....

....

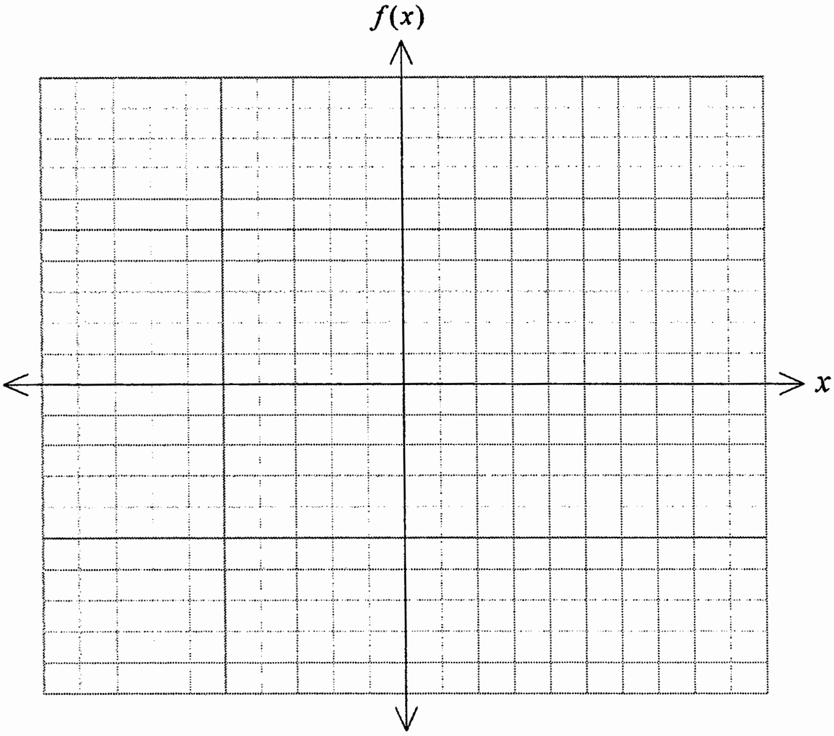
....

....

....

....

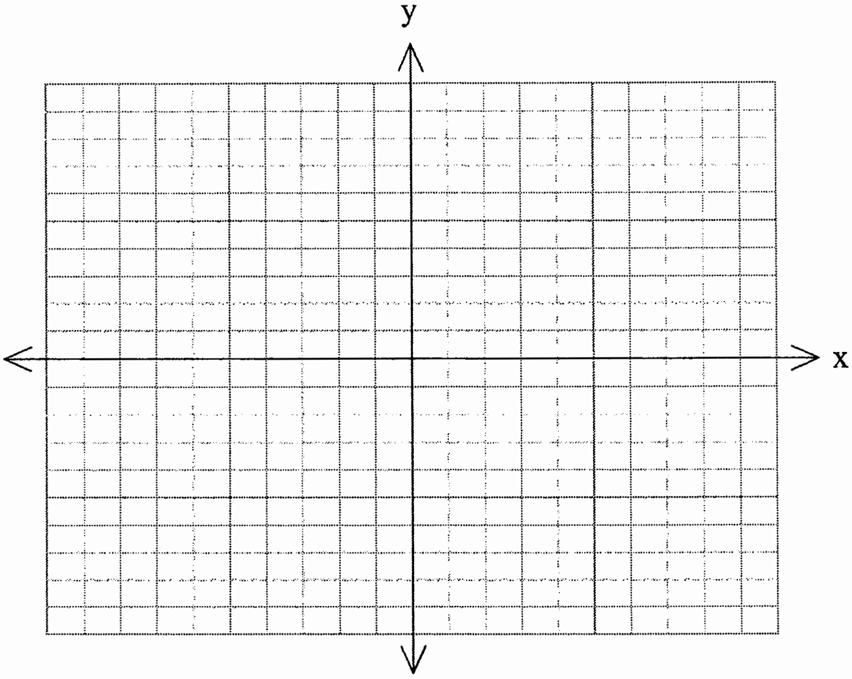
(iii) Sketch the function showing all intercepts. **2**



End of Question 12

Question 13 (18 Marks)

a) (i) Sketch the graphs of $f(x) = 2x - 2x^2$ and $g(x) = x - 1$ on the same number plane. **2**

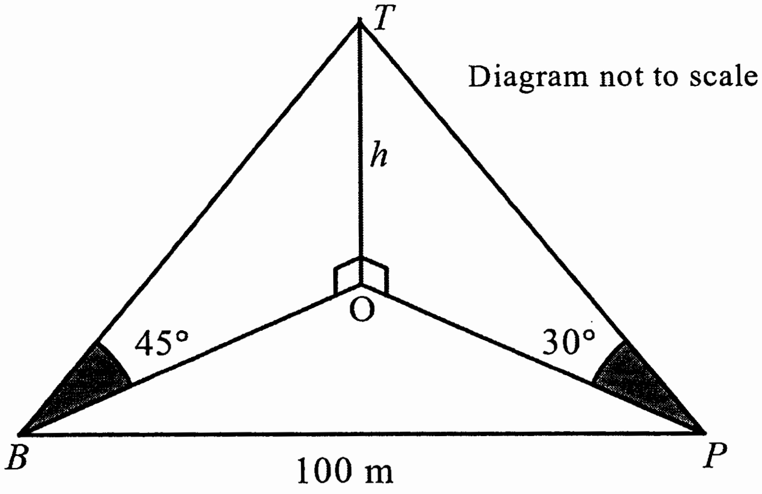


(ii) Using your graphs from part (i), or otherwise solve the inequality **2**

$$x - 1 < 2x - 2x^2.$$

Question 13 continued on the next page

b) A surveyor stands at a point P, which is due east of the tower OT, of height h metres. The angle of elevation of the top of the tower T from P is $30°$. The surveyor then walks 100 metres to point B, which is on a bearing of $150°$ from the foot of tower O. The angle of elevation of the top of the tower from B is now $45°$.



(i) Express the length of OP in terms of h. **1**

Question 13 continued on the next page

(ii) Show that $(100)^2 = h^2 + \dfrac{h^2}{\tan^2 30°} - \dfrac{h^2}{\tan 30°}$. **2**

(iii) Hence find the height of the tower. Answer correct to 1 decimal place. **1**

Question 13 continued on the next page

c) The following information shows a group of people's waist measurements and weights.

Waist (cm)x	72	67	85	96	80	90	98	105
Weight (kg) y	58	50	72	85	70	79	82	84

(i) Calculate the correlation coefficient, r, for their waist and weight measurements correct to 3 decimal places and hence describe the strength of the relationship. **2**

(ii) Find the equation of the Least -Squares Regression Line. **1**

Question 13 continued on the next page

d) Given the function $f(x) = \ln(x^2 + 1)$.

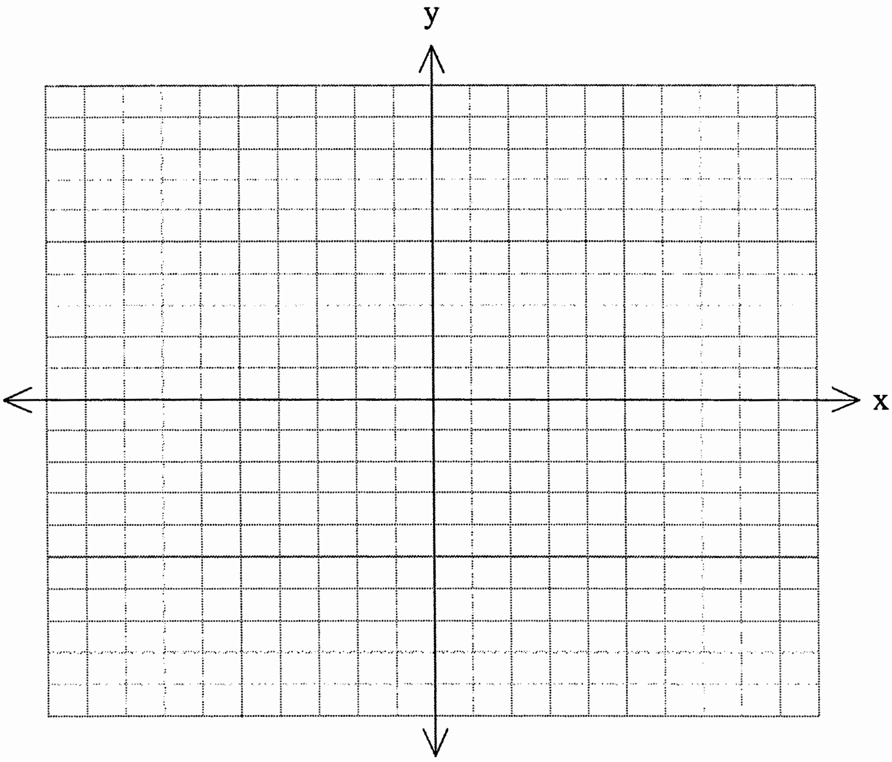
(i) Find the domain of $f(x)$. **1**

(ii) Find any stationary point(s) and determine their nature. **2**

Question 13 continued on the next page

(iii) Find any point(s) of inflection. **2**

(iv) Sketch the graph of $f(x) = \ln(x^2 + 1)$ showing all features from part (ii) and (iii). **2**



End of Question 13

Question 14 (14 marks)

a) (i) Prove the following identity **1**

$$(1 + \tan x)^2 = 2\tan x + \sec^2 x$$

(ii) Hence find the area bounded by $y = (1 + \tan x)^2$ and the x –axis between **3**

$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

Question 14 continued on the next page

b) Given $y = 2\sin\left(2x - \frac{\pi}{3}\right)$

(i) State the amplitude and period. **2**

....

