FH
AW
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AT

Name: ____________________

Class: 12MT2__ or 12MTX__

Teacher: ____________________

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2004

YEAR 12

AP4 EXAMINATION

MATHEMATICS

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.

***Each page must show your name and your class. ***

QUESTION 1 **MARKS**

(a) Evaluate correct to 3 significant figures $\dfrac{67.78 - 14.43}{(6.8 \times 5.4)^2}$ **2**

(b) Solve for x

$$|3x - 4| = 2$$ **2**

(c) For what values of k does the equation $x^2 + (k - 2)x + 1 = 0$ have real roots? **2**

(d) Find the second derivative of $\dfrac{1}{\sqrt{x}}$. Express the answer as a surd. **2**

(e) Solve the simultaneous equations **2**

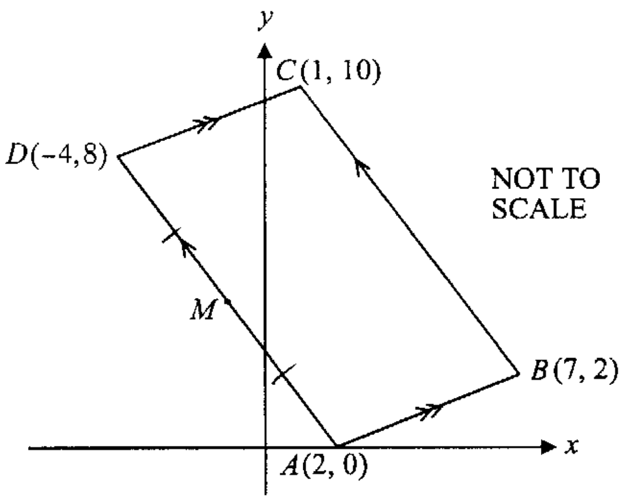
$$x - 3y = 5$$
$$2x + y = 3.$$

(f) Connor bought a new car costing \$19 600. The value of the car depreciates by 15% of the new price in its first year and at 12.5% of the previous year's value in each succeeding year. What is the value of the car (to the nearest \$10) after 8 years? **2**

QUESTION 2 *(Start a new page)* **MARKS**

(a) Find the slope of the normal to the curve $y = 3\sin x$ at the point $(\pi, 0)$. **2**

(b)



In the diagram above, $ABCD$ is a parallelogram with the corner points $A(2, 0)$, $B(7, 2)$, $C(1, 10)$ and $D(-4,8)$. The point M is the midpoint of AD.

(i) Show that the gradient of AD is $\frac{-4}{3}$. **1**

(ii) Find the length of AD. **1**

(iii) Show that the coordinates of M, the midpoint of AD, are (-1, 4). **1**

(iv) Show that the equation of BC is $4x + 3y - 34 = 0$. **2**

(v) Find the perpendicular distance from M to BC. **2**

(vi) Find the area of the parallelogram $ABCD$. **1**

(c) Show that $\dfrac{2\cos\theta\sin^2\theta + 2\cos^3\theta}{4\sin\theta} = \dfrac{1}{2}\cot\theta$. **2**

QUESTION 3 *(Start a new page)* **MARKS**

(a) Differentiate

(i) $\cos\left(x^2+1\right)$ **2**

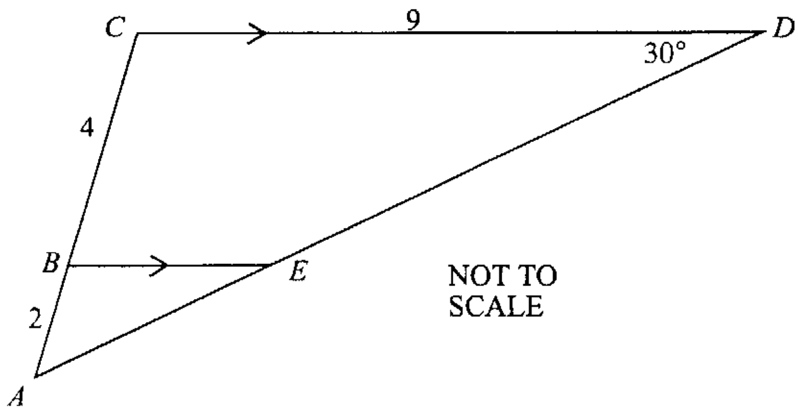
(ii) $\dfrac{e^x}{x^2}$ **1**

(iii) $\left(4x^8+5\right)^3$. **2**

(b) Find $\int\left(4-e^{3x}\right)dx$. **1**

(c) Evaluate $\int_{-4}^{1}\dfrac{2}{x+5}\,dx$. **2**

(d)



In the diagram, ACD is a triangle where $AB = 2$ cm, $BC = 4$ cm, $CD = 9$ cm and $\angle CDE = 30^\circ$. Also, BE is parallel to CD.

(i) Find the size of $\angle BED$. Give a reason for your answer. **1**

(ii) Find the length of BE. Give reasons for your answer. **3**

QUESTION 4 *(Start a new page)* **MARKS**

(a) Solve for x

$$\log_2 x + \log_2(x+7) = 3 \text{ , for } x > 0.$$

2

(b) Consider the parabola $6y = 5 - 8x - x^2$.

(i) Find the coordinates of the vertex **2**

(ii) Find the coordinates of the focus **2**

(c) Solve for x

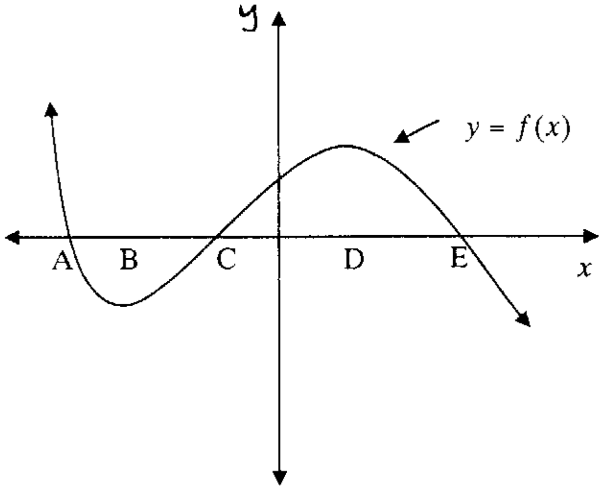
$$e^{\ln 5x} = 10$$ **1**

(d) Find the values of k if $\int_1^k (x+1)dx = 6$ **2**

(e) Evaluate $\sum_{n=1}^{10} 5 \times 2^{n-1}$. **3**

QUESTION 5 *(Start a new page)* **MARKS**

(a)



The above diagram shows a sketch of the function $y = f(x)$.
Copy this diagram into your writing booklet, and on the same diagram sketch the gradient function $y = f'(x)$. **2**

(b) John and Robert play a tennis match against each other. The probability in any set that John wins is $\frac{3}{5}$. The first player to win 2 sets wins the match.

