# Caringbah High School

## Year 12 2017
## **Mathematics**
## HSC Course
## Assessment Task 4 (Trial HSC)

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

**Total marks – 100**

**Section I** **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

**Section II** **90 marks**
Attempt Questions 11-16
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

**Name:** ______________________________ **Class:** ____________

| Marker's Use Only | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Section I** | **Section II** | | | | | | **Total** | |
| **Q 1-10** | **Q11** | **Q12** | **Q13** | **Q14** | **Q15** | **Q16** | | |
| /10 | /15 | /15 | /15 | /15 | /15 | /15 | /100 | % |

## Section I

**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–5

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**1.** Factorise $8x^6 - 27$

(A) $(2x^2 - 3)(4x^4 - 6x^2 + 9)$

(B) $(2x^2 + 3)(4x^4 - 6x^2 + 9)$

(C) $(2x^2 - 3)(4x^4 + 6x^2 + 9)$

(D) $(2x^2 + 3)(4x^4 + 6x^2 + 9)$

**2.** The quadratic equation $3x^2 + 5x - 2 = 0$ has roots $\alpha$ and $\beta$.
What is the value of $\alpha^2\beta + \alpha\beta^2$?

(A) $-\frac{10}{9}$

(B) $-\frac{9}{10}$

(C) $\frac{9}{10}$

(D) $\frac{10}{9}$

**3.** Which of the following is true about $y = 4\sin\frac{x}{2}$?

(A) Amplitude = 4 and period = $\frac{1}{2}$

(B) Amplitude = 4 and period = $2\pi$

(C) Amplitude = $\frac{1}{2}$ and period = 4

(D) Amplitude = 4 and period = $4\pi$

**4.** The equation $x^2 + y^2 + 6y = 7$ describes a circle with:

(A) Centre (0,3) and radius 4 units

(B) Centre (0, –9) and radius 6 units

(C) Centre (0, –3) and radius 4 units

(D) Centre (0,3) and radius 6 units

**5.** The terms $x,\ 1-x,\ x^2-2$ form an arithmetic series.
Which of the following are possible values for the common difference?

(A) 1 and 9

(B) –1 and 9

(C) 1 and –9

(D) –1 and –9

**6.** What is the derivative of $\dfrac{e^{-x}}{x}$?

(A) $\dfrac{-xe^{-x} - e^{-x}}{x^2}$

(B) $\dfrac{-xe^{-x} + e^{-x}}{x^2}$

(C) $\dfrac{e^{-x} + xe^{-x}}{x^2}$

(D) $\dfrac{e^{-x} - xe^{-x}}{x^2}$

**7.** What is the value of $\int_{-3}^{2} |x+1|\, dx$ ?

(A) $\dfrac{5}{2}$

(B) $\dfrac{11}{2}$

(C) $\dfrac{13}{2}$

(D) $\dfrac{17}{2}$

**8.** What is the value of $\lim_{x \to 10} \dfrac{x^2 - 100}{x - 10}$?

(A) Undefined

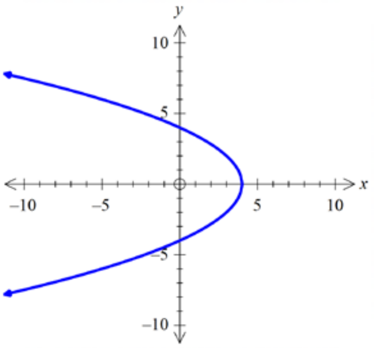
(B) 0

(C) 8

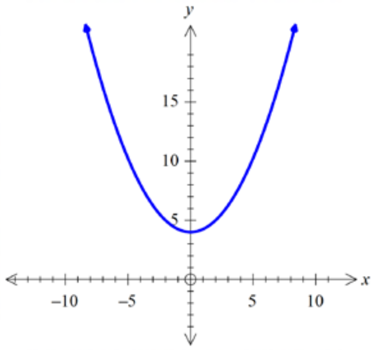
(D) 20

**9.** Which of the following graphs represents the parabola $y^2 = -4x + 16$?

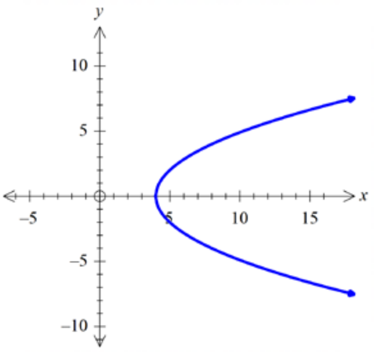
(A)



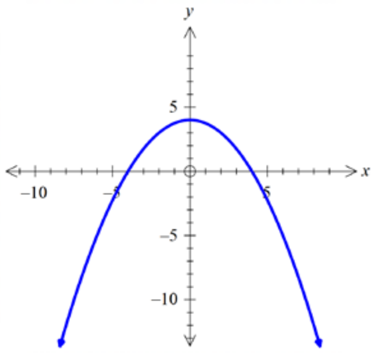

(B)




(C)




(D)




**10.** If $y = 10^x$, which of the following is true?

(A) $x = \sqrt[10]{y}$

(B) $x = \log_e y$

(C) $\dfrac{dy}{dx} = 10^x$

(D) $\dfrac{dy}{dx} = (\log_e 10) \times 10^x$

**End of Section I**

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours 45 minutes for this section**

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

**(a)** Express $\dfrac{1}{\sqrt{5}+2}$ in the form $a+b\sqrt{5}$ where $a$ and $b$ are integers. **2**

**(b)** Consider the table of values for $f(x)$ below. **2**

| $x$ | 3 | 3.25 | 3.5 | 3.75 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | 1.0 | 0.8 | 0.65 | 0.55 | 0.5 |

Find an approximation for the definite integral $\int_3^4 f(x)\ dx$ using Simpson's Rule and the above table. Give your answer correct to 3 significant figures.