....

....

....

....

....

....

....

(ii) Find the exact values of all intercepts of **2**

$y = 2\sin\left(2x - \frac{\pi}{3}\right)$ with the axes for $0 \leq x \leq \pi$.

....

....

....

....

....

....

....

....

....

....

....

....

....

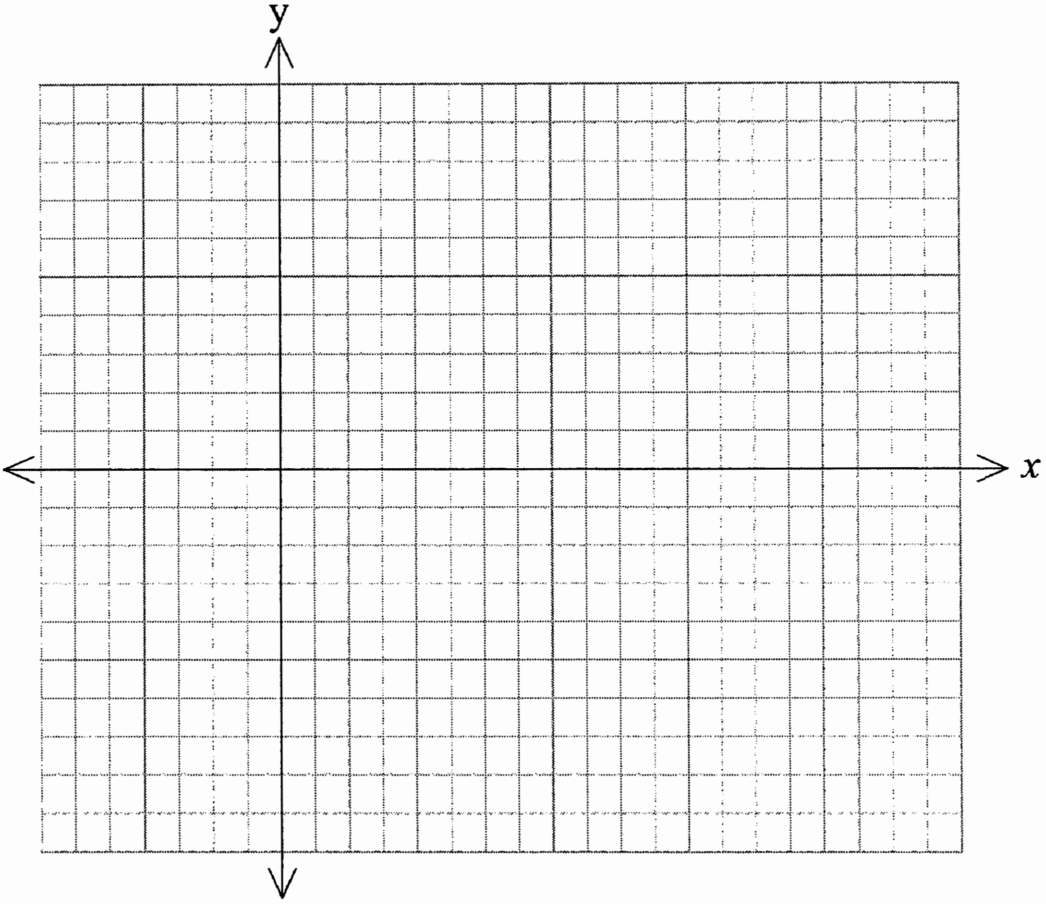
....

....

Question 14 continued on the next page

(iii) Hence sketch the graph of $y = 2\sin\left(2x - \frac{\pi}{3}\right)$ for $0 \le x \le \pi$, **2**

showing all features from part (i) and (ii) and the global maximum and minimum.



Question 14 continued on the next page

c) A bag contains three red balls and four black balls. Two balls are selected at random without replacement from the bag.

Let X be the number of black balls drawn.

(i) Fill in the following table and hence find exact value of $E(X)$. **2**

x	0	1	2
$P(X = x)$			

....

....

....

....

....

....

....

....

(ii) Find $E(X^2)$ and hence find $\mathrm{Var}(X)$ and standard deviation σ. **2**

....

....

....

....

....

....

....

....

....

....

....

....

....

....

End of question 14

Question 15 (16 marks)

a) The velocity v of a particle in metres per second is given by the formula

$v = 5(1 + e^{-t})$, where t is the time in seconds.

(i) Find the initial velocity of the particle. **1**

....

....

....

....

....

(ii) Is the particle ever stationary? Justify your answer. **1**

....

....

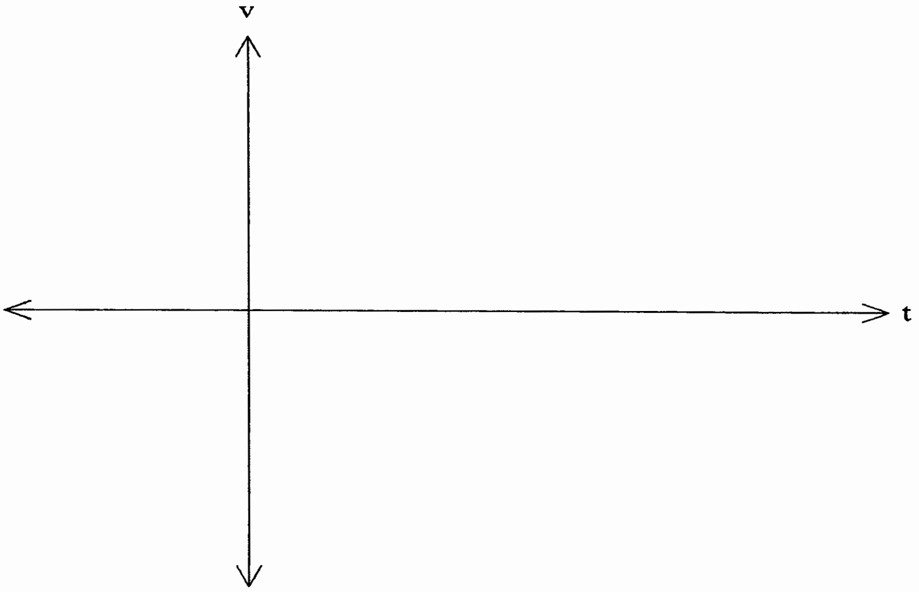
....

....

....

....

(iii) Sketch the graph of the velocity. **2**



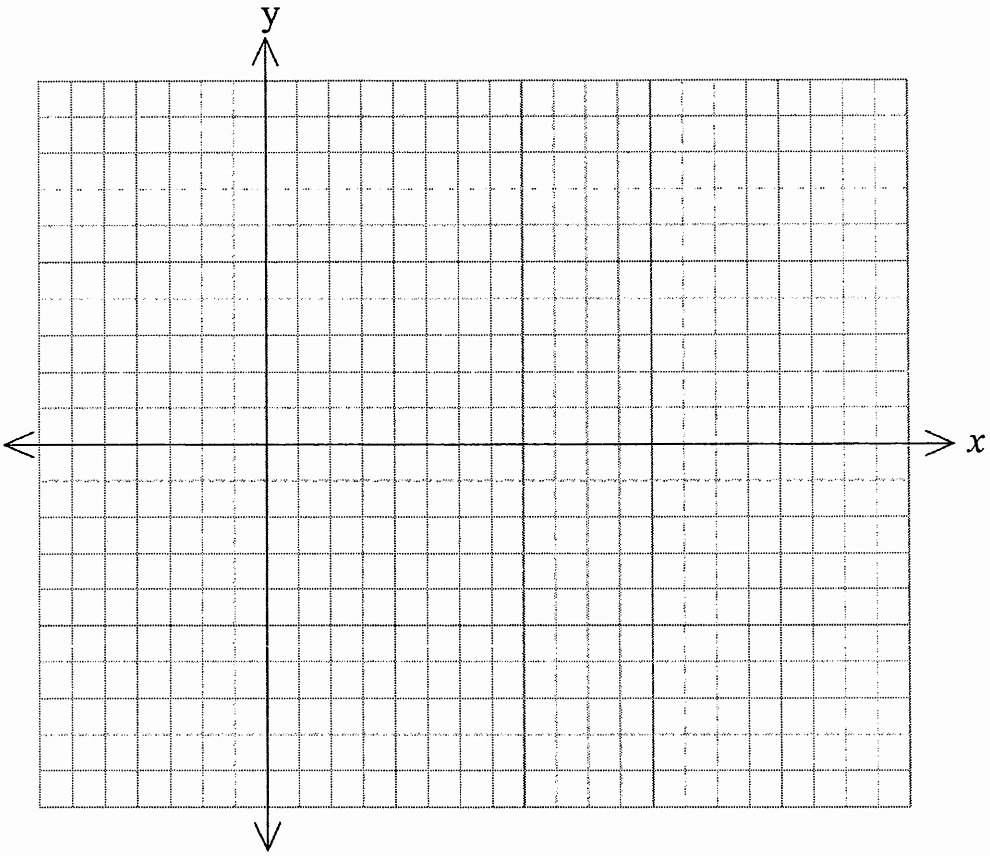
Question 15 continued on the next page

(iv) Find the total distance travelled by the particle in the first 5 seconds. **2**