(i) Find the probability that the game finishes at the end of the second set. **1**

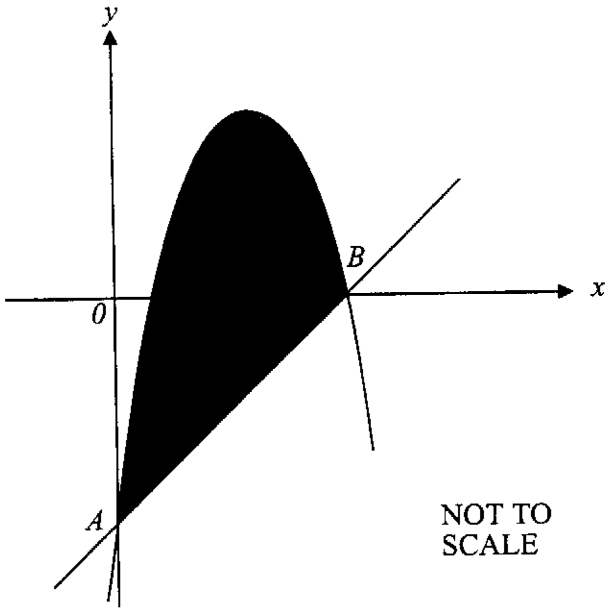
(ii) Find the probability that John wins the match. **2**

(iii) Find the probability that the person who wins the first set goes on to win the match. **2**

Question 5 is continued on the next page

 MARKS

(c)



The graphs of $y = x - 5$ and $y = -x^2 + 6x - 5$ intersect at the points A and B as shown on the diagram.

(i) Find the x-coordinates of point A and point B. **2**

(ii) Find the area of the shaded region bounded by $y = -x^2 + 6x - 5$ and $y = x - 5$. **3**

QUESTION 6 *(Start a new page)* **MARKS**

(a) The curve $y = f(x)$ passes through the origin and has a gradient function given by

$$f'(x) = x(x-2)^2 .$$

(i) Find the equation of the curve $y = f(x)$. **2**

(ii) Find the coordinates of the stationary points of the curve. **1**

(iii) Determine the nature of the stationary points. **2**

(iv) Show that $(\frac{2}{3}, \frac{44}{81})$ is a point of inflexion. **2**

(v) Sketch the curve $y = f(x)$, labelling clearly the stationary points and the y-intercept. **2**

(b) (i) Sketch the graphs $y = \sin x + 1$ *and* $y = \cos x$ on the ***same*** number plane for $0 \le x \le 2\pi$. **2**

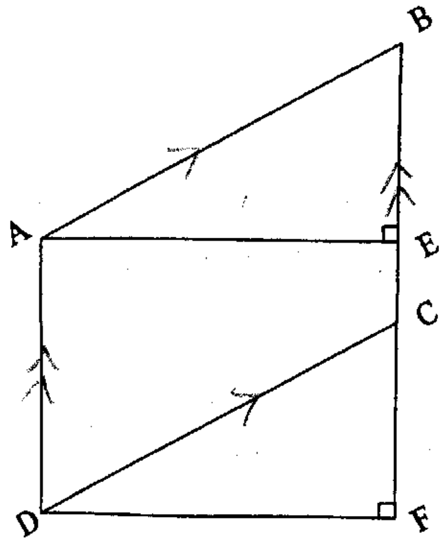
(ii) Using the graph, or otherwise, find all solutions to $\sin x - \cos x + 1 = 0$ in the domain $0 \le x \le 2\pi$. **1**

QUESTION 7 *(Start a new page)* **MARKS**

(a) Find the limiting sum of the geometric series

$$1+\frac{\sqrt{2}}{\sqrt{2}+1}+\frac{2}{(\sqrt{2}+1)^2}+\frac{2\sqrt{2}}{(\sqrt{2}+1)^3}+\ldots\ldots$$ **2**

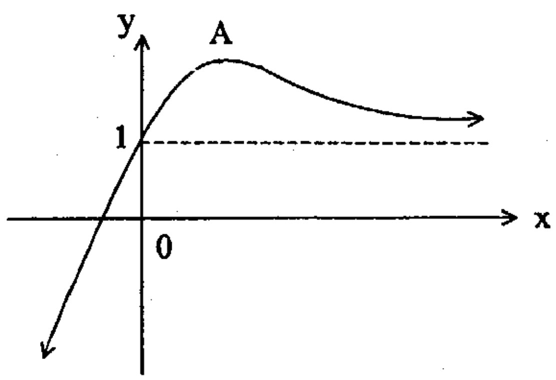
(b) ABCD is a parallelogram. $A\hat{E}B = D\hat{F}C = 90^\circ$.



(i) Show that $\Delta ABE \equiv \Delta DCF$. Give reasons. **3**

(ii) Show that AEFD is a rectangle. Give reasons. **3**

(c) The diagram shows a sketch of the curve $y = xe^{-x} + 1$
The curve has a maximum turning point at A.



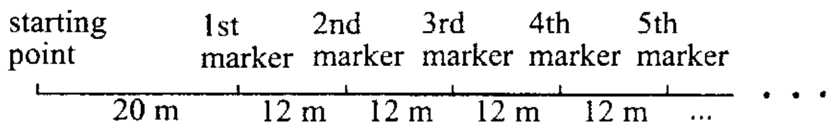
(i) Find the coordinates of A **2**
(ii) For what values of x is the curve concave up? **2**

QUESTION 8 *(Start a new page)* **MARKS**

(a) (i) Show that $\frac{d}{dx}\left[\ln\sqrt{\frac{2+x}{2-x}}\right] = \frac{2}{4-x^2}$. **2**

(ii) Hence or otherwise, evaluate $\int_0^1 \frac{4}{4-x^2}\,dx$. **2**

(b) In a training drill at an athletics club an athlete must run from a starting point to the first marker and back to the starting point then out to the second marker and back to the starting point, then out to the third marker and back to the starting point and so on. The markers and the starting point are in a straight line with the first marker 20 m from the starting point and each marker 12 m apart thereafter as indicated in the diagram below.



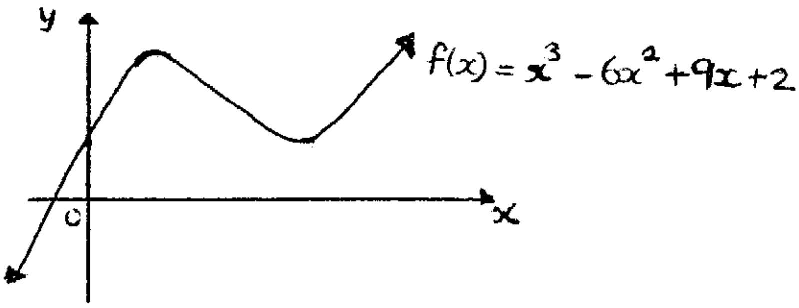
NOT TO SCALE

Tonight the coach has had 18 markers laid out.