**(c)** Find $\int_{\frac{\pi}{2}}^{\pi} \sec^2 \dfrac{x}{3}\, dx$, leaving your answer in exact form. **3**

**(d)** Differentiate $\dfrac{\sin x}{e^x}$ with respect to $x$, leaving your answer in simplified form. **2**

**(e)** Find the area bounded by the curve $y = x^3 - 4x$ and the $x$-axis. **3**

**(f)** Solve $\log_2 x + \log_2(x+7) = 3$ for $x > 0$. **3**

**End of Question 11**

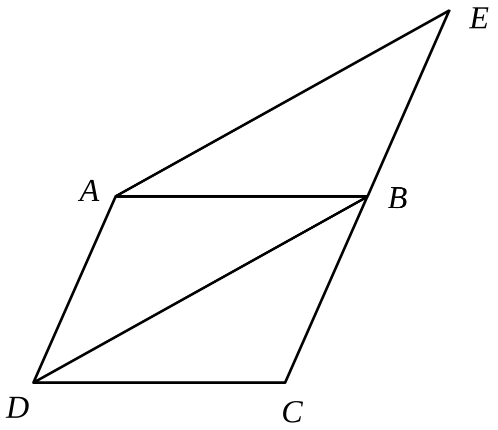
**Question 12** (15 marks) Use a SEPARATE writing booklet.

**(a)** Solve $|2x-5|=3$. **2**

**(b)** If $\tan\theta = \dfrac{7}{9}$ and $\cos\theta < 0$, find the exact value of $\sin\theta$. **2**

**(c)** For what values of $x$ is the function $f(x) = -x^2(x+3)$ decreasing? **2**

**(d)** *ABCD* is a rhombus. *CB* is produced to *E* such that *CB* = *BE*.




NOT TO SCALE

Copy the diagram into your answer booklet.

**(i)** Prove that $\Delta ABE \equiv \Delta DCB$. **3**

**(ii)** Hence explain why *AE*//*DB*. **1**

**(iii)** What type of quadrilateral is *AEBD*? Justify your answer. **1**

**(e)** Given that $\log_a m = 1.75$ and $\log_a n = 2.25$, find the value of:

**(i)** $\log_a mn$ **1**

**(ii)** $\log_a \dfrac{n}{m}$ **1**

**(iii)** $\sqrt[5]{mn^2}$ in terms of $a$ **2**

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

**(a)** The tangents of the parabola $y = x^2 + ax - 3$ at $x = 0$ and $x = 1$ are known to be perpendicular. Find the value of $a$. **2**

**(b)** Given that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, find $\int \tan(2x)\, dx$. **2**

**(c)** Consider the graph of $y = x^3 + 3x^2 - 9x$.

**(i)** Find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **2**

**(ii)** Find the coordinates of any stationary points and determine their nature. **3**

**(iii)** Show that there is a point of inflexion at $(-1, 11)$. **1**

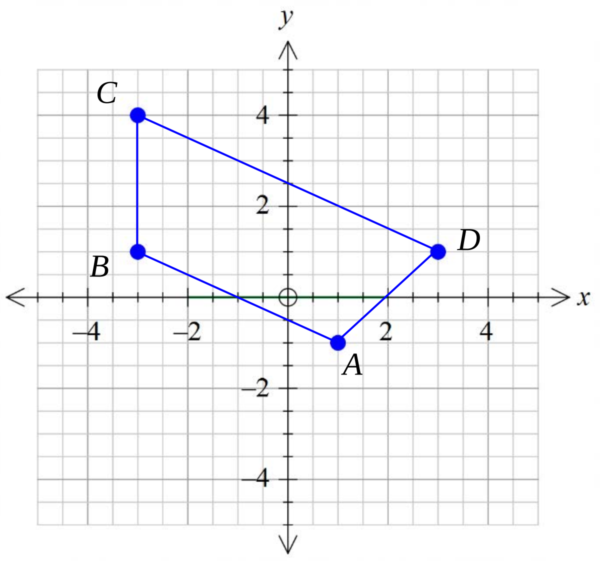
**(iv)** Sketch the graph of $y = x^3 + 3x^2 - 9x$, labelling the above features and the $y$-intercept. **2**

**(d)** Given that $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^2} + 2e^{2x}$ and that $\dfrac{dy}{dx} = e^2$ at $\left(1, \dfrac{e^2}{2}\right)$, find the expression for $y$ in terms of $x$. **3**

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

**(a)** $A(1,-1)$, $B(-3,1)$, $C(-3,4)$ and $D(3,1)$ are points on the Cartesian plane with $AB \| CD$.




**(i)** Find the distances $BA$ and $CD$. **2**

**(ii)** Show that the equation of $CD$ is $x+2y-5=0$. **2**

**(iii)** Find the perpendicular distance of $A$ to $CD$. **2**

**(iv)** Hence, or otherwise, find the area of the quadrilateral $ABCD$. **1**

**(b)** $A$ and $B$ are points on a circle with centre $O$. **3**

The area of the sector $AOB$ is $3\pi$ cm$^2$ while the length of arc $AB$ is $\dfrac{3\pi}{2}$ cm.

Find the value of $\theta$ and $r$.