Question 15 continued on the next page

b) The line $y = mx$ is a tangent to the curve $y = \ln(2x - 1)$ at a point P.

(i) Sketch the line and the curve on the same diagram, clearly indicating the point P. **2**



Question 15 continued on the next page

(ii) Show that the coordinates of P are $\left(\dfrac{2+m}{2m}, \dfrac{2+m}{2}\right)$. **2**

(iii) Hence show that $2 + m = \ln\left(\dfrac{4}{m^2}\right)$. **2**

Question 15 continued on the next page

c) Given the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the cumulative distribution function $F(x)$. **2**

(ii) Hence find the median. **2**

End of question 15

Question 16 (14 marks)

a) Michelle borrows \$450 000 to be repaid by regular monthly repayments of \$$M$ over a period of 25 years at 6% per annum reducible monthly. Interest is calculated and charged just before each repayment.

Let A_n be the amount owing after n –repayments.

(i) Show that the expression for the amount owing after two repayments is **1**

$$A_2 = 450\ 000(1.005)^2 - M(1.005) - M.$$

(ii) Show that the amount owing after n –repayments is **2**

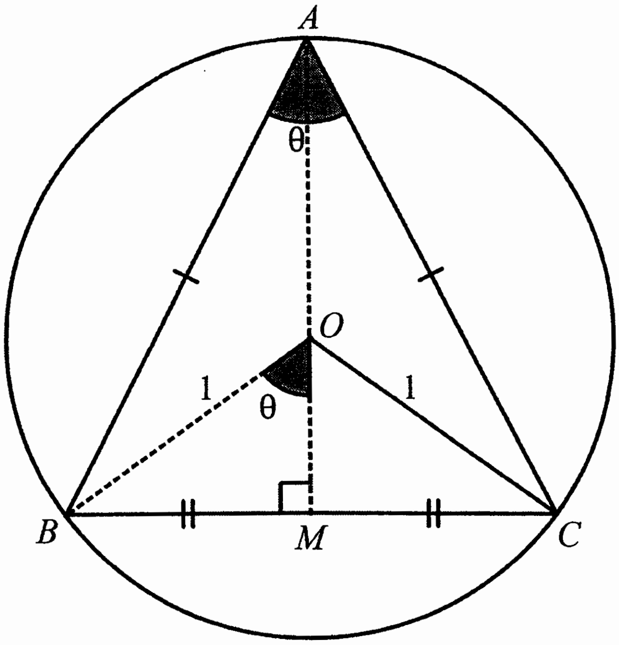
$$A_n = 450\ 000(1.005)^n - M\,\frac{(1.005)^n - 1}{0.005}.$$

Question 16 continued on the next page

(iii) Calculate the amount of each regular monthly repayment. **2**

Question 16 continued on the next page

b) An isosceles triangle ΔABC is inscribed within a unit circle centred at O, as shown in the diagram below. Let M be the midpoint of BC , $\angle BAC = \theta$ and $\angle \mathrm{BOM} = \theta$.



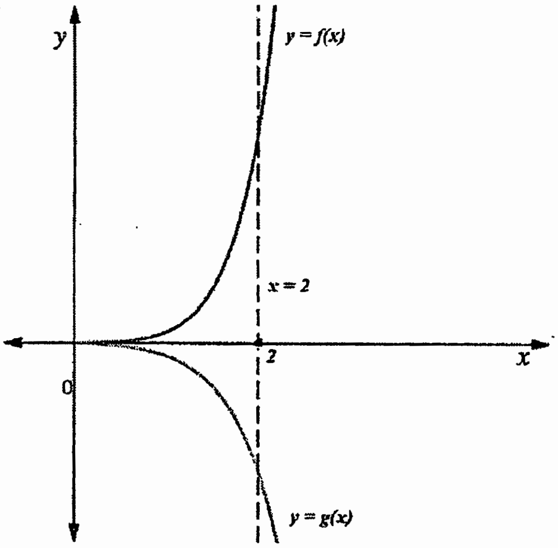
(i) Show that the area of ΔABC is $A = \sin\theta(1 + \cos\theta)$. **2**

Question 16 continued on the next page

(ii) Hence prove that the area of the isosceles triangle ΔABC is maximum when it is equilateral. **3**

Question 16 continued on the next page

c) The graph of $f(x) = x^2e^{kx}$ and $g(x) = -\frac{2xe^{kx}}{k}$ and the line $x = 2$ is drawn below, where k is a positive constant. $f(x) = g(x)$ at only one point, that is at $(0, 0)$.



Let A be the area of the region bounded by the curve $y = f(x)$, $y = g(x)$ and the line $x = 2$.

(i) Write down a definite integral that gives the value of A. **1**

(ii) The function $f(x)$ from part (i) is given by $f(x) = x^2e^{kx}$, where k is a positive constant. Show that $f'(x) = xe^{kx}(kx + 2)$. **1**

Question 16 continued on the next page

(iii) Using the results of part (i) and (ii), or otherwise, find the value of k such that $A = \dfrac{16}{k}$. **2**

End of Exam

Answers – Multiple Choice (2020)
Yr.12 TRIAL – Maths Advanced

① $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$ Ⓑ

$= \tan x - x + C$

② $\frac{d}{dx} \log_e \frac{4x^2-9}{2x-3} = \frac{d}{dx} \log_e \frac{\cancel{(2x-3)}(2x+3)}{\cancel{(2x-3)}}$ Ⓑ

$= \frac{d}{dx} \log_e (2x+3) = \frac{2}{2x+3}$

③ $f'(x) = \frac{x}{e^{x^2-8}} \therefore f(x) = \int \frac{x}{e^{x^2-8}} dx$

$f(x) = \int x \cdot (e^{x^2-8})^{-1} dx = \int x e^{8-x^2} dx$

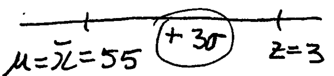
$= -\frac{1}{2} \int \underbrace{-2x}_{g'(x)} \cdot e^{\overset{g(x)}{\boxed{8-x^2}}} dx = \frac{-1}{2} e^{8-x^2} + C$

but C - can be any constant ∴ Ⓓ

④ curve is decreasing $\therefore \frac{dy}{dx} < 0$

curve is concave up $\therefore \frac{d^2y}{dx^2} > 0$ Ⓓ

⑤



$\therefore$ score $= 55 + 3 \times 6 = 73$ Ⓑ

⑥ Area $= 34 = \left| \int_1^2 K(x-2)^3 dx \right| + \int_2^4 K(x-2)^3 dx$

$34 = K \left| \int_1^2 (x-2)^3 dx \right| + K \int_2^4 (x-2)^3 dx$

$34 = K \left| \left[\frac{(x-2)^4}{4} \right]_1^2 \right| + K \left[\frac{(x-2)^4}{4} \right]_2^4$

$34 = K \left| 0 - \frac{1}{4} \right| + K \left[\frac{16}{4} - 0 \right]$

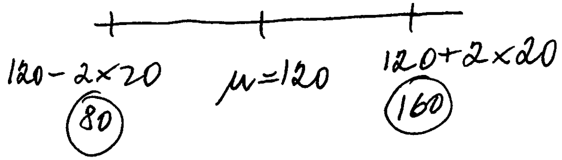
$34 = K(\frac{1}{4} + 4)$ Ⓑ

$\therefore K = 8$

⑦ $\mu = 120 \quad \sigma = 20 \; (Var(T) = \sigma^2)$

middle 95% is 2σ around μ.