(i) How far from the starting point would the 18th marker have been placed? **2**

(ii) How far in total would an athlete run in completing the drill tonight? **2**

(c)

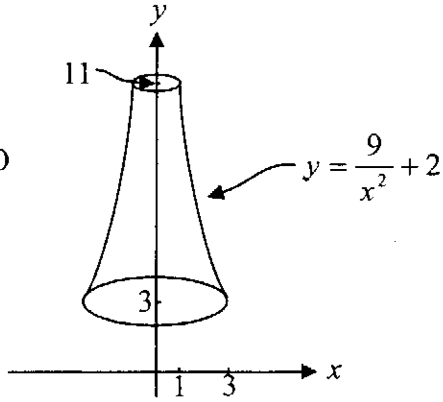


(i) Apply the Trapezoidal Rule with 7 function values to find an approximation to an area between $f(x) = x^3 - 6x^2 + 9x + 2$ and the x-axis between $x = 0$ and $x = 6$. **2**

(ii) Find the exact size of the area in part (i). **2**

QUESTION 9 *(Start a new page)* **MARKS**

(a) **4**



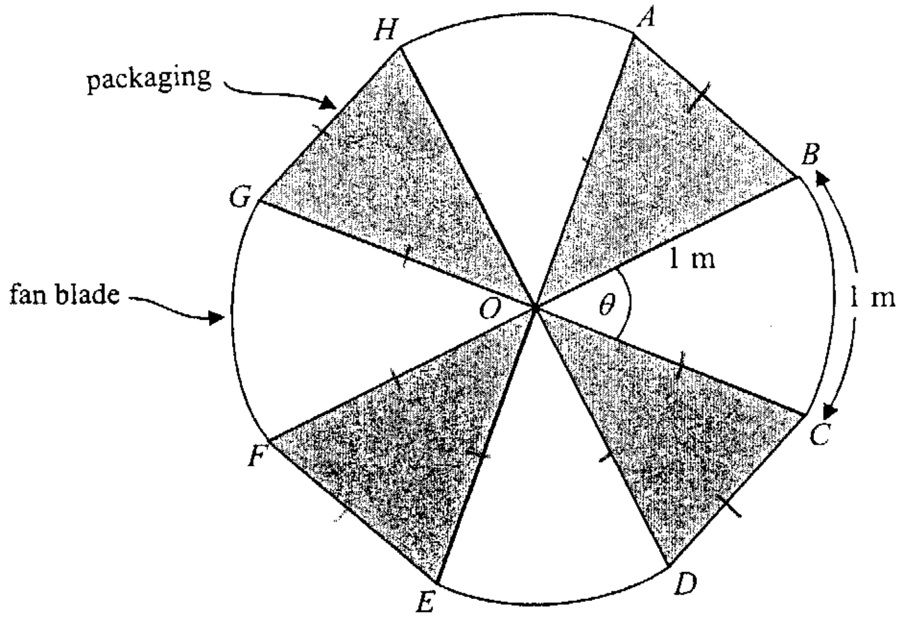
The part of the curve $y = \frac{9}{x^2} + 2$ between $x = 1$ and $x = 3$ is rotated around the y-axis.

Find the volume of the solid of revolution.

Question 9 is continued on the next page

QUESTION 9 continued **MARKS**

(b)



A large industrial fan blade is made up of four identical sectors *BOC, DOE, FOG* and *HOA* and *AE, BF, CG* and *DH* are straight lines with *O* as their midpoint. The fan blade rests in packaging so that the congruent triangles of packaging material *AOB, COD, EOF* and *GOH* are visible and are shaded in the diagram above. The arc length of the sector *BOC* is 1 metre and the length of *BO* is 1 metre.

(i) Show that $\angle AOB = \frac{\pi}{2} - \theta$. **2**

(ii) Use the cosine rule to show that $AB = \sqrt{2(1-\sin\theta)}$. **2**

(iii) Find the angle θ in radians. **1**

(iv) Find the total area covered by the fan blade and the visible packaging material. Express your answer correct to 2 significant figures. **3**

QUESTION 10 *(Start a new page)* **MARKS**

(a) Ethel pays a single sum of \$40 000 to a financial institution to provide an annual payment of \$5 000 which is to be divided between her grandchildren. The account attracts interest of 5% per annum compounding yearly. The first payment is made to the grandchildren one year after Ethel sets up the account and just after the interest has been calculated.

(i) How much is left in the account after the first payment has been made? **1**

(ii) Let the amount in the account after n payments have been made, be $\$A_n$. **2**

Show that

$$A_n = 100000 - 60000 \times (1 \cdot 05)^n$$

(iii) Another type of account offers identical features except that the interest paid is 5% per annum compounding six monthly. Payments of \$5 000 are still to be made annually. Let the amount in this account after n years be $\$B_n$. **2**

Show that

$$B_n = 98765 \cdot 43 - 58765 \cdot 43 \times 1 \cdot 025^{2n}.$$

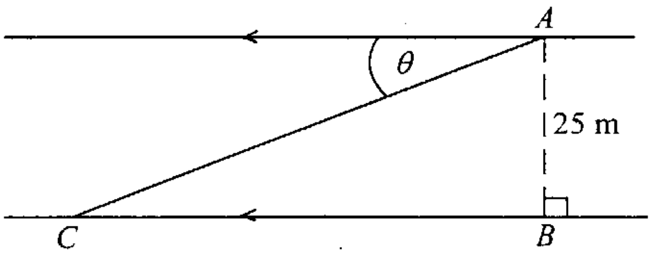
Question 10 is continued on the next page

QUESTION 10 continued **MARKS**

(b) George is walking along a straight section of a river which is 25m wide and which has parallel riverbanks.
When George is at point A, he spots Mildred directly opposite on the other side of the river at point B.

George loves Mildred and so immediately dives into the river and swims in a straight line at an angle of θ to the riverbank. George swims at $1\,\text{ms}^{-1}$.

Meanwhile Mildred hasn't seen George and has continued walking along her side of the riverbank towards point C at $2\,\text{ms}^{-1}$.
George reaches Mildred's side of the riverbank at point C.



(i) Find an expression in terms of θ for the time taken by Mildred to walk from point B to point C. **1**

(ii) Find an expression in terms of θ for the time taken by George to swim from point A to point C. **1**

(iii) Show that George doesn't arrive at point C at the same time as Mildred. **1**

(iv) Find the least time by which George can miss Mildred at point C.
Justify your answer and express it to the nearest second. **4**

END OF EXAM

Solutions to Mathematics Trial HSC Exam 2004

Question 1

(a) $\frac{67.78-14.43}{(6.8\times5.4)^2}=0.0395666\ldots$

$=0.0396$ *(3sig figs)*

2 marks	For correct answer
1 mark	for 0.0395666

(b) $3x-4=2$ or $3x-4=-2$

$3x=6$ $\quad$ $3x=2$

$x=2$ $\quad$ $x=\frac{2}{3}$

2 marks	For solving both equations correctly
1 mark	For solving one equation correctly

(c) $x^2+(k-2)x+1=0$

Real roots when $\Delta=b^2-4ac\geq0$

i.e. $(k-2)^2-4(1)(1)\geq0$

$k^2-4k+4-4\geq0$

$k^2-4k\geq0$

$k(k-4)\geq0$

$\therefore k\leq0,\ k\geq4$

2 marks	for values of k
1 mark	for correct statement $(k-2)^2-4(1)(1)\geq0$

(d)

$y=x^{-\frac{1}{2}}$

$y'=-\frac{1}{2}x^{\frac{-3}{2}}$

$y''=\frac{3}{4}x^{\frac{-5}{2}}$

$y''=\frac{3}{4\sqrt{x^5}}$

2 marks	for correct second derivative in surd form
1 mark	for correct first derivative