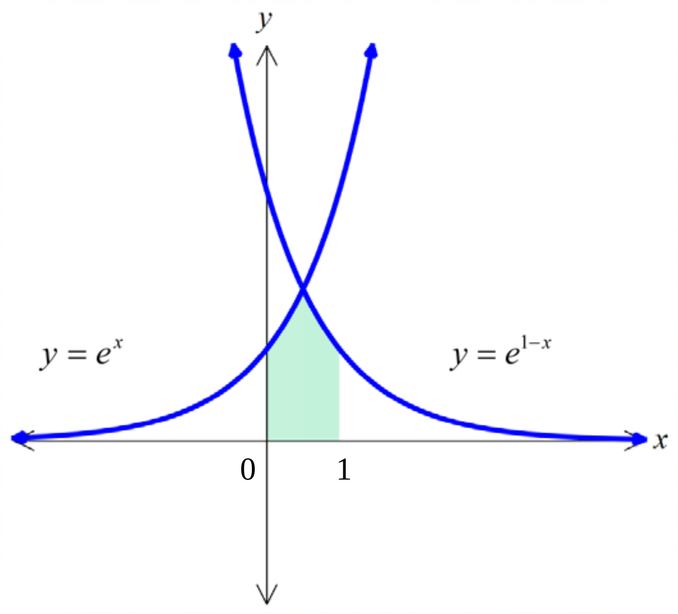
**(c)** **(i)** Sketch the curve $y = \log_e x$ and shade the region bounded by the curve, the $y$-axis, $y = 1$ and $y = 2$. **2**

**(ii)** Find, in exact form, the volume of the solid of revolution formed when the above region is rotated about the $y$-axis. **3**

**End of Question 14**

**Question 15** (15 marks) Use a SEPARATE writing booklet.

**(a)** Consider the curves $y = e^x$ and $y = e^{1-x}$ below.




**(i)** Find the $x$-coordinate of point of intersection between the two curves. **1**

**(ii)** Find the exact value of the shaded area bounded by the two curves, the coordinate axes and $x = 1$. **3**

**(b)** The acceleration of a particle travelling in a straight line is given by $\frac{d^2x}{dt^2} = 8 - 6t$.

The particle is initially at the origin and travelling at 5 m/s to the right.

**(i)** Find the equations for the velocity and displacement of the particle at any time $t$ seconds. **2**

**(ii)** At what time does the particle return to the origin? Find the velocity of the particle at that time. **3**

**Question 15 continues on page 10**

**(c)** A car company offers a loan of $20 000 to purchase a new car for which it charges interest at 1% per month. As a special deal, the company does not charge interest for the first 6 months; however, the monthly repayments start at the end of the first month. Wayne takes out a loan and agrees to repay the loan over 5 years by making 60 equal monthly repayments of $*M*.

Let $A_n$ be the amount owing at the end of the *n*th month.

**(i)** Find an expression for $A_4$. **1**

**(ii)** Show that $A_8 = (20000 - 6M)(1.01)^2 - M(1 + 1.01)$ **1**

**(iii)** Find an expression for $A_{60}$. **1**

**(iv)** Find the value of $M$. **3**

**End of Question 15**

**Question 16** (15 marks) Use a SEPARATE writing booklet.

**(a)** **(i)** Differentiate $x^2 \log_e x$ **1**

**(ii)** Hence, or otherwise, find $\int x \log_e x \; dx$ **2**

**(b)** A bacteria culture of $N$ bacteria is increasing exponentially so that $\dfrac{dN}{dt} = kN$ where $t$ is time in minutes and $k$ is a constant.
The number of bacteria increases from 100 to 400 in 2 minutes.

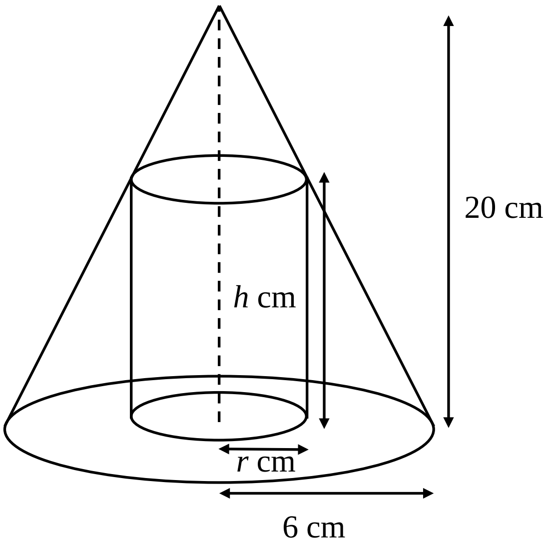
**(i)** Show that $N = Ae^{kt}$ is a solution to the above differential equation, where $A$ is a constant. **1**

**(ii)** Find the exact value of the constant $k$ in simplest form. **2**

**(iii)** Find the number of bacteria present after 10 minutes. **1**

**(iv)** After how many minutes and seconds will there be 1000 bacteria? **2**

**(c)** A cylinder of radius $r$ cm and height $h$ cm is inscribed in a cone with base radius 6 cm and height 20 cm, as shown in the diagram below.




NOT TO SCALE

**(i)** Show that the volume $V$ of the cylinder is given by $V = \dfrac{10\pi r^2 (6-r)}{3}$. **3**

**(ii)** Hence, find the values of $r$ and $h$ such that the cylinder has a maximum volume. **3**

**End of Paper**

# MATHEMATICS (2U) TRIAL 2017

1. $8x^6 - 27 = (2x^2)^3 - 3^3$

$= (2x^2 - 3)(4x^4 + 6x^2 + 9)$ C

2. $3x^2 + 5x - 2 = 0$.