$80 \le x \le 160$ Ⓑ



⑧ $I \doteqdot \frac{h}{2}$ (first value + last value)

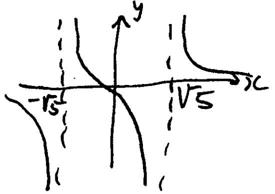
$= \frac{1}{2} \left(\frac{\ln 1}{2} + \frac{\ln 2}{2} \right) = \frac{1}{4} \ln 2 = 0.17328\ldots$ (smaller)

Exact $I = \frac{1}{2} (\ln 2)^2 = 0.2402265$

$\therefore \% = \frac{\text{approx. } I - \text{Exact } I}{\text{Exact } I} = \frac{\frac{1}{2}(\ln 2)^2 - \frac{1}{4} \ln 2}{\frac{1}{2}(\ln 2)^2} \times 100$

$\% = 27.865\% \doteqdot 28\%$ smaller Ⓐ

⑨ $f(-x) = \frac{-x}{(-x)^2 - 5} = \frac{-x}{x^2-5} = -f(x) \therefore$ odd

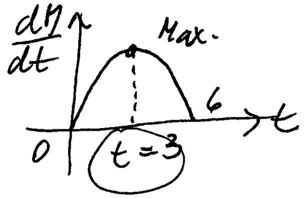


by horiz. line test ∴ many to one Ⓓ

⑩ $M = 9t^2 - t^3$

$\frac{dM}{dt} = 18t - 3t^2 = 3t(6-t)$

$t = 3 \therefore$ Ⓓ



Section II- Extended Response

Attempt Questions 11-16.

Allow about 75 minutes for this section.

Question 11(14 Marks)

a) Differentiate the following

(i) $y = (4x - 5)(4x + 5) = 16x^2 - 25$

Marks 1

$\frac{dy}{dx} = 32x$

1 - correct soln

OR $\frac{dy}{dx} = 4(4x+5) + (4x-5) \times 4$

$= 16x + 20 + 16x - 20 = 32x$

(ii) $y = \sin^2 x$ **2**

$\frac{dy}{dx} = 2\sin x \cdot \cos x$

2 - correct soln.

1 - correctly diff. $\sin x$

1 - $\frac{dy}{dx} = 2\sin x$

b) In AP. $T_3 = 5$ and $T_{10} = 26$. **2**

Find the sum of S_{14}.

AP: $T_3 = 5 \quad T_{10} = 26$

$T_3 = 5 = a + 2d$

$T_{10} = 26 = a + 9d$

$21 = 7d \therefore d = 3, \quad a = -1$

$\therefore S_{14} = \frac{14}{2}(2a + (14-1) \times 3) = 259$

2 - correct soln.

1 - correctly finds a or d

1 - applies S_n for A.P correctly

Question 11 continued on the next page

c) Evaluate **2**

$$\int_1^4 5(9x - 4)^4 dx$$

$$= \left[\cancel{5} \times \frac{(9x-4)^5}{\cancel{5} \times 9} \right]_1^4$$

$$= \frac{1}{9}\left[(9x-4)^5\right]_1^4$$

$$= \frac{1}{9}\left[(9(4)-4)^5 - (9-4)^5\right]$$

$$= \frac{1}{9}\left[32^5 - 5^5\right] = 3727923$$

2 - correct soln.

1 - correct integral

1 - correct answer from incorrect integral

d) $e^{2x} + 3e^x - 10 = 0$

let $m = e^x$

$m^2 + 3m - 10 = 0$

$(m+5)(m-2) = 0$

$m = -5 \qquad m = +2$

$e^x = -5 \qquad e^x = 2$

no solns. $\qquad x = \ln 2$

$\therefore$ solution $x = \ln 2$ (only)

2 - correct soln.

1 - correctly reduces to quadratic eqn & solves it correctly

Question 11 continued on the next page

e)

(i) Show that $\frac{d}{dx}(\sec^2 x) = 2\tan x \sec^2 x$ **2**

$LHS = \frac{d}{dx}\left(\sec^2 x\right) = \frac{d}{dx}\left(\frac{1}{\cos^2 x}\right)$

$= \frac{d}{dx}\left(\cos^{-2} x\right) = -2\cos^{-3} x \cdot (-\sin x)$

$= \frac{2\sin x}{\cos^3 x} = \frac{2\sin x}{\cos x} \times \frac{1}{\cos^2 x}$

$= 2\tan x \cdot \sec^2 x$

$= RHS \therefore$ shown

2 – correct soln.
1 – differentiates correctly $\sec^2 x$
1 – applies trig. identities correctly

(ii) Hence find $\int \tan x \sec^2 x dx$ **1**

from (i) $\frac{d}{dx}(\sec^2 x) = 2\tan x \sec^2 x$

$\therefore \sec^2 x = \int 2\tan x \sec^2 x dx$

$\therefore \frac{1}{2}\sec^2 x = \int \tan x \sec^2 x dx$

$\therefore \int \tan x \sec^2 x dx = \frac{1}{2}\sec^2 x + C$

$OR = \frac{1}{2}\left(1 + \tan^2 x\right) + C$

1 – correct soln.
– ignore +c

Question 11 continued on the next page

f) Given a function $f(x) = \begin{cases} 6x - 6x^2 & 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$

(i) Show that $f(x)$ represents probability density function. **2**

f(x) represents PDF if

- $\int_0^1 f(x) = 1$ and $\underbrace{f(x) \ge 0}_{\text{on domain}}$ true

$\therefore \int_0^1 6x - 6x^2 dx$

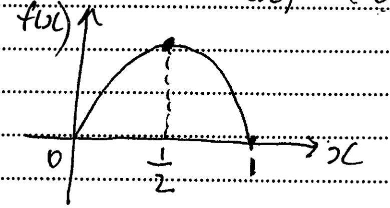
$= \left[3x^2 - 2x^3\right]_0^1 =$

$= \left[(3-2) - 0\right] = 1 \therefore$ Yes it's PDF

2 – correct soln.
1 – finds $\int f(x)dx$ correctly

(ii) Find the mode of the probability density function $f(x)$ **1**

By sketching $f(x) = 6x - 6x^2$
$f(x) = 6x(1-x)$



Mode is $x = \frac{1}{2}$ with the highest value of f(x)

1 – correct soln.

End of Question 11

Question 12 (13 Marks)

a) Find the value(s) of m such that $y = 2x + m$ is a tangent to the parabola ②

$$y = 2x^2 + 6x - 5.$$

2 - correct soln.

$y = 2x + m$

$2 \Rightarrow$ gradient of tangent

1 - finds point of contact by using calculus

$\therefore y' = 4x + 6$ where $m = 2$

$\therefore 2 = 4x + 6$

$-1 = x$ $\quad \therefore y = 2(-1)^2 + 6(-1) - 5$

$y = -9$

1 - finds Δ correctly by using simult. eqns.

$\therefore$ pt. of contact $(-1, -9)$

sub. $(-1, -9)$ into $y = 2x + m$

$-9 = 2(-1) + m$

$\therefore \boxed{m = -7}$

1 - finds gradient function correctly & x-coord. of pt. of contact

OR by simultaneous eqn.

$2x + m = 2x^2 + 6x - 5$

$2x^2 + 4x - 5 - m = 0$

$\Delta = 0$ (since tangent $\therefore$ only one solution)

$0 = b^2 - 4ac$

$0 = 4^2 - 4(2)(-5 - m)$

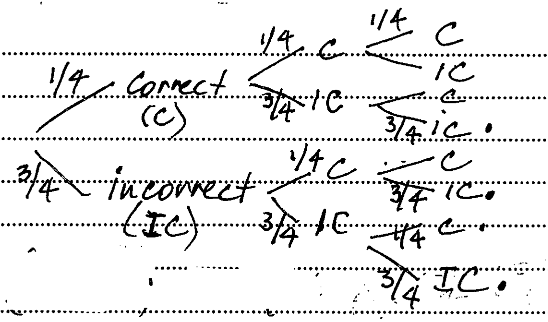
$0 = 16 + 40 + 8m$

$\boxed{m = -7}$

1 - creates quadr. eqn. correctly by solving simult. eqns, & attempts to solve $\Delta = 0$

Question 12 continued on the next page

b) Angela guesses three questions in her multiple choice test, which has four options per question. Find the probability that Angela gets



i) Only one correct **1**

$P = P(Cii) + P(iCi) + P(iiC)$

$= \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)$

$= 3 \times \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right)^2 = \frac{27}{64}$

1 - correct soln.