Question 1 continued

(e) $x-3y=5 \quad -(A)$

$2x+y=3 \quad -(B)$

$(B)\times3 \quad 6x+3y=9 \quad -(C)$

$(A)+(C) \quad 7x=14$

$x=2$

In (A) $\quad 2-3y=5$

$-3y=3$

$y=-1$

So $x=2$ and $y=-1$

2 marks	Correct answer
1 mark	Correct value of y or x

(f) Value at end of 1[st] year $=85\%\times19600=\$16\,660$

Value after 8 years $=16660(1-0.125)^7=16660\times0.875^7$

$=\$6542.31$

$=\$6540$

2 marks	Correct answer rounded to nearest \$10
1 mark	Correct substitution into depreciation formula

Question 2

(a) $y=3\sin x$

$\frac{dy}{dx}=3\cos x$

At $x=\pi$, $\frac{dy}{dx}=3\cos\pi$

$=-3$

The gradient of the tangent at $x=\pi$ is -3 so the gradient of the normal at $x=\pi$ is $\frac{1}{3}$.

2 marks	Obtaining correct gradient of normal
1 mark	Obtaining correct derivative of 3sin$x$

(b) (i) gradient of $AD=\frac{0-8}{2--4}$

$=\frac{-8}{6}$

$=\frac{-4}{3}$ as required

1 mark	Correct answer

Question 2 continued

(ii) distance $AD = \sqrt{(2--4)^2+(0-8)^2}$

$= \sqrt{36+64}$

$= \sqrt{100}$

$= 10$ units

1 mark	Correct answer

(iii) $M = \left(\frac{2-4}{2}, \frac{0+8}{2}\right)$

$= \left(\frac{-2}{2}, \frac{8}{2}\right)$

$= (-1,4)$ as required

1 mark	Correct answer

(iv) From part (i), gradient of AD is $\frac{-4}{3}$ so gradient of BC is $\frac{-4}{3}$.

Equation of BC is

$y-2 = \frac{-4}{3}(x-7)$

$3y-6 = -4x+28$

$3y+4x-34 = 0$

(Alternatively, using the point (1, 10), we have

$y-10 = \frac{-4}{3}(x-1)$

$3y-30 = -4x+4$

$3y+4x-34 = 0$)

2 marks	Correct answer
1 mark	Substituting correctly the gradient and coordinates into the straight line formula

Question 2 continued

(v) Perpendicular distance from $M(-1,4)$ to the line $3y+4x-34=0$ (which for ease of substitution should be expressed as $4x+3y-34=0$) is given by

$\frac{|4\times -1+3\times 4-34|}{\sqrt{4^2+3^2}}$

$= \frac{|-26|}{\sqrt{25}}$

$= \frac{26}{5}$ units

2 marks	Correct answer (ignore units)
1 mark	Correct substitution into formula

(vi) Area of parallelogram = base × perpendicular height

$= AD \times$ perpendicular height

$= 10 \times \frac{26}{5}$ (from parts (ii) and (v) respectively)

$= 52$ square units

1 mark	Correct answer (ignore units)

(c) $\frac{2\cos\theta\sin^2\theta + 2\cos^3\theta}{4\sin\theta} = \frac{1}{2}\cot\theta$

LHS $= \frac{2\cos\theta\sin^2\theta + 2\cos^3\theta}{4\sin\theta}$

$= \frac{2\cos\theta(\sin^2\theta+\cos^2\theta)}{4\sin\theta}$

$= \frac{2\cos\theta}{4\sin\theta}$

$= \frac{1}{2}\cot\theta$

= RHS $\quad \therefore \frac{2\cos\theta\sin^2\theta + 2\cos^3\theta}{4\sin\theta} = \frac{1}{2}\cot\theta$

2 marks	Completing the proof
1 mark	For using the identity

Question 3

(a) (i) $y = \cos(x^2 - 1)$

$$\frac{dy}{dx} = -2x\sin(x^2 - 1)$$

2 marks	Correct answer
1 mark	For 2x or -sin

(ii) $y = \dfrac{e^x}{x^2}$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2},$$

where $u = e^x$, $v = x^2$, $u' = e^x$ and $v' = 2x$.

$$\frac{dy}{dx} = \frac{(e^x)(x^2) - (e^x)(2x)}{x^4}$$

$$= \frac{e^x(x-2)}{x^3} \qquad \left(\text{acc: } \frac{x^2e^x - 2xe^x}{x^4}\right)$$

1 mark	Correct answer

(iii) $y = (4x^8 + 5)^3$

$$\frac{dy}{dx} = 3(4x^8 + 5)^2 \times 32x^7$$

$$= 96x^7(4x^8 + 5)^2$$

2 marks	Correct answer
1 mark	$3(4x^8 + 5)^2$ or $32x^7$

(b) $\displaystyle\int (4 - e^{3x})\,dx = 4x - \frac{e^{3x}}{3} + c$

1 mark	Correct answer (ignore "c")

(c) $\displaystyle\int_{-4}^{1} \frac{2}{x+5}\,dx = 2\left[\ln(x+5)\right]_{-4}^{1}$

$$= 2\{\ln 6 - \ln 1\}$$

$$= 2\ln 6$$

2 marks	Correct answer
1 mark	Correct integration or correct answer from incorrect integration

Question 3 continued

(d) (i)

$\angle BED + \angle CDE = 180°$ (co-interior angles are supplementary, $BE \parallel CD$)

$\angle BED + 30° =$

$\angle BED = 150°$

1 mark	Correct answer with correct reason

(ii)

$\angle BEA = \angle CDE$ (corr. ∠'s =, $BE \parallel CD$) ✓(1)

$\angle BAE = \angle CAD$ (common angle)

$\therefore \triangle BAE \,|||\, \triangle CAD$ (equiangular) ✓(1)

$\dfrac{BE}{CD} = \dfrac{BA}{AC}$ (corresponding sides similar Δ's are in propn)

$\therefore \dfrac{BE}{9} = \dfrac{2}{6}$

$BE = 3$ cm ✓(1) (must have reason)

Question 4

(a) $\log_2 x + \log_2(x+7) = 3$

$$\log_2[x(x+7)] = 3$$

$$2^3 = x(x+7)$$

$$8 = x^2 + 7x$$

$$x^2 + 7x - 8 = 0$$

$$(x-1)(x+8) = 0$$

Since $x > 0$, $x = 1$

2 marks	Correct answer
1 mark	For establishment of $2^3 = x(x+7)$

Question 4 continued

(b) (i)

$$6y = 5 - 8x - x^2$$
$$x^2 + 8x + 16 = -6y + 5 + 16$$

$$(x+4)^2 = -6\left(y - \frac{7}{2}\right)$$

$\therefore$ *vertex is* $\left(-4, \frac{7}{2}\right)$

2 marks	Gives the correct answer
1 mark	Uses an appropriate method to find the vertex

① may use the derivative
• solve $\frac{dy}{dx} = 0$, $x = -4$ (1 mark)
• find y value (1 mark)
OR
② might use • $x = \frac{-b}{2a}$, $x = -4$ (1 mark)
• find y value (1 mark)

(ii) Focal length is $\frac{6}{4}$ *or* $\frac{3}{2}$

Parabola is concave down.