$\begin{cases} \alpha + \beta = -5/3 \\ \alpha\beta = -2/3 \end{cases}$

$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta)$

$= \frac{-5}{3} \times \frac{-2}{3}$

$= \frac{10}{9}$ D

3. $y = 4\sin(x/2)$

$A = 4$

$T = \frac{2\pi}{1/2} = 4\pi$ D

4. $x^2 + y^2 + 6y = 7$

$x^2 + (y+3)^2 = 16$

$\begin{cases} C(0, -3) \\ r = 4 \end{cases}$ C

5. AP: $d = T_2 - T_1 = T_3 - T_2$

$1 - x - x = x^2 - 2 - (1 - x)$

$1 - 2x = x^2 - 2 - 1 + x$

$0 = x^2 + 3x - 4$

$(x+4)(x-1) = 0$

$x = -4, 1.$

i.e. $\begin{cases} -4, 5, 13 \\ 1, 0, -1 \end{cases}$ OR

$\therefore d = 9, -1.$ B

6. $\dfrac{d}{dx}\left(\dfrac{e^{-x}}{x}\right) = \dfrac{(e^{-x})' \cdot x - e^{-x} \cdot (x)'}{x^2}$

$= \dfrac{-e^{-x} \cdot x - e^{-x} \cdot 1}{x^2}$

$= \dfrac{-xe^{-x} - e^{-x}}{x^2}$

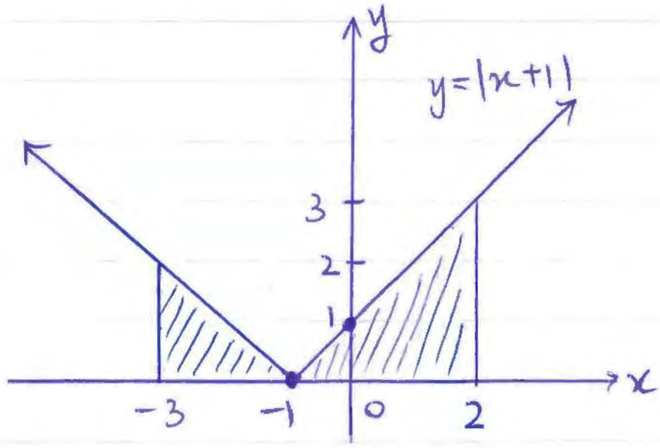
(A)

7. $\int_{-3}^{2} |x+1|\, dx$

$= \frac{1}{2}(2)(2) + \frac{1}{2}(3)(3)$

$= \dfrac{13}{2}$

(C)




8. $\lim_{x \to 10} \dfrac{x^2-100}{x-10} = \lim_{x \to 10} \dfrac{\cancel{(x-10)}(x+10)}{\cancel{x-10}}$

$= \lim_{x \to 10} x+10$

$= 10+10$

$= 20$

(D)

9. $y^2 = -4x + 16.$

(A)

10. $y = 10^x$

- $x = \log_{10} y$
- $y' = 10^x . \ln 10.$

(D)

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| C | D | D | C | B | A | C | D | A | D |

Question 11.

(a) $\dfrac{1}{\sqrt{5}+2} \times \dfrac{\sqrt{5}-2}{\sqrt{5}-2} = \dfrac{\sqrt{5}-2}{(\sqrt{5})^2-2^2}$

$= \dfrac{\sqrt{5}-2}{5-4}$

$= \dfrac{\sqrt{5}-2}{1}$

$= \sqrt{5}-2$

$= -2+\sqrt{5}$

(b) $\int_3^4 f(x)\,dx \approx \dfrac{0.25}{3}\left[1.0 + 4(0.8+0.55) + 2(0.65) + 0.5\right]$

$= 0.6833\ldots$

$= 0.683$ (3 sig fig)

(c) $\int_{\pi/2}^{\pi} \sec^2\left(\frac{x}{3}\right) dx = \dfrac{1}{1/3}\left[\tan\frac{x}{3}\right]_{\pi/2}^{\pi}$

$= 3\left[\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right]$

$= 3(\sqrt{3} - 1/\sqrt{3})$

$= 3 \cdot \dfrac{3-1}{\sqrt{3}}$

$= \dfrac{6}{\sqrt{3}}$

(d) $\dfrac{d}{dx}\left(\dfrac{\sin x}{e^x}\right) = \dfrac{\cos x . e^x - \sin x . e^x}{(e^x)^2}$

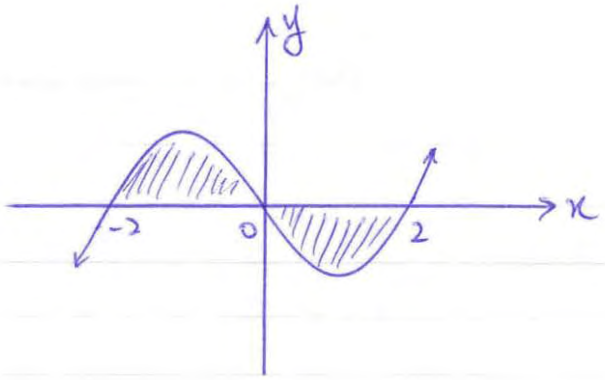
$= \dfrac{e^x(\cos x - \sin x)}{(e^x)^2}$

$= \dfrac{\cos x - \sin x}{e^x}$

(e) $y = x^3 - 4x$.

$= x(x^2-4)$

$= x(x+2)(x-2)$




$$\text{Area} = 2\left|\int_0^2 x^3 - 4x \, dx\right|$$

$$= 2\left|\left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_0^2\right|$$

$$= 2\left|\left(\frac{2^4}{4} - 2(2)^2 - 0\right)\right|$$

$$= 2|-4|$$

$$= 8 \text{ units}^2$$

(f) $\log_2 x + \log_2 (x+7) = 3$ $\qquad x > 0$.