(ii) At least one correct **1**

P(atleast one correct)

$= 1 - P$(no correct)

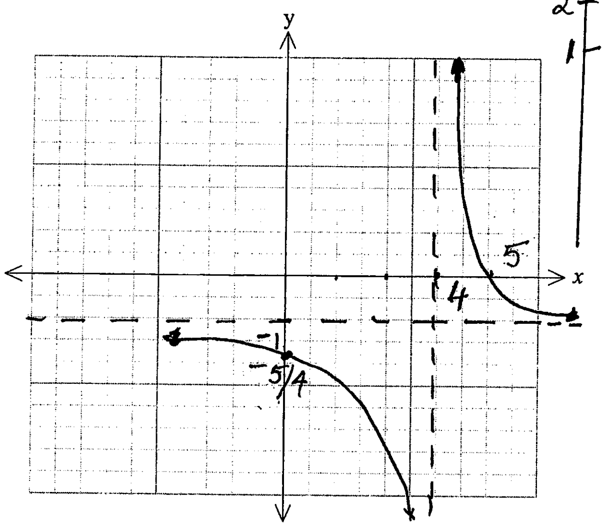
$= 1 - \left(\frac{3}{4}\right)^3 = \frac{37}{64}$

1 - correct soln.

Question 12 continued on the next page

c)

(i) Sketch the hyperbola by shifting $y = \frac{1}{x-1}$ horizontally 3 units to the right and 1 unit down. **2**



2 - correct graph
1 - correct shape and one of the asymptotes correct

(ii) State the equation of the shifted hyperbola, then find all the intercepts of the shifted hyperbola with the axes and mark them on your graph in part (i). **2**

$y = \frac{1}{x-4} - 1$

$x = 0 \therefore y = \frac{1}{0-4} - 1 = -1\frac{1}{4}$ or $-5/4$

$\therefore$ y-int $\left(0, -\frac{5}{4}\right)$

$y = 0 \therefore 0 = \frac{1}{x-4} - 1$

$1 = \frac{1}{x-4}$ $\therefore$ x-int $(5, 0)$

$x - 4 = 1$ $\therefore x = 5$

2 - correct soln.
1 - correct eqn.
1 - one of the intercepts correct and labeled on the graph

Question 12 continued on the next page

(d) Consider the piece -wise defined function.

$$f(x) = \begin{cases} x^2 - 1 & x \leq 1 \\ 4 - x^2 & x > 1 \end{cases}$$

(i) Find $f(1)$ **1**

$f(1) = 1^2 - 1 = 0$

1 - correct soln.

(ii) Find x if $f(x) = 0$ **2**

$f(x) = 0 \begin{cases} x^2 - 1 = 0 \therefore x = \pm 1 \\ 4 - x^2 = 0 \therefore x = \pm 2 \end{cases}$

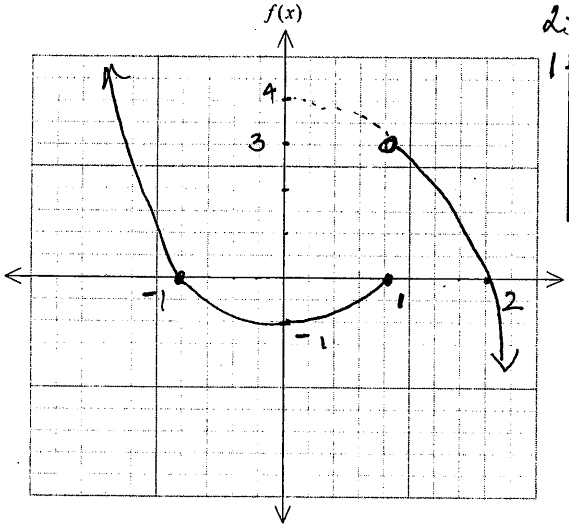
but $x > 1 \therefore x = 2$

2 - correct answers
1 - all 4 solns.
correct withou excluding $x = -$
1 → $x = \pm 1$

(iii) Sketch the function showing all intercepts. **2**

at $x = 1$

$f(x) = 4 - 1^2 = 3$



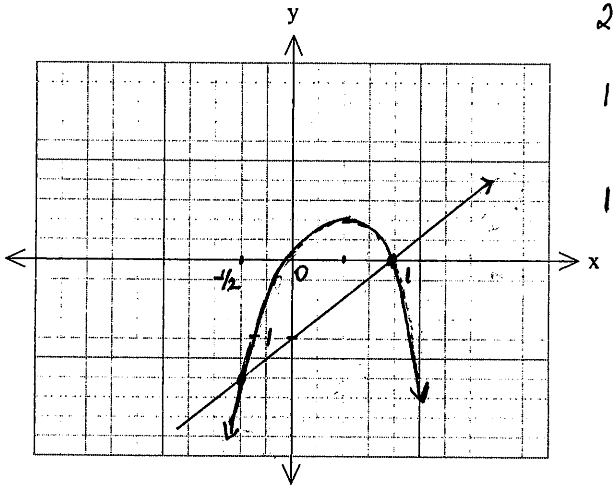
2 - correct grap
1 - correct shap without excluding p
(1, 3)

End of Question 12

Question 13 (18 Marks)

a) (i) Sketch the graphs of $f(x) = 2x - 2x^2$ and $g(x) = x - 1$ on the same number plane. 2

$f(x)=2x(1-x)$



2 - both graphs correct
1 - correct graph of $g(x)$ with correct intercept
1 - $f(x)$ correct graph with correct x-intercepts

(ii) Using your graphs from part (i), or otherwise solve the inequality 2

$$x - 1 < 2x - 2x^2.$$

$f(x) \cap g(x) =$ pts. of intersection

$x-1 = 2x-2x^2$

$2x^2 - x - 1 = 0$

$(2x+1)(x-1)=0$

$x = -\frac{1}{2} \quad x = 1$

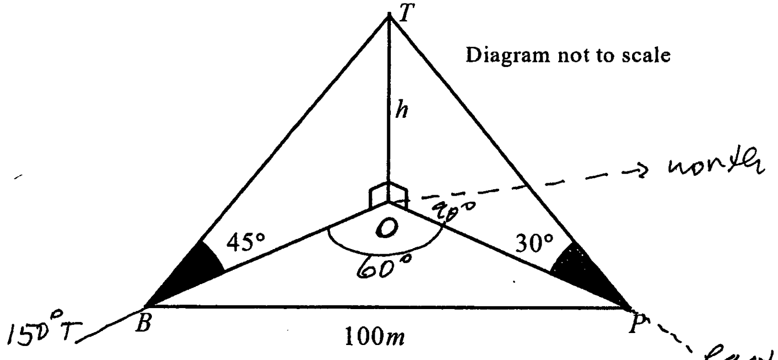
$\therefore$ answer graphically (line below parabola)
$g(x)$ $f(x)$

$-\frac{1}{2} < x < 1$

2 - correct solns.
1 - finds pts. of intersection $f(x) \cap g(x)$ correctly

Question 13 continued on the next page

b) A surveyor stands at a point P, which is due east of the tower OT, of height h metres. The angle of elevation of the top of the tower T from P is $30°$. The surveyor then walks 100 metres to point B, which is on a bearing of $150°$ from the foot of tower O. The angle of elevation of the top of the tower from B is now $45°$.



(i) Express the length of OP in terms of h. 1

from $\triangle$ OPT

$\tan 30° = \frac{h}{OP}$

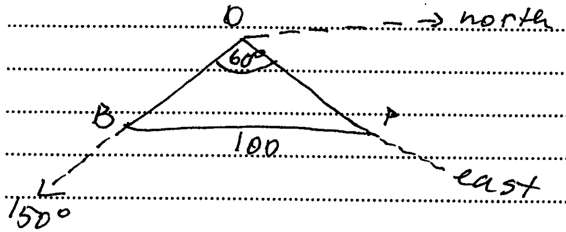
$OP = \frac{h}{\tan 30°}$

OR $OP = \sqrt{3}h$

1 - correct expression with $\tan 30°$
1 - correct answer $OP = \sqrt{3}h$

Question 13 continued on the next page

(ii) Show that $(100)^2 = h^2 + \frac{h^2}{\tan^2 30°} - \frac{h^2}{\tan 30°}$. **2**



from △ BOT ∴ $\tan 45° = \frac{h}{BO}$

∴ $\boxed{BO = h}$

cosine rule: $BP^2 = BO^2 + OP^2 - 2BO \cdot OP \cdot \cos 60°$

∴ $100^2 = h^2 + \frac{h^2}{\tan^2 30°} - \cancel{2} h \times \frac{h}{\tan 30} \times \frac{1}{\cancel{2}}$

∴ $100^2 = h^2 + \frac{h^2}{\tan^2 30°} - \frac{h^2}{\tan 30°}$

2 – correct soln ✓

1 – finds expression for OB and applies cosine rule for △ OPB correctly

1 – finds expression for OB and ∠ BOP = 60°

(iii) Hence find the height of the tower. Answer correct to 1 decimal place. **1**

$100^2 = h^2\left(1 + \frac{1}{\tan^2 30°} - \frac{1}{\tan 30°}\right)$

$h^2 = \frac{100^2}{\left(1 + \frac{1}{\tan^2 30°} - \frac{1}{\tan 30°}\right)}$

$h^2 = 4409.269852$

∴ $h = 66.4$m (1dp)

1 – correct answer (ignore rounding)

Question 13 continued on the next page

c) The following information shows a group of people's waist measurements and weights.