Focus $x = -4$, $y = \frac{7}{2} - \frac{3}{2} = 2$

$\therefore$ *the focus is* $(-4, 2)$

2 marks	Gives the correct answer or gives a correct answer from their working in (i)
1 mark	Finds a correct focal length from their (i) and a correct concavity OR Substantially correct working to find the focus using the focal length

(c)

$$e^{\ln 5x} = 10$$
$$\ln e^{\ln 5x} = \ln 10$$
$$\therefore 5x = 10$$
$$\therefore x = 2$$

1 mark	Correct answer

(d) $\left[\frac{x^2}{2} + x\right]_1^k = 6 \quad \therefore \left[\frac{k^2}{2} + k\right] - \left[\frac{1}{2} + 1\right] = 6$

$\therefore k^2 + 2k - 15 = 0 \quad \therefore (k+5)(k-3) = 0$

$\therefore k = 3$ *or* $k = -5$

2 marks	Gives the correct answer k=-5 or k=3
1 mark	Substitute correctly to get $\left[\frac{k^2}{2} + k\right] - \left[\frac{1}{2} + 1\right] = 6$

Question 4 continued

(e) $\sum_{r=1}^{10} 5 \times 2^{r-1} = 5 \times 2^0 + 5 \times 2^1 + 5 \times 2^2 + \ \ldots\ + 5 \times 2^9$

$= 5 + 10 + 20 + \ \ldots$

$= \frac{a(r^n - 1)}{r - 1}$

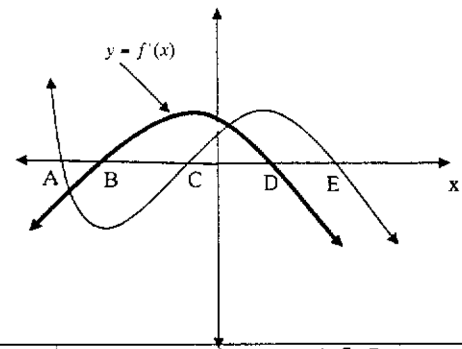
$= \frac{5(2^{10} - 1)}{2 - 1}$

$= 5115$

3 marks	Correct answer
2 marks	Correct use of geometric sum formula
1 mark	Correct expansion of geometric sum

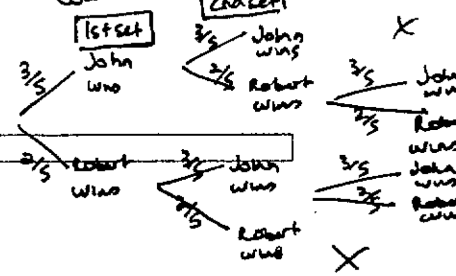
Question 5

(a)



2 marks	Correct shape and x intercepts at B, D
1 mark	Correct shape or x-intercepts

The correct tree diagram would be



(b) $P(J) = \frac{3}{5}$ and $P(R) = \frac{2}{5}$

(i) $P(J,J) + P(R,R) = \left(\frac{3}{5} \times \frac{3}{5}\right) + \left(\frac{2}{5} \times \frac{2}{5}\right) = \frac{13}{25}$

1 mark	Correct working

Question 5 continued

(ii) $P(J,J)+P(R,J,J)+P(J,R,J)$

$= \left(\frac{3}{5}\times\frac{3}{5}\right)+\left(\frac{2}{5}\times\frac{3}{5}\times\frac{3}{5}\right)+\left(\frac{3}{5}\times\frac{2}{5}\times\frac{3}{5}\right)$

$= \frac{81}{125}$

1 mark if they have a diagram (tree or otherwise) and work on with an incorrect idea from their nearly correct tree etc

2 marks	Correct working

(iii) $P(J,J)+P(J,R,J)+P(R,R)+P(R,J,R)$

$= \left(\frac{3}{5}\times\frac{3}{5}\right)+\left(\frac{3}{5}\times\frac{2}{5}\times\frac{3}{5}\right)+\left(\frac{2}{5}\times\frac{2}{5}\right)+\left(\frac{2}{5}\times\frac{3}{5}\times\frac{2}{5}\right)$

$= \frac{19}{25}$

1 mark same as above

2 marks	Correct working

(c) (i) The lines intersect when

$x-5=-x^2+6x-5$

$x^2-5x=0$

$x(x-5)=0$

$x=0$ or $x=5$

The x-coordinate of A is 0 and the x-coordinate of B is 5.

2 marks	Correct answer
1 mark	Obtaining the equation $x^2-5x=0$

Question 5 continued

(ii) Area required $= \int_0^5 \left(\left(-x^2+6x-5\right)-(x-5)\right)dx$

$= \int_0^5 \left(-x^2+5x\right)dx$

$= \left[\frac{-x^3}{3}+\frac{5x^2}{2}\right]_0^5$

$= \left(\frac{-125}{3}+\frac{125}{2}\right)-(0+0)$

$= \frac{-250+375}{6} = \frac{125}{6}$

$= 20\frac{5}{6}$ square units

3 marks	Correct answer (units not necessary) $125/6$ or $20\frac{5}{6}$
2 marks	Correct integration OR correct evaluation of an expression containing a minor mistake
1 mark	Correct terminals of integration and integrand

Question 6

(a)

(i) $f'(x) = x(x-2)^2$

$= x\left(x^2-4x+4\right)$

$= x^3-4x^2+4x$

$f(x) = \int\left(x^3-4x^2+4x\right)dx$

$= \frac{x^4}{4}-\frac{4x^3}{3}+\frac{4x^2}{2}+c$

Since $y=f(x)$ passes through the origin, we have

$0=0-0+0+c$

$c=0$

$f(x)=\frac{x^4}{4}-\frac{4x^3}{3}+2x^2$

2 marks	Correct answer
1 mark	Correct integration but leaves out c OR expands incorrectly but completes the integration including c correctly

Question 6 continued

(ii) A stationary point occurs when $f'(x) = 0$ that is, when $x(x-2)^2 = 0$
so when $x = 0$ or $x = 2$.
Now, $f(0) = 0$
So one stationary point occurs at $(0,0)$.

Also $f(2) = \frac{16}{4} - \frac{4 \times 8}{3} + 2 \times 4$

$= \frac{4}{3}$

The other stationary point occurs at $(2, \frac{4}{3})$.

(nothing extra awarded for (0,0))

1 mark	Correct coordinates of $(2, \frac{4}{3})$.