$\log_2 (x(x+7)) = 3$

$x(x+7) = 2^3$

$x^2 + 7x - 8 = 0$

$(x-1)(x+8) = 0$

$x = 1, -8$

But $x > 0$, $\therefore x = 1$ only.

## Question 12

(a) $|2x-5| = 3$

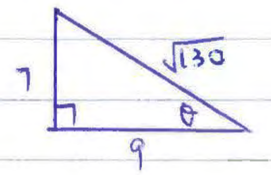
$2x-5=3 \qquad 2x-5=-3$

$2x=8 \qquad 2x=2$

$x=4 \qquad x=1$

$\therefore x=1, 4.$

(b) $\tan\theta = 7/9 > 0, \quad \cos\theta < 0.$



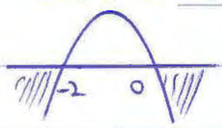

$\sin\theta = \dfrac{-7}{\sqrt{130}}$

(c) $f(x) = -x^2(x+3)$

$= -x^3 - 3x^2$

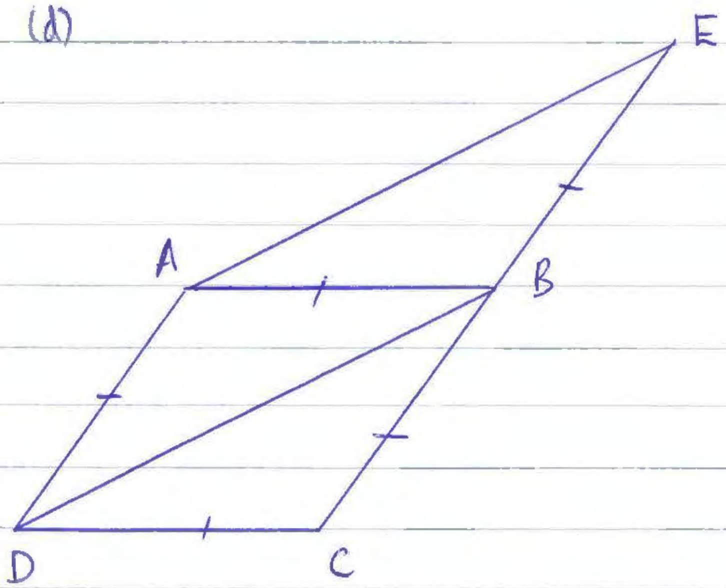
$f'(x) = -3x^2 - 6x$

$= -3x(x+2) < 0$ for decreasing.




$\therefore x < -2, \ x > 0.$

(d)



(i) In $\triangle ABE$ & $\triangle DCB$,

$AB = DC$ (opp. sides of rhombus are equal)

$AB \parallel DC$ (opp. sides of rhombus are parallel)

$\angle ABE = \angle DCB$ (corres. $\angle$s equal, $AB \parallel DC$)

$CB = BE$ (given)

$\therefore \triangle ABE \equiv \triangle DCB$ (SAS)

(ii) $\angle AEB = \angle DBC$ (corres. $\angle$s in cong. $\triangle$s are equal)

$\therefore AE \parallel DB$ (corres. $\angle$s equal on parallel lines)

(iii) $AE = DC$ (corres. sides in cong. $\triangle$s are equal).

$\therefore AEBD$ is a parallelogram (opp. sides are equal & parallel

(e) $\log_a m = 1.75$, $\log_a n = 2.25$.

(i) $\log_a(mn) = \log_a m + \log_a n$

$= 1.75 + 2.25$

$= 4$

(ii) $\log_a\left(\frac{n}{m}\right) = \log_a n - \log_a m$

$= 2.25 - 1.75$

$= 0.5$

(iii) $\sqrt[5]{mn^2} = \left[a^{1.75} \times (a^{2.25})^2\right]^{1/5}$

$= \left[a^{1.75} \times a^{4.5}\right]^{1/5}$

$= \left[a^{6.25}\right]^{1/5}$

$= a^{1.25}$

$= a^{5/4}$

$\log_a m = 1.75$

$m = a^{1.75}$

$\log_a n = 2.25$

$n = a^{2.25}$

# Question 13.

(a) $y = x^2 + ax - 3$

$y' = 2x + a$

At $x = 0$, $m_1 = 2(0) + a = a$

At $x = 1$, $m_2 = 2(1) + a = 2 + a$

$m_1 \times m_2 = -1$

$a \times (2 + a) = -1$

$a^2 + 2a + 1 = 0$

$(a+1)^2 = 0$

$\therefore a = -1$,

(b) $$\int \tan(2x)\,dx = \int \frac{\sin 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{-2\sin 2x}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\int \frac{(\cos 2x)'}{\cos 2x}\,dx$$

$$= \frac{-1}{2}\ln|\cos 2x| + c.$$

(c) $y = x^3 + 3x^2 - 9x$

(i) $$\frac{dy}{dx} = 3x^2 + 6x - 9$$

$$\frac{d^2y}{dx^2} = 6x + 6.$$

(ii) SP when $\frac{dy}{dx} = 0$

$3(x^2 + 2x - 3) = 0$

$3(x-1)(x+3) = 0$

$x = 1, -3$

At $x=1$, $\quad y = 1^3 + 3(1)^2 - 9(1) = -5$

$$\frac{d^2y}{dx^2} = 6(1) + 6 = 12 > 0$$

$\therefore$ min SP at $(1, -5)$

At $x=-3$, $\quad y = (-3)^3 - 3(-3)^2 - 9(-3) = 27$

$$\frac{d^2y}{dx^2} = 6(-3) + 6 = -12 < 0$$

$\therefore$ max SP at $(-3, 27)$.