Waist (cm)x	72	67	85	96	80	90	98	105
Weight (kg) y	58	50	72	85	70	79	82	84

(i) Calculate the correlation coefficient, r, for their waist and weight measurements and hence describe the strength of the relationship. **2**

$r = 0.9592$ (calculator)

∴ strong correlation positive

2 – correct soln.

1 – correct r

1 – from their "r" correct conclusion for strength of the relationship

(ii) Find the equation of the Least -Squares Regression Line. **1**

from calculator

$A = -8.2368$

$B = 0.93203$

$y = A + Bx$

$y = -8.2368 + 0.93203x$

1 – correct solns.

Question 13 continued on the next page

d) Given the function $f(x) = \ln(x^2 + 1)$.

(i) Find the domain of $f(x)$. 1

$x^2+1 > 0$

which is always

∴ Domain = all real x

1- correct solns.

(ii) Find any stationary point(s) and determine their nature. 2

$f'(x) = \frac{2x}{x^2+1}$

$0 = \frac{2x}{x^2+1}$

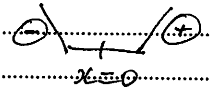
$x=0$, $y = \ln(0+1) = 0$

∴ $(0,0)$ is stationary point

2- correct solns.
1- finds stationary point correctly
1- determines the nature of st. point correctly

Nature by f'' or table

x	-1	0	1
f'	-1	0	1



∴ $(0,0)$ is minimum turning pt.

or $f''(x) = \frac{2-2x^2}{(x^2+1)^2}$

$f''(0) = 2 > 0$ ∴ $(0,0)$ is min t.p.

Question 13 continued on the next page

(iii) Find any point(s) of inflection. 2

$f''(x) = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$

$f''(x) = \frac{2-2x^2}{(x^2+1)^2} = 0$

$2-2x^2 = 0$ ∴ $x = \pm 1$

∴ $(1, \ln 2)$, $(-1, \ln 2)$ possible pts. of inflexion

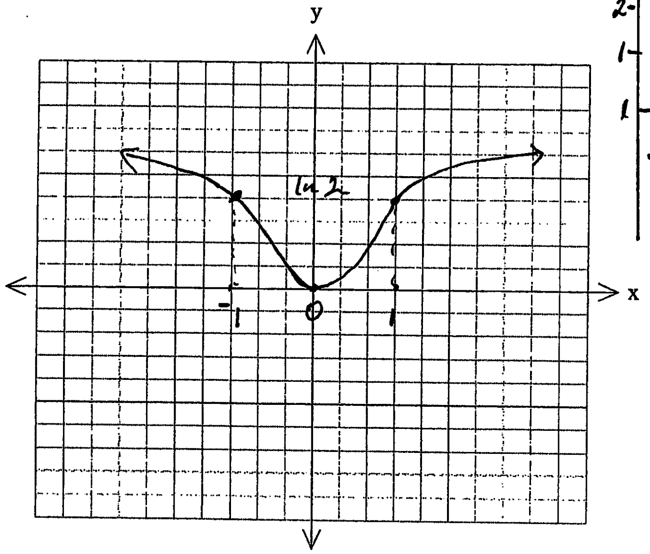
x	-2	-1	0	1	2
f''	$\frac{-6}{25}$	0	2	0	$-\frac{6}{25}$

concavity changes at $x = \pm 1$

∴ $(1, \ln 2)$, $(-1, \ln 2)$ are points of inflexion

2- correct solns.
1- solves for $f''=0$ correctly
1- justifies pt. of inflexion correctly

(iv) Sketch the graph of $f(x) = \ln(x^2 + 1)$ showing all features from part (ii) and (iii). 2



2- correct graph showing all features
1- correct shape
1- shows correctly features from part(ii) & (iii)

End of Question 13

Question 14 (14 marks)

a) (i) Prove the following identity

$$(1 + \tan x)^2 = 2\tan x + \sec^2 x$$

1 - correct solns

$LHS = (1+\tan x)^2$

$= 1 + 2\tan x + \tan^2 x$

$= \underbrace{1+\tan^2 x} + 2\tan x$

$= \sec^2 x + 2\tan x$

$= RHS$

(ii) Hence find the area bounded by $y = (1 + \tan x)^2$ and the x-axis between $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ 3

since $(1+\tan x)^2 \geq 0$ for $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$

$\text{Area} = \int_{-\frac{\pi}{4}}^{\pi/4} (1+\tan x)^2 \, dx$

$= \int_{-\frac{\pi}{4}}^{\pi/4} 2\tan x + \sec^2 x \, dx$

$= \left[-2\ln|\cos x| + \tan x\right]_{-\frac{\pi}{4}}^{\pi/4}$

$= \left[\left(-2\ln\left|\frac{1}{\sqrt{2}}\right| + 1\right) - \left(-2\ln\left|\cos\frac{-\pi}{4}\right| + (-1)\right)\right]$

$= 2$

2 - correct solns.

1 - correctly integrates tan x

1 - correctly uses part (i)

1 - correctly evaluates the definite integral

Question 14 continued on the next page

b) given $y = 2\sin\left(2x - \frac{\pi}{3}\right)$

(i) State the amplitude and period. 2

amplitude = 2

period $T = \frac{2\pi}{2} = \pi$

2 - correct solns.

1 - correct amplitude

1 - correct period

(ii) Find exact values of all intercepts of $y = 2\sin\left(2x - \frac{\pi}{3}\right)$ with the axes for $0 \le x \le \pi$ 2

y-int: $x = 0$

$y = 2\sin\left(2(0) - \frac{\pi}{3}\right)$

$y = -2\frac{\sqrt{3}}{2} = -\sqrt{3}$

$\therefore (0, -\sqrt{3})$ y-int.

x-int: $y = 0$

$0 = 2\sin\left(2x - \frac{\pi}{3}\right)$

$2x - \frac{\pi}{3} = 0, \pi, 2\pi$

$\therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \cancel{\frac{7\pi}{6}} \rightarrow$ out of domain

$\therefore \left(\frac{\pi}{6}, 0\right) \left(\frac{2\pi}{3}, 0\right)$ are x-int.

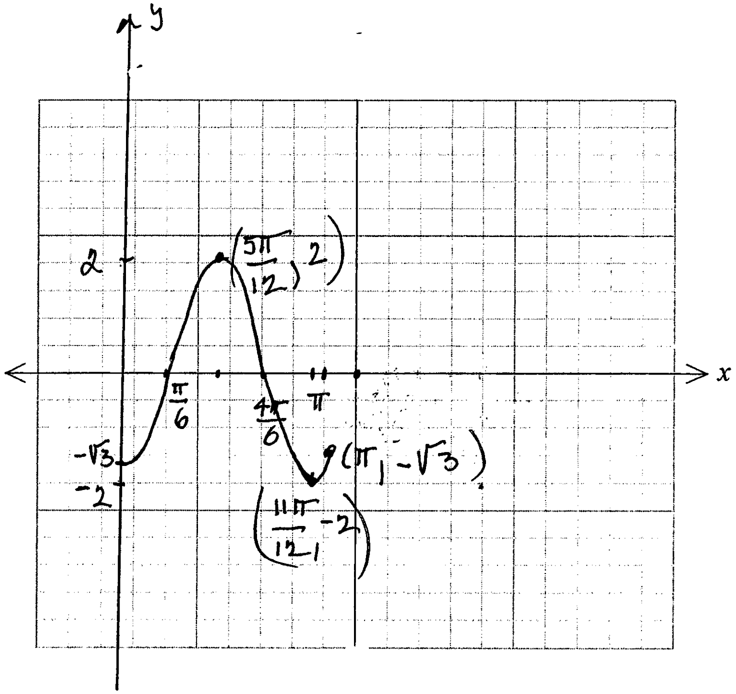
2 - correct solns.

1 - correct y-int.

Question 14 continued on the next page

(iii) Hence sketch the graph of $y = 2\sin\left(2x - \frac{\pi}{3}\right)$ for $0 \le x \le \pi$ **2**

showing all features from part (i) and (ii) and global maximum and minimum.



2	correct graph
1	correct shape and x, y-intercepts
1	showing correct Max / Min

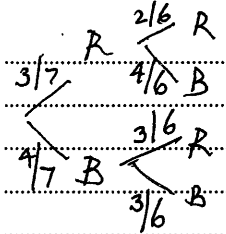
Question 14 continued on the next page

c) A bag contains three red balls and four black balls. Two balls are selected at random without replacement from the bag.

Let X be the number of black balls drawn.