(iii) First derivative test

	-1		1		3
x	0^-	0	0^+ / ½	2	2^+
$f'(x)$	-9	0	1	0	3
slope	\	—	/	—	/

(1)

∴ Relative minimum turning pt at (0,0)
∴ Stationary pt of inflection at $(2, \frac{4}{3})$ (1)

OR 1 test + 1 pt = 1 mark

OR Second derivative test
$f''(x) = 3x^2 - 8x + 4$
at $x = 0$
$f''(x) = 4 > 0$
∴ rel. min. turning pt at (0,0) (1)

at $x = 2$
$f''(x) = 0$
poss. inflection (must do one of)

x	2^-	2	2^+
	$x=1$		$x=3$
$f'(x)$	1	0	3
slope	/	—	/

x	2^-	2	2^+
	$x=1$		$x=3$
$f''(x)$	-1	0	7

there is a change in sign of $f''(x)$ ∴ change in concavity

stationary point of inflection at $(2, \frac{4}{3})$ (1)

$f''(x) = 3x^2 - 8x + 4$
For point of inflexion $f''(x) = 0$
$3x^2 - 8x + 4 = 0$
$(3x-2)(x-2) = 0$
$x = \frac{2}{3}, 2$ ← (already classified)

When $x = \frac{2}{3}$ $\quad f(x) = \frac{\left(\frac{2}{3}\right)^4}{4} - \frac{4}{3} \times \left(\frac{2}{3}\right)^3 + 2 \times \left(\frac{2}{3}\right)^2 = \frac{44}{81}$

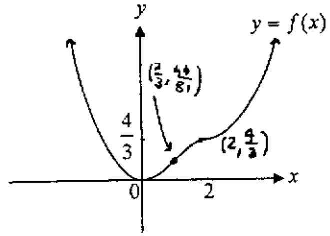
x	$\frac{2}{3}^-$ (0)	$\frac{2}{3}$	$\frac{2}{3}^+$ (1)
$f''(x)$	4	0	-1

there is a change in sign of $f''(x)$ ∴ change in concavity
POI at $\left(\frac{2}{3}, \frac{44}{81}\right)$

2 marks	correct answer with working + includes test
1 mark	Second derivative (may have been found in (iii))

Question 6 continued

(v)

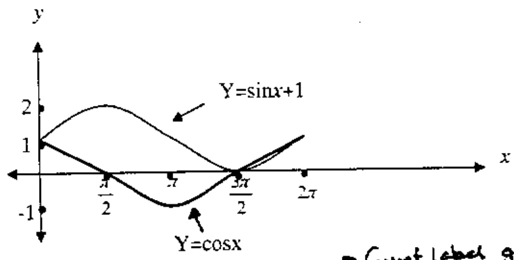


no marks deducted if extra inflection not graphed (not asked for in question)

2 marks	Correct graph with the points $(0,0)$ and $\left(2, \frac{4}{3}\right)$ clearly marked.
1 mark	Correct graph with no points marked

(b)

(i)



(must label graphs – hard to guess!)

2 marks	Both graphs correct
1 mark	One graph correct / or both graphs drawn no label

(ii)

$x = 0, \frac{3\pi}{2}, 2\pi$

1 mark	Correct answer (must have all 3)

If using 'CFM' – graphs in b(i) must be 'substantial' to allow 'CFM' to follow
(eg. straight line/parabola shapes would be too simple to earn CFM)

Question 7

(a)This is a geometric series with a limiting sum with:

$$r = \frac{\sqrt{2}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = 2-\sqrt{2}$$

$$S_\infty = \frac{1}{1-(2-\sqrt{2})} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$$

2 marks	Correct answer
1 mark	Correct r or correct S from incorrect r

(b) (i) Consider $\Delta ABE, \Delta DCF$

$A\hat{E}B = D\hat{F}C = 90°$ (given)

AB=DC (opposite sides of parallelogram ABCD are equal)

$D\hat{C}F = A\hat{B}F$ (corresp. angles, $AB \| DC$)

$\therefore \Delta ABE \equiv \Delta DCF \, (AAS)$

3 marks	Correct proof for congruency
2 marks	AB=DC and $D\hat{C}F = A\hat{B}F$ with correct reasons
1 mark	AB=DC or $D\hat{C}F = A\hat{B}F$ with correct reasons

(ii)AE=DF (corresp. sides in congruent triangles are equal)

Also $AE \| DF$ since $A\hat{E}B = D\hat{F}C$ are corresponding angles.

$\therefore$ AEFD is a rectangle since $\angle DFE = 90°$ (given) and two opposite sides are parallel and equal.

3 marks	Correct proof for rectangle
2 marks	Two correct properties with reasons
1 mark	One correct property with reason

other solutions possible
must have correct geometry / reasons

Question 7 continued

(c) (i) $\frac{dy}{dx} = e^{-x} - xe^{-x} = (1-x)e^{-x}$

For stationary points $\frac{dy}{dx} = 0$

$e^{-x}(1-x) = 0$

$1-x = 0 \quad (e^{-x} \neq 0)$

$\therefore x = 1$

For $x = 1$, $y = e^{-1} + 1 = \frac{1}{e} + 1$

$\therefore A(1, \frac{1}{e} + 1)$

2 marks	Correct coordinates of A
1 mark	Correct first derivative

(ii) $\frac{d^2y}{dx^2} = -e^{-x}(1-x) - e^{-x} = (x-2)e^{-x}$

for concave up $(x-2)e^{-x} > 0$

$e^{-x} > 0$ always

$\therefore x-2$ must be positive

$\therefore x > 2$

2 marks	Correct solution
1 mark	Second derivative

Question 8

(a)

$$\frac{d}{dx}\ln\sqrt{\frac{2+x}{2-x}} = \frac{d}{dx}\left(\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)\right)$$

$$= \frac{d}{dx}\frac{1}{2}\left[\ln(2+x) - \ln(2-x)\right]$$

$$= \frac{1}{2}\left(\frac{1}{2+x} - \frac{-1}{2-x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2+x} + \frac{1}{2-x}\right)$$

$$= \frac{1}{2}\left(\frac{4}{4-x^2}\right) = \frac{2}{4-x^2}$$

2 marks	for $\ln\sqrt{\frac{2+x}{2-x}} = \frac{1}{2}[\ln(2+x) - \ln(2-x)]$, and simplification
1 mark	for $\ln\sqrt{\frac{2+x}{2-x}} = \frac{1}{2}[\ln(2+x) - \ln(2-x)]$

(ii)

$$\int_0^1 \frac{4}{4-x^2}dx = 2\left[\ln\sqrt{\frac{2+x}{2-x}}\right]_0^1$$

$$\int_0^1 \frac{4}{4-x^2}dx = 2\ln\sqrt{3} = \ln 3$$

accept either

2 marks	for $\int_0^1 \frac{4}{4-x^2}dx = 2\left[\ln\sqrt{\frac{2+x}{2-x}}\right]_0^1$, and simplification
1 mark	for $\int_0^1 \frac{4}{4-x^2}dx = 2\left[\ln\sqrt{\frac{2+x}{2-x}}\right]_0^1$,

(b) (i) We have an arithmetic sequence with $a = 20$ and $d = 12$.

$$t_n = a + (n-1)d$$
$$= 20 + (18-1)\times 12$$
$$= 224$$

The 18th marker would have been placed 224m from the starting point.

2 marks	Correct answer
1 mark	Correct formula with incorrect substitution OR correct values of a, n and d but incorrect formula

Question 8 continued

b (ii) Method 1

Find the sum of the distances from the starting point to each of the markers up to the 18th and double the result since the players run up to the markers and back!