(iii) IP when $\frac{d^2y}{dx^2} = 0$

$$6x + 6 = 0$$
$$6x = -6$$
$$x = -1$$

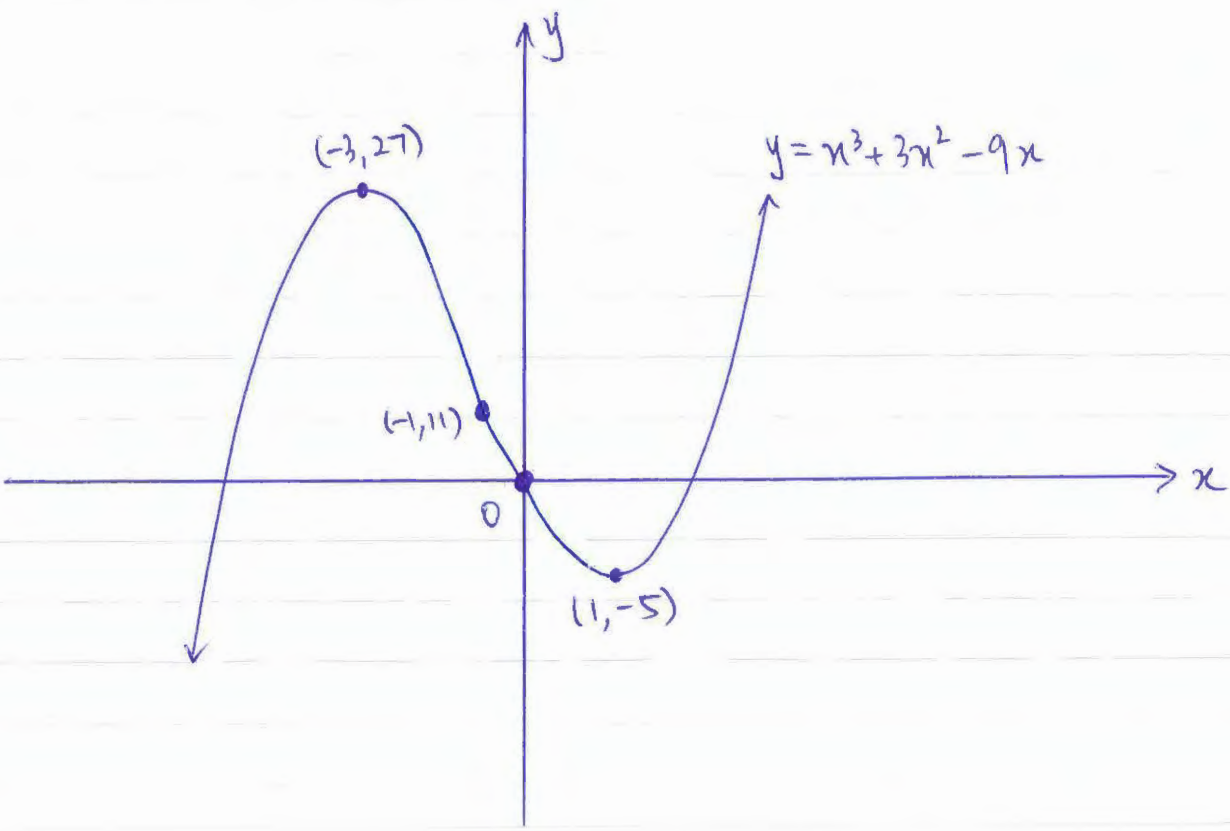
Check for concavity change

| $x$ | $-1.1$ | $-1$ | $-0.9$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-0.6$ | $0$ | $0.6$ |

At $x=-1$, $\quad y = (-1)^3 + 3(-1)^2 - 9(-1) = 11$

$\therefore$ IP at $(-1, 11)$.

(iv)



(d) $\frac{d^2y}{dx^2} = \frac{2}{x^2} + 2e^{2x}$

$= 2x^{-2} + 2e^{2x}$

$\frac{dy}{dx} = \int 2x^{-2} + 2e^{2x}\,dx$

$= \frac{2x^{-1}}{-1} + \frac{2e^{2x}}{2} + C_1$

$= -2x^{-1} + e^{2x} + C_1$

Sub $x=1$, $\frac{dy}{dx} = e^2$

$e^2 = -2(1)^{-1} + e^{2(1)} + C_1$

$= -2 + e^2 + C_1$

$C_1 = 2$

$\frac{dy}{dx} = -2x^{-1} + e^{2x} + 2$

$y = \int -2x^{-1} + e^{2x} + 2\,dx$

$= -2\ln x + \frac{e^{2x}}{2} + 2x + C_2$

Sub $x=1$, $y = \frac{e^2}{2}$

$\frac{e^2}{2} = -2\ln(1) + \frac{e^{2(1)}}{2} + 2(1) + C_2$

$0 = 0 + 2 + C_2$

$C_2 = -2$

$\therefore\ y = -2\ln x + \frac{e^{2x}}{2} + 2x - 2.$

Question 14.

(a) (i) $BA = \sqrt{(-3-1)^2 + (1--1)^2}$

$= \sqrt{20}$

$= 2\sqrt{5}$

$CD = \sqrt{(3--3)^2 + (1-4)^2}$

$= \sqrt{45}$

$= 3\sqrt{5}$

A(1, −1)
B(−3, 1)
C(−3, 4)
D(3, 1).