(i) Fill in the following table and hence find exact value of $E(X)$. **2**

x	0	1	2
$P(X = x)$	RR - 1/7	RB or BR 4/7	BB 2/7



$P(0) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$

$P(1) = 2 \times \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$

$P(2) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

2	correct soln.
1	finds correct $E(X) = \mu$ from incorrect table of values

$E(X) = \sum x p(x) = 0 \times \frac{1}{7} + 1 \times \frac{4}{7} + 2 \times \frac{2}{7} = \boxed{\frac{8}{7}}$

(ii) Find $E(X^2)$ and hence find $\mathrm{Var}(X)$ and σ **2**

$E(X^2) = \sum x^2 p(x) = 0 \times \frac{1}{7} + 1^2 \times \frac{4}{7} + 2^2 \times \frac{2}{7}$

$\therefore E(X^2) = \frac{12}{7}$

$\mathrm{Var}(X) = E(X^2) - \mu^2$

$= \frac{12}{7} - \left(\frac{8}{7}\right)^2 = \frac{20}{49}$

2	correct soln.
1	correct $E(X^2)$
1	finds $\mathrm{Var}(X^2)$

$\mathrm{Var}(X) = \frac{20}{49} \quad \therefore \sigma = 0.638876\ldots$

$\therefore \sigma = 0.64396$

End of question 14

Question 15 (16 marks)

a) The velocity v of a particle in metres/seconds is given by the formula

$v = 5(1 + e^{-t})$, where t is time in seconds.

(i) Find the initial velocity of the particle. **1**

$t = 0$

$V = 5(1+e^{-0}) = 10$ m/s

1 - correct soln.

(ii) Is the particle ever stationary? Justify your answer. **1**

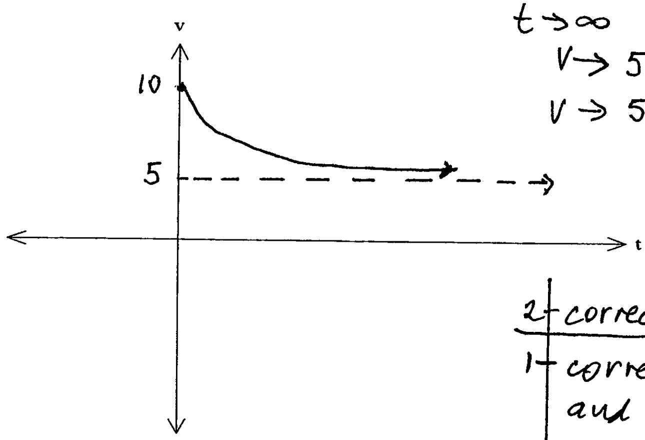
$V = 0$?

$0 = 5(1+e^{-t})$

∴ no soln. ∴ $V \neq 0$

1 - correct soln.

(iii) Sketch the graph of the velocity. **2**



$t \to \infty$

$V \to 5(1+0)$

$V \to 5$

2 - correct graph

1 - correct shape and y-intercept

Question 15 continued on the next page

(iv) Find the total distance travelled by the particle in the first 5 seconds. **2**

$d = \int_0^5 5(1+e^{-t})\,dt$

$= 5\left[t - e^{-t}\right]_0^5$

$= 5\left[5 - e^{-5} - (0 - e^{0})\right]$

$= 5(5 - e^{-5} + 1)$

$= 30 - 5e^{-5}$ (metres) exact

OR $d = 29.97$ (2d.p)

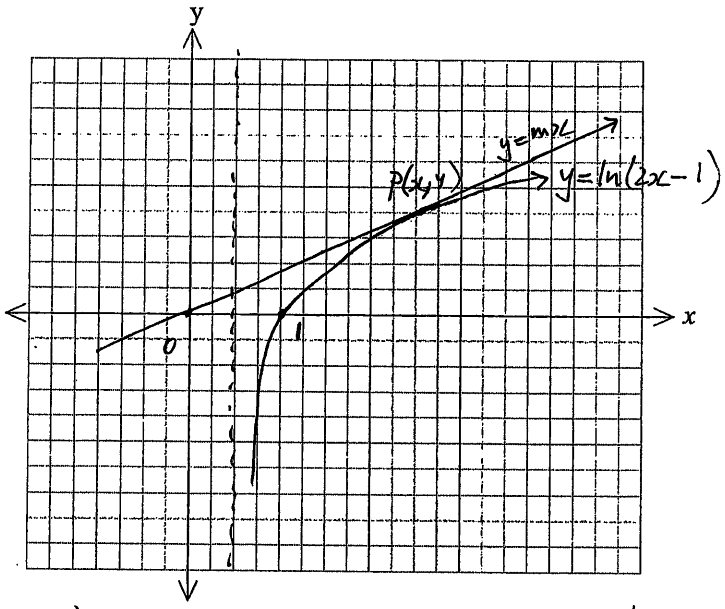
2 - correct solns.

1 - correct integration

Question 15 continued on the next page

b) The line $y = mx$ is a tangent to the curve $y = \ln(2x - 1)$ at a point P.

(i) Sketch the line and the curve on the same diagram, clearly indicating the point P. **2**



$y = \ln(2x-1)$

$2x - 1 > 0$ $\quad$ $2x - 1 = 1$

$x > \frac{1}{2}$ $\quad$ $x = 1$ (x-int)

$y = mx \rightarrow$ passing through $(0,0)$

2-	correct graphs
1-	correct log. graph
1-	correct graph of the line passing through (0,0) & clearly indicating point of contact P(x,y).

Question 15 continued on the next page

(ii) Show that the coordinates of P are $\left(\frac{2+m}{2m}, \frac{2+m}{2}\right)$. **2**

$y = \ln(2x-1)$

$\therefore y' = \frac{2}{2x-1}$

if $y = mx$ is a tangent to $y = \ln(2x-1)$

$\therefore m = \frac{2}{2x-1}$ (at point $P(x,y)$)

$2x - 1 = \frac{2}{m}$

$2x = \frac{2}{m} + 1 \therefore x_p = \frac{1}{m} + \frac{1}{2} = \frac{2+m}{2m}$

sub. $x_{coord.}$ into $y = mx$

$\therefore y_p = \cancel{m} \times \frac{2+m}{2\cancel{m}} = \frac{2+m}{2}$

2-	correct solns.
1-	equates $y' = m$ and solves for x
1-	substitutes x-value into one of the eqns.

(iii) Hence show that $2 + m = \ln\left(\frac{4}{m^2}\right)$. **2**

since $P\left(\frac{2+m}{2m}, \frac{2+m}{2}\right)$ (part ii)

and P lies on $y = \ln(2x-1)$

$\therefore$ coord. of P satisfy equation

$\therefore$ sub. in $\therefore y = \ln(2x-1)$

$$\frac{2+m}{2} = \ln\left(\cancel{2}\left(\frac{2+m}{\cancel{2}m}\right) - 1\right)$$

$$\frac{2+m}{2} = \ln\left(\frac{2+m}{m} - 1\right) = \ln\left(\frac{2}{m} + \frac{m}{m} - 1\right)$$

$$\frac{2+m}{2} = \ln\left(\frac{2}{m} + 1 - 1\right) = \ln\left(\frac{2}{m}\right)$$

$$\therefore 2+m = 2\ln\left(\frac{2}{m}\right) = \ln\left(\frac{2}{m}\right)^2 = \ln\frac{4}{m^2}$$

2-	correct solns.
1-	sub. coordinates of P into $y = \ln(2x-1)$ and attempts to solve it

Question 15 continued on the next page

$\therefore 2+m = \ln\left(\frac{4}{m^2}\right) \therefore$ shown

c) Given the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the cumulative distribution function $F(x)$. **2**

$$F(x) = \int_0^x 2e^{-2x}\,dx$$

$$F(x) = 2x - \frac{1}{2}\left[e^{-2x}\right]_0^x$$

$$F(x) = -\left[e^{-2x} - e^0\right]$$

$$\therefore F(x) = -e^{-2x} + 1$$

2 - correct solns.
1 - correctly integrates f(x)

(ii) Hence find the median. **2**

let m be median

$$\therefore F(m) = \frac{1}{2}$$

$$\frac{1}{2} = -e^{-2x} + 1 \quad \boxed{x = m}$$

$$e^{-2x} = 1 - \frac{1}{2}$$

$$\ln\left(e^{-2x}\right) = \ln\frac{1}{2}$$

$$-2x = \ln\frac{1}{2}$$

$$x = -\frac{1}{2}\ln\frac{1}{2} \quad (\text{median})$$

$$\therefore \text{median} = -\frac{1}{2}\ln\frac{1}{2} \text{ or } \ln\sqrt{2} \text{ or } 0.347$$

$$\text{OR } -\frac{1}{2}(-\ln 2) = \frac{1}{2}\ln 2$$

2 - correct solns.
1 - equates $F(x) = \frac{1}{2}$ and shows significant progress to find the value of the median.