Using $a = 20$, $n = 18$, $d = 12$

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$
$$= \frac{1}{2}\times 18(2\times 20 + (18-1)\times 12)$$
$$= 9(40 + 204)$$
$$= 2196$$

[OR using the answer from part (i),

$$S_n = \frac{1}{2}n(a + l)$$
$$= \frac{1}{2}\times 18(20 + 224)$$
$$= 2196]$$

So, an athlete would run $2\times 2196\text{m} = 4392\text{m}$
$= 4\cdot 392\text{km}$

Method 2

Sum each of the distances up and back.
Use $a = 40$, $n = 18$, $d = 24$

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$
$$= \frac{1}{2}\times 18(2\times 40 + (18-1)\times 24)$$
$$= 9(80 + 408)$$
$$= 4392$$

So, an athlete would run 4 392m or 4.392km. accept appropriate rounded answers

2 marks	Correct answer
1 mark	Uses Method 1 but forgets to double the distance OR uses Method 2 with either correct formula or correct substitution

Question 8 continued

(c) (i)

x	0	1	2	3	4	5	6
y	2	6	4	2	6	22	56

$$A \approx \frac{h}{2}\{f(0)+2f(1)+2f(2)+2f(3)+2f(4)+2f(5)+f(6)\}$$

$$A \approx \frac{1}{2}\{2+12+8+4+12+44+56\}$$

$A \approx 69$ units^2

2 marks	Correct answer with working
1 mark	Correct substitution into rule

no marks deducted for units missing

(ii) $A = \int_0^6 (x^3 - 6x^2 + 9x + 2)dx$

$$= \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} + 2x\right]_0^6$$

$= (324 - 432 + 162 + 12) - (0)$

$= 66$ units^2

2 marks	Correct answer with working
1 mark	Correct integral

Question 9

(a) volume $= \pi \int x^2\, dy$

Now $y = \dfrac{9}{x^2} + 2$

$y - 2 = \dfrac{9}{x^2}$

$x^2(y-2) = 9$

So, $x^2 = \dfrac{9}{y-2}$

So volume $= \pi \displaystyle\int_3^{11} \frac{9}{y-2}\, dy$

$= \pi[9\log_e(y-2)]_3^{11}$

(b) $= 9\pi\{\log_e 9 - \log_e 1\}$

$= 9\pi \log_e 9$ cubic units

since $\log_e 1 = 0$

Note that the terminals are 11 and 3 and NOT 3 and 1 since we are rotating about the y-axis.

4 marks	Correct answer ignore units
3 marks	Correct integrand, terminals and integration but incorrect evaluation OR correct integrand, terminals but incorrect integration and subsequent correct evaluation OR correct evaluation of correct integrand with incorrect terminals
2 marks	Correct integrand in terms of y with incorrect terminals
1 mark	Correct integrand OR correct limits OR gives $v = \pi \int_1^3 \left(\frac{9}{x^2}+2\right)^2 dx$

(b) (i) $\angle BOC + \angle AOB + \angle HOA + \angle GOH = \pi$

(CG is a straight line (given) and angles along a straight line add to give 180° or π.)

So $\theta + \angle AOB + \theta + \angle AOB = \pi$

(Sectors BOC and HOA are identical and Δ's AOB and GOH are congruent (given)).

So $2\times\angle AOB + 2\theta = \pi$

$2\times\angle AOB = \pi - 2\theta$

$\angle AOB = \dfrac{\pi}{2} - \theta$

as required.

2 marks	Correctly reasoned result
1 mark	States first result with explanation OR states second result with explanation

Question 9 continued

(ii) From part (i), $\angle AOB = \frac{\pi}{2} - \theta$

In ΔAOB,

$$(AB)^2 = 1^2 + 1^2 - 2\times1\times1\times\cos\left(\frac{\pi}{2}-\theta\right) \quad \text{(cosine rule)}$$

$$= 2 - 2\cos\left(\frac{\pi}{2}-\theta\right)$$

$$= 2 - 2\sin\theta \qquad \text{since } \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$

$$(AB)^2 = 2(1-\sin\theta)$$

$$AB = \sqrt{2(1-\sin\theta)}$$

as required.

2 marks	Correctly derived formula
1 mark	Correct substitution into cosine rule

(iii) Method 1

Use the formula for the length of an arc of a sector.

$l = r\theta$

$1 = 1\times\theta$

$\theta = 1^c$ Note : θ is in radians

Method 2

1 radian is the angle subtended by an arc of length 1 unit in a unit circle. From the diagram, in the sector BOC, we have BO and CO equal to 1 unit and $BC = 1$ so θ must be 1 radian.

1 mark	Correct answer

(iv) Total area = $4\times$ area $\Delta AOB + 4\times$ area of sector BOC

$$= 4\times\frac{1}{2}\times1\times1\sin\left(\frac{\pi}{2}-\theta\right) + 4\times\frac{1}{2}\times1^2\times\theta$$

$$= 2\sin\left(\frac{\pi}{2}-1\right) + 2\times1$$

$$= 3\cdot1\text{m}^2 \quad \text{(correct to 2 significant figures)}$$

3 marks	Correct answer
2 marks	Correct answer without correct rounding to 2 significant figures OR incorrect area for triangles but correct area for sectors and correct subsequent rounding OR incorrect area for sectors but correct area for triangles and correct subsequent rounding.
1 mark	Correct expression for area of triangles OR correct expression for area of sectors OR correct rounding

Question 10

(a) (i) The amount in the account after the first payment has been made is

$40000\times1\cdot05 - 5000$

$= 37000$

There is \$37 000 in the account.

1 mark	Correct answer

(ii) Establish the pattern.

$A_1 = 40000\times1\cdot05 - 5000$ from part (i)

$A_2 = [40000\times1\cdot05 - 5000]\times1\cdot05 - 5000$

$= 40000\times1\cdot05^2 - 5000\times1\cdot05 - 5000$

$= 40000\times1\cdot05^2 - (5000 + 5000\times1\cdot05)$

$A_3 = [40000\times1\cdot05^2 - (5000+5000\times1\cdot05)]\times1\cdot05 - 5000$

$= 40000\times1\cdot05^3 - 5000\times1\cdot05 - 5000\times1\cdot05^2 - 5000$

$= 40000\times1\cdot05^3 - 5000(1 + 1\cdot05 + 1\cdot05^2)$

Continuing the pattern,

$$A_n = 40000\times(1\cdot05)^n - 5000(1 + 1\cdot05 + 1\cdot05^2 + \ldots + 1\cdot05^{n-1})$$

Now for $1 + 1\cdot05 + 1\cdot05^2 + \ldots + 1\cdot05^{n-1}$

$$S_n = \frac{1((1\cdot05)^n - 1)}{1\cdot05 - 1}$$

$$= \frac{(1\cdot05)^n - 1}{0\cdot05}$$

$$= 20((1\cdot05)^n - 1)$$

So $A_n = 40000\times(1\cdot05)^n - 5000\times20((1\cdot05)^n - 1)$

$= 40000\times(1\cdot05)^n - 100000\times(1\cdot05)^n + 100000$

$= 100000 - 60000\times(1\cdot05)^n$

as required

2 marks	Correct answer which has been correctly reasoned with no fudging (for example the geometric series goes to $1\cdot05^{n-1}$ not $1\cdot05^n$ and is not expressed as just $1+1\cdot05+1\cdot05^2+\ldots$)
1 mark	Correctly established series for A_n without successfully finding the sum of the geometric series OR correct methods applied with an arithmetic or algebraic mistake made