(ii) $m_{CD} = \frac{1-4}{3--3} = \frac{-3}{6} = \frac{-1}{2}$

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{-1}{2}(x-3)$

$2y - 2 = -x + 3$

$\therefore CD: \quad x + 2y - 5 = 0.$

(iii) $d_\perp = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \frac{|1(1) + 2(-1) - 5|}{\sqrt{1^2 + 2^2}}$

$= \frac{|-6|}{\sqrt{5}}$

$= \frac{6}{\sqrt{5}}$

(iv) ABCD is a trapezium (AB ∥ CD)

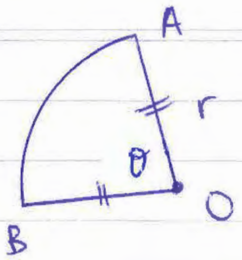
$\text{Area} = \frac{1}{2}h(a+b)$

$= \frac{1}{2} \times \frac{6}{\sqrt{5}} \times (2\sqrt{5} + 3\sqrt{5})$

$= \frac{3}{\sqrt{5}} \times 5\sqrt{5}$

$= 15 \text{ units}^2$

(b) $\text{Area} = \frac{1}{2}r^2\theta$




$3\pi = \frac{1}{2}r^2\theta$

$6\pi = r^2\theta$ —— ①

$l = r\theta$

$\frac{3\pi}{2} = r\theta$ —— ②

Take ① ÷ ②

$\frac{6\pi}{3\pi/2} = \frac{r^2\theta}{r\theta}$

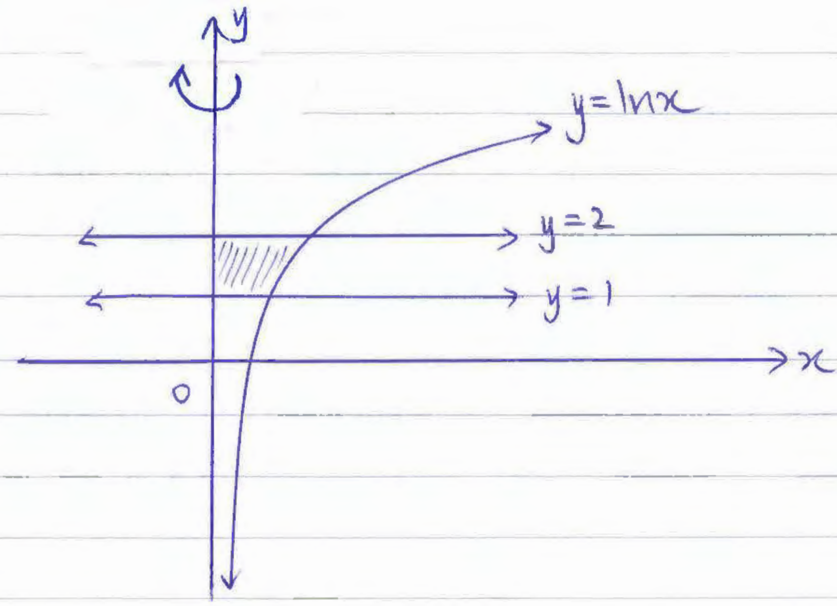
$\therefore r = 4$

Sub $r=4$ in ①

$6\pi = 16\theta$

$\therefore \theta = \frac{3\pi}{8}$

(c) (i)




$y = \ln x$

$x = e^y$

$x^2 = e^{2y}$

(ii) $V = \pi\int_a^b x^2\,dy$

$= \pi\int_1^2 e^{2y}\,dy$

$= \pi\left[\frac{e^{2y}}{2}\right]_1^2$

$= \pi\left(\frac{e^4}{2} - \frac{e^2}{2}\right)$

$= \frac{\pi(e^4 - e^2)}{2}$

Question 15.

(a) (i) $y = e^x$ — ①

$y = e^{1-x}$ — ②

POI: Sub ① in ②

$$e^x = e^{1-x}$$

$$x = 1 - x$$

$$2x = 1$$

$$\therefore x = \frac{1}{2}$$

(ii) $$\text{Area} = \int_0^{1/2} e^x \, dx + \int_{1/2}^{1} e^{1-x} \, dx$$

$$= 2\int_0^{1/2} e^x \, dx$$

$$= 2\left[e^x\right]_0^{1/2}$$

$$= 2\left(e^{1/2} - e^0\right)$$

$$= 2(\sqrt{e} - 1)$$

(b) (i) $$\frac{d^2x}{dt^2} = 8 - 6t$$

$$\frac{dx}{dt} = \int 8 - 6t \, dt$$

$$= 8t - \frac{6t^2}{2} + C_1$$

Sub $t = 0$, $x = 5$

$$5 = 8(0) - 3(0)^2 + C_1$$

$$C_1 = 5$$

$$\frac{dx}{dt} = 8t - 3t^2 + 5$$

$x = \int 8t - 3t^2 + 5 \, dt$

$= \frac{8t^2}{2} - \frac{3t^3}{3} + 5t + C_2$

Sub $t=0$, $x=0$.

$0 = 4(0)^2 - 0^3 + 5(0) + C_2$

$C_2 = 0$

$\therefore x = 4t^2 - t^3 + 5t$

(ii) $x = 4t^2 - t^3 + 5t = 0$

$-t(t^2 - 4t - 5) = 0$

$-t(t-5)(t+1) = 0$

$t = 0, 5, -1$

But $t \geq 0$, $\therefore t = 0, 5$ Returns at $t = 5$ sec.

At $t=5$,

$\frac{dx}{dt} = 8(5) - 3(5)^2 + 5$

$= -30$ m/s.

(c)(i) $A_1 = 20000 - M$

$A_2 = A_1 - M$

$= 20000 - 2M$

$A_3 = 20000 - 3M$

$\therefore A_4 = 20000 - 4M$

(ii) $A_6 = 20000 - 6M$

$A_7 = A_6 \times 1.01 - M$

$= (20000 - 6M) \times 1.01 - M$

$A_8 = A_7 \times 1.01 - M$

$= [(20000 - 6M) \times 1.01 - M] \times 1.01 - M$

$= (20000 - 6M) \times 1.01^2 - M \times 1.01 - M$

$= (20000 - 6M) \times 1.01^2 - M(1 + 1.01)$.