End of question 15

Question 16 (14 marks)

a) Michelle borrows ~~\$50 000~~ \$ 450 000 to be repaid by regular monthly repayments of \$$P$ over a period of 25 years at 6% per annum reducible monthly. Interest is calculated and charged just before each repayment.

Let A_n be the amount owing after n –repayments.

(i) Show that the expression for the amount owing after two repayments is **1**

$$A_2 = 450\ 000(1.005)^2 - P(1.005) - P$$

$$A_1 = 450000\left(1 + \frac{6 \div 12}{100}\right) - P$$

$$= 450000(1.005) - P$$

$$A_2 = A_1(1.005) - P$$

$$= 450\,000 \times 1.005^2 - P(1.005) - P$$

$\therefore$ shown

1 - correct solns.

(ii) Show that the amount owing after n –repayments is **2**

$$A_n = 450\ 000(1.005)^n - P\frac{(1.005)^n - 1}{0.005}$$

following pattern from (i)

$$A_n = 450000(1.005)^n - P(1.005)^{n-1} - \ldots - P$$

$$\therefore A_n = 450000(1.005)^n - P\underbrace{(1.005^{n-1} + \ldots + 1)}_{G.P\ a=1\ r=1.005\ n=n}$$

$$\therefore A_n = 45000(1.005)^n - P\frac{r^n - 1}{r - 1}$$

$$A_n = 450000(1.005)^n - P\frac{(1.005)^n - 1}{1.005 - 1}$$

$$\therefore A_n = 450000(1.005)^n - P\frac{(1.005)^n - 1}{0.005}$$

$\therefore$ shown

2 - correct soln.
1 - correctly applies pattern from (i)
1 - correctly applies the Sum of GP formula

Question 16 continued on the next page

(iii) Calculate the amount of each repayments P. 2

after 25 years = $25 \times 12 = 300 = n$

$\therefore A_{300} = 0 = 450\,000(1.005)^{300} - P\frac{1.005^{300} - 1}{0.005}$

$P\frac{1.005^{300} - 1}{0.005} = 450000\,(1.005)^{300}$

$P = \frac{450000\,(1.005)^{300}}{1.005^{300} - 1} \times 0.005$

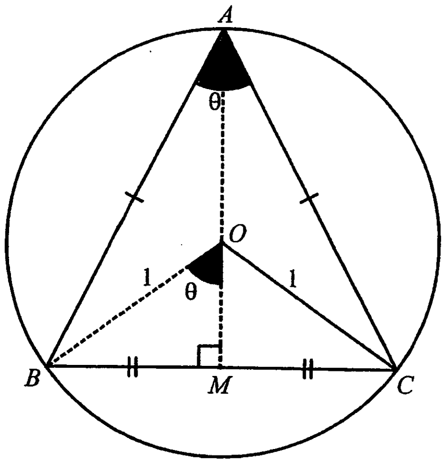
$P = \$2899.3563\ldots$

$\therefore \boxed{P = \$2899.36}$

2 – correct soln.
1 – using part (i)
Correctly equates $A_{300} = 0$ and attempts to solve it for P

Question 16 continued on the next page

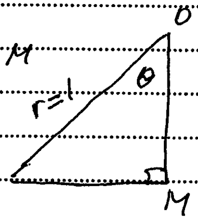
b) An isosceles triangle ΔABC is inscribed within a unit circle centred at O, as shown in the diagram below. Let M be the midpoint of BC, $\angle BAC = \theta$ and $\angle BOM = \theta$.



(i) Show that the area of ΔABC is $A = \sin\theta(1 + \cos\theta)$. 2

$\text{Area}_{\Delta ABC} = \frac{1}{2} BC \times AM \quad (AM \perp BC)$
base height

from ΔOBM



$\sin\theta = \frac{BM}{1} = BM$

$\cos\theta = OM$

$AM = AO + OM = 1 + \cos\theta$

$BC = 2 \times BM = 2 \times \sin\theta$

$\therefore \text{Area} = \frac{1}{2} \times BC \times AM = \frac{1}{\cancel{2}} \times \cancel{2}\sin\theta \times (1 + \cos\theta)$

$\therefore A = \sin\theta(1 + \cos\theta)$ ∴ shown

2 – correct soln.
1 – finds BC or AM in terms of θ

Question 16 continued on the next page

(iii) Hence prove that the area of the isosceles triangle is maximum when it is equilateral. 3

$A = \sin\theta(1 + \cos\theta)$

$A' = \cos\theta(1 + \cos\theta) + \sin\theta(0 - \sin\theta)$

$\therefore A' = \cos\theta + \cos^2\theta - \sin^2\theta$

$A' = 0$

$0 = \cos\theta + \cos^2\theta - \sin^2\theta$

$\theta = \cos\theta + \cos^2\theta - (1 - \cos^2\theta)$

$0 = 2\cos^2\theta + \cos\theta - 1$

$\theta = (2\cos\theta - 1)(\cos\theta + 1)$

$\cos\theta = \frac{1}{2}$ $\quad\quad \cos\theta = -1$

$\theta = \boxed{180^\circ}$

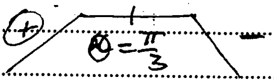
$\theta = \frac{\pi}{3}, \boxed{\frac{5\pi}{3}}$ - impossible in triangle

θ - acute

$\therefore \theta = \frac{\pi}{3}$ (the only solution)

Nature

θ	1	$\frac{\pi}{3}$	1.1
A'	0.124	0	-0.135



$\therefore$ at $\theta = \frac{\pi}{3}$ Area is Maximum

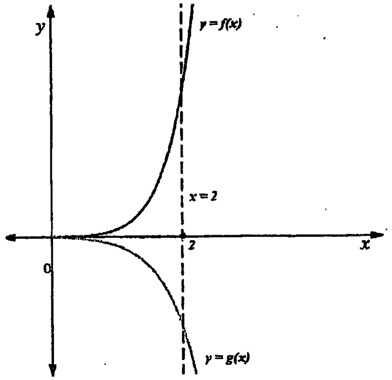
but if $\theta = \frac{\pi}{3}$ and given Δ ABC is isosceles $\}$ ∴ Δ ABC is equilateral

3 – correct soln.
2 – finds stationary points correctly
1 – differentiates Area formula correctly & attempts to soln $A' = 0$
1 – determines the nature of st. pt. and concludes for Δ to be equilateral

Question 16 continued on the next page

---

c) The graph of $f(x) = x^2e^{kx}$ *and* $g(x) = -\frac{2xe^{kx}}{k}$ and the line $x = 2$ is drawn below. $f(x) = g(x)$ at only one point, that is at $(0, 0)$.



Let A be the area of the region bounded by the curve $y = f(x)$, $y = g(x)$ and the line $x = 2$.

(i) Write down a definite integral that gives the value of A. 1

$A = \int_0^2 f(x) - g(x)\, dx$

OR

$\therefore A = \int_0^2 x^2e^{kx} - \left(\frac{-2xe^{kx}}{k}\right) dx$

1 – correct soln.

(ii) The function $f(x)$ from part (i) is given by $f(x) = x^2e^{kx}$ where k is a positive constant. Show that $f'(x) = xe^{kx}(kx + 2)$ 1

$f(x) = x^2e^{kx}$ $\quad k > 0$

$f'(x) = 2xe^{kx} + x^2 \cdot ke^{kx}$

$\therefore f'(x) = xe^{kx}(2 + kx)$

∴ shown

1 – correct soln.

Question 16 continued on the next page

(iii) Using the results of part (i) and (ii), or otherwise, find the value of k such that $A = \frac{16}{k}$. **2**

from (i) $A = \int_0^2 x^2 e^{kx} - \frac{-2xe^{kx}}{k}\,dx$

$\therefore A = \int_0^2 x^2 e^{kx} + \frac{2xe^{kx}}{k}\,dx$

$= \frac{1}{k}\int_0^2 kx^2 e^{kx} + 2xe^{kx}\,dx$

$= \frac{1}{k}\int_0^2 \underbrace{xe^{kx}(kx+2)}_{\text{but } = f'(x)}\,dx$

$\therefore A = \frac{1}{k}\Big[\underbrace{x^2 e^{kx}}_{f(x)}\Big]_0^2$

2 - correct solns.

1 - simplifies integrand $A = \frac{1}{k}\int_0^2 f'(x)\,dx$

if $A = \frac{16}{k}$

$\therefore \frac{16}{\cancel{k}} = \frac{1}{\cancel{k}}\Big[x^2 e^{kx}\Big]_0^2$

$16 = [2^2 e^{2k} - 0]$

$16 = 4e^{2k}$

$4 = e^{2k}$

$\ln 4 = 2k$

End of Exam

$\underline{\underline{k = \tfrac{1}{2}\ln 4}}$ or $k = \ln\sqrt{4} = \underline{\underline{\ln 2}}$