Question 10 continued

(iii) Establish the pattern.

$$B_{\frac{1}{2}} = 40000 \times 1 \cdot 025$$

$$B_1 = (40000 \times 1 \cdot 025) \times 1 \cdot 025 - 5000$$

$$= 40000 \times 1 \cdot 025^2 - 5000$$

$$B_{1\frac{1}{2}} = (40000 \times 1 \cdot 025^2 - 5000) \times 1 \cdot 025$$

$$= 40000 \times 1 \cdot 025^3 - 5000 \times 1 \cdot 025$$

$$B_2 = [40000 \times 1 \cdot 025^3 - 5000 \times 1 \cdot 025] \times 1 \cdot 025 - 5000$$

$$= 40000 \times 1 \cdot 025^4 - 5000 \times 1 \cdot 025^2 - 5000$$

$$= 40000 \times 1 \cdot 025^4 - 5000(1 + 1 \cdot 025^2)$$

$$B_{2\frac{1}{2}} = [40000 \times 1 \cdot 025^4 - 5000(1 + 1 \cdot 025^2)] \times 1 \cdot 025$$

$$= 40000 \times 1 \cdot 025^5 - 5000(1 \cdot 025 \times 1 \cdot 025^3)$$

$$B_3 = [40000 \times 1 \cdot 025^5 - 5000(1 \cdot 025 \times 1 \cdot 025^3)] \times 1 \cdot 025 - 5000$$

$$= 40000 \times 1 \cdot 025^6 - 5000(1 \cdot 025^2 \times 1 \cdot 025^4) - 5000$$

$$= 40000 \times 1 \cdot 025^6 - 5000(1 + 1 \cdot 025^2 + 1 \cdot 025^4)$$

Continuing the pattern,

$$B_n = 40000 \times 1 \cdot 025^{2n} - 5000(1 + 1 \cdot 025^2 + \ldots + 1 \cdot 025^{2n-2})$$

Now for $1 + 1 \cdot 025^2 + 1 \cdot 025^4 + \ldots + 1 \cdot 025^{2n-2}$

$$S_n = \frac{1((1 \cdot 025^2)^n - 1)}{1 \cdot 025^2 - 1}$$

$$= 19 \cdot 753086(1 \cdot 025^{2n} - 1)$$

So, $B_n = 40000 \times 1 \cdot 025^{2n} - 5000(19 \cdot 753086(1 \cdot 025^{2n} - 1))$

$$= 40000 \times 1 \cdot 025^{2n} - 98765.43 \times 1 \cdot 025^{2n} + 98765.43$$

$$B_n = 98765.43 - 58765 \cdot 43 \times 1 \cdot 025^{2n}$$

as required

2 marks	Correct answer which has been correctly reasoned with no fudging (for example the geometric series goes to $1 \cdot 025^{2n-2}$ not $1 \cdot 025^{2n}$ and is not expressed as just $1 + 1 \cdot 025 + 1 \cdot 025^2 + \ldots$)
1 mark	Correctly established series for B_n without successfully finding the sum of the geometric series OR correct methods applied with an arithmetic or algebraic mistake made

Question 10 continued

(b) (i) In ΔABC, $\angle ACB = \theta$ (Alternative angles in parallel lines are equal)

So $\tan\theta = \dfrac{25}{BC}$

$$BC = \frac{25}{\tan\theta}$$

Now $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$$2 = \frac{25}{\tan\theta} \div \text{time}$$

$$\text{time} = \frac{25}{2\tan\theta}$$

So the time taken by Mildred to walk from B to C is given by $\dfrac{25}{2\tan\theta}$.

1 mark	Correct expression

(ii) Similarly, $\sin\theta = \dfrac{25}{AC}$

$$AC = \frac{25}{\sin\theta}$$

Now, $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

$$1 = \frac{25}{\sin\theta} \div \text{time}$$

$$\text{time} = \frac{25}{\sin\theta}$$

So the time taken for George to swim from A to C is given by $\dfrac{25}{\sin\theta}$.

1 mark	Correct expression

(iii) If George arrives at point C at the same time as Mildred then

$$\frac{25}{2\tan\theta} = \frac{25}{\sin\theta}$$

$$\sin\theta = 2\tan\theta$$

$$\sin\theta = \frac{2\sin\theta}{\cos\theta}$$

$$1 = \frac{2}{\cos\theta} \qquad \sin\theta \neq 0$$

$$\cos\theta = 2$$

There is no solution to this equation and hence George cannot arrive at point C at the same time as Mildred.

1 mark	Correct reasoning

Question 10 continued

(iv) We want the difference in time taken for George and Mildred to get to point C, to be a minimum. Let the difference in time be y.

So $\quad y = \dfrac{25}{\sin\theta} - \dfrac{25}{2\tan\theta}$

and $\quad \dfrac{dy}{d\theta} = 25 \times -1(\sin\theta)^{-2} \times \cos\theta - \dfrac{25}{2} \times -1(\tan\theta)^{-2} \times \sec^2\theta$

$$= \frac{-25\cos\theta}{\sin^2\theta} + \frac{25\sec^2\theta}{2\tan^2\theta}$$

$$= \frac{-25\cos\theta}{\sin^2\theta} + \frac{25}{2} \times \frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{-25\cos\theta}{\sin^2\theta} + \frac{25}{2\sin^2\theta}$$

$$= \frac{-50\cos\theta + 25}{2\sin^2\theta}$$

Now y is a minimum when $\dfrac{dy}{d\theta} = 0$

So $\quad \dfrac{-50\cos\theta + 25}{2\sin^2\theta} = 0$

So $\quad -50\cos\theta + 25 = 0$

$$\cos\theta = \frac{1}{2}$$

$$\theta = 60^\circ$$

Test for a minimum.

When $\theta = 30^\circ$ $\quad \dfrac{dy}{d\theta} = \dfrac{-50\cos 30^\circ + 25}{2(\sin^2 30^\circ)}$

$= -36 \cdot 6$ (to 1 decimal place)

When $\theta = 70^\circ$ $\quad \dfrac{dy}{d\theta} = \dfrac{-50\cos 70^\circ + 25}{2(\sin^2 70^\circ)}$

$= 4 \cdot 5$ (to 1 decimal place)

θ	30°	60°	70°
$\frac{dy}{d\theta}$	$-$	0	+
slope	\	—	/

or test could be second derivative !

So we have a minimum at $\theta = 60^\circ$.

So when $\theta = 60^\circ$

$$y = \frac{25}{\sin 60^\circ} - \frac{25}{2\tan 60^\circ}$$

$= 21 \cdot 65$ (to 2 decimal places)

The least time that George can miss Mildred by at point C is 22 seconds.

4 marks	Correct answer with justification of minimum
3 marks	Correct answer without justification of minimum OR correct expression for $\frac{dy}{d\theta}$, the value of θ when $\frac{dy}{d\theta} = 0$ and justification of minimum
2 marks	Correct expression for $\frac{dy}{d\theta}$ and the value of θ when $\frac{dy}{d\theta} = 0$
1 mark	Correct expression for $\frac{dy}{d\theta}$