(iii) $A_9 = A_8 \times 1.01 - M$

$$= \left[(20000-6M)\times 1.01^2 - M(1+1.01)\right]\times 1.01 - M$$

$$= (20000-6M)\times 1.01^3 - M(1+1.01)\times 1.01 - M$$

$$= (20000-6M)\times 1.01^3 - M(1+1.01+1.01^2)$$

$\vdots$

$$A_{60} = (20000-6M)\times 1.01^{54} - M\underbrace{(1+1.01+1.01^2+\ldots+1.01^3)}_{GP:\ a=1,\ r=1.01,\ n=54}$$

enough for 1 mark

$$= (20000-6M)\times 1.01^{54} - M\times \frac{1(1.01^{54}-1)}{1.01-1}$$

$$= (20000-6M)\times 1.01^{54} - 100M(1.01^{54}-1)$$

(iv) $A_{60} = 0$

$$0 = (20000-6M)\times 1.01^{54} - 100M(1.01^{54}-1)$$

$$= 20000\times 1.01^{54} - 6M\times 1.01^{54} - 100M\times 1.01^{54} + 100M$$

$$= 20000\times 1.01^{54} - M(106\times 1.01^{54} - 100)$$

$$M(106\times 1.01^{54} - 100) = 20\,000\times 1.01^{54}$$

$$M = \frac{20000\times 1.01^{54}}{106\times 1.01^{54} - 100}$$

$$= 420.4448\ldots$$

$\therefore M = \$420.44$ (nrst ¢).

Question 16.

(a) (i) $\frac{d}{dx}(x^2 \ln x) = 2x.\ln x + x^2.\frac{1}{x}$

$= 2x\ln x + x$

(ii) $\int x\ln x \, dx$

From (i), $\frac{d}{dx}(x^2 \ln x) = 2x\ln x + x$

$x^2 \ln x = \int 2x\ln x + x \, dx$

$= \int 2x\ln x \, dx + \int x \, dx$

$= 2\int x\ln x \, dx + \frac{x^2}{2}$

$2\int x\ln x \, dx = x^2 \ln x - \frac{x^2}{2}$

$\therefore \int x\ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c.$

(b) (i) $\frac{dN}{dt} = kN$ sub $N = Ae^{kt}$

LHS $= \frac{dN}{dt}$

$= \frac{d}{dt}(Ae^{kt})$

$= k.Ae^{kt}$

$= kN$

$=$ RHS

$\therefore N = Ae^{kt}$ is a solution

(ii) At $t=0$, $N=100$.

$100 = Ae^{k(0)}$

$= A$

$N = 100e^{kt}$

At $t=2$, $N=400$.

$400 = 100e^{2k}$

$4 = e^{2k}$

$2k = \ln 4$

$k = \frac{1}{2}\ln 4$

$= \ln 4^{1/2}$

$\therefore k = \ln 2$

(iii) At $t=10$,

$N = 100e^{10\ln 2}$

$= 102400$.

(iv) $N=1000$, $t=?$

$1000 = 100e^{t.\ln 2}$

$10 = e^{t.\ln 2}$

$t.\ln 2 = \ln 10$

$t = \frac{\ln 10}{\ln 2}$

$= 3.3219\ldots$

$=$ 3 mins 19 sec (nrst sec).

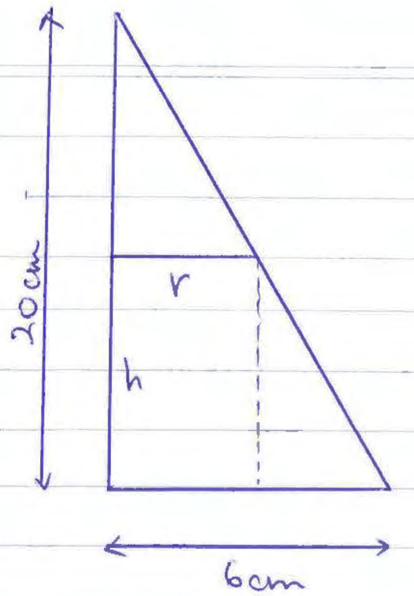
(c)(i) Similar triangles (equiangular)

$$\frac{r}{6} = \frac{20-h}{20}$$ (corres. sides in sim. $\Delta$s in same ratio)

$20r = 120 - 6h$

$6h = 120 - 20r$

$\therefore h = 20 - \frac{10}{3}r$




Then, $V = \pi r^2 h$

$= \pi r^2 \cdot \left(20 - \frac{10}{3}r\right)$

$= \pi r^2 \left(\frac{60 - 10r}{3}\right)$

$= \frac{10\pi r^2(6-r)}{3}$

(ii) $\frac{dV}{dr} = \frac{10\pi}{3}\left[2r\cdot(6-r) + r^2\cdot(-1)\right]$

$= \frac{10\pi}{3}\left[12r - 2r^2 - r^2\right]$

$= \frac{10\pi(12r - 3r^2)}{3}$

$= \frac{\cancel{30}^{10}\pi r(4-r)}{\cancel{3}}$

$= 0$ for max/min $V$.

$r = 0, 4.$

But $r > 0$, $\therefore r = 4$ only.

TEST for max V.

| $r$ | 3 | 4 | 5 |
|---|---|---|---|
| $dV/dr$ | 94 | 0 | -157 |
| | ↗ | → | ↘ |

$\therefore$ max V at $r = 4$.

When $r = 4$, $h = 20 - \frac{10}{3}(4)$

$= \frac{20}{3}$