FORT STREET HIGH SCHOOL

TRIAL HIGHER SCHOOL EXAMINATION
(ASSESSMENT TASK 4)

2003

MATHEMATICS

Reading Time : Five Minutes
WorkingT ime : Three Hours

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work
- Standard integrals are supplied.
- Board approved calculators may be used.
- Each question is to be started on a **new page**

Name :________________________
Class Teacher :____________________

Q	1	2	3	4	5	6	7	8	9	10	Total	%
Mk	/12	/12	/12	/12	/12	/12	/12	/12	/12	/12	/120	/100

Question One (12 marks) | Marks

a) Write in simplest form the expression — 2

$$2x-(x-2)$$

b) Solve each of the following:-

i) $-3 \le 1-x < 4$ — 2

ii) $|x+1| = 4$ — 2

c) Rationalise the denominator and express in simple surd form:- — 2

$$\frac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

d) A function is defined by the following rule:-

$$f(x) = \begin{cases} 0 & \text{if } x \le -2 \\ -1 & \text{if } -2 < x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

i) f(-2)+f(-1)+f(0) — 2

ii) $f(a^2)$ — 2

Question Two (12 marks)

a) Find the value of sin 1.7 to 3 decimal places. (N.B. radians) — 1

b) Simplify and evaluate to 3 decimal places.

i) $\cot 65^o \times \sin 65^o$ — 2

ii) $\sin^2 20^o + \sin^2 70^o$ — 2

c) Draw on separate sketches (showing the main features - NOT on graph paper) of:-

i) $x^2 + y^2 = 36$ — 2

ii) $y = x^2 + 4$ — 2

iii) $y = 2^x$ — 2

iv) $xy = 1$ — 1

Question Three (12 marks) | Marks

a)

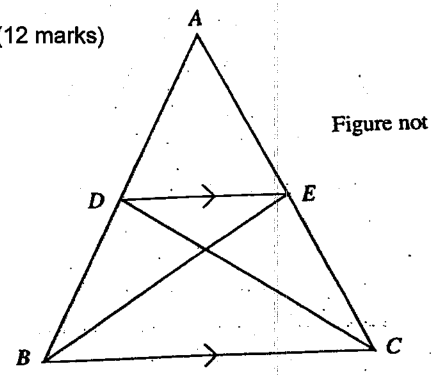


Figure not to scale.

In the diagram ABC is an isosceles triangle where AB = AC and DE is parallel to BC.

i) Show that ADE is an isosceles triangle. — 3

ii) Show that DB = EC — 1

iii) Show that the triangles DBC and ECB are congruent. — 4

b) Find all values of θ such that

i) cos θ = ½ and $0^o \le \theta \le 360^o$ — 2

ii) Also find the exact value of sin θ for each of the values of θ. — 2

Question Four (12 marks)

a) Differentiate with to respect to x :-

i) $\dfrac{2x-1}{3x+2}$ — 2

ii) $x\sqrt{1+x}$ — 2

b) The first term of an arithmetic series is 4, and the fifth term Is four times the third term. Find the common difference. — 4

c) A geometric series has second term 6 and the ratio of the seventh term to the sixth term is 3.

i) Find the common ratio. — 1

ii) What is the first term? — 1

iii) Calculate the sum of the first 12 terms. — 2

Question Five (12 marks) — Marks

a) Two identical perfect cubes (similar to dice) each having faces numbered 0, 1, 2, 3, 4, 5 are rolled. A score for the roll is determined as the <u>product</u> of two numbers on the uppermost faces.

i) The cubes are rolled once. What is the probability that the score is

(α) 0? — 2

(β) at least 16? — 1

ii) If the cubes are rolled twice and the scores for each roll are added, what is the probability of a combined score of at least 41? — 2

b) One hundred tickets are sold in a raffle. Two different tickets are to be drawn out for first and second prizes respectively. A man buys ten tickets. Find the probabilities that:

i) he wins the first prize; — 1

ii) he wins both prizes; — 1

iii) he wins neither prize; — 1

iv) he wins at least one prize. — 1

c) Solve the equation $x+\frac{1}{x}=3$. (Leave answers in surd form) — 3

Question Six (12 marks) — Marks

a) Draw on separate sketches (-NOT on graph paper) of:-

i) $y=2\cos x$ for $-2\pi \le x \le 2\pi$ — 1

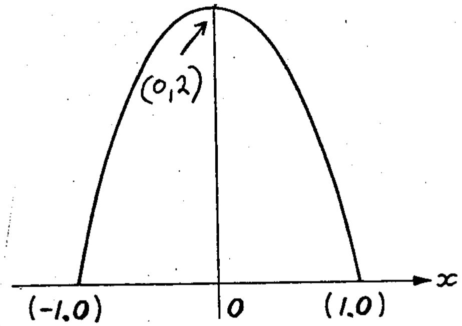
ii) $y=2+2\cos x$ for $-2\pi \le x \le 2\pi$ — 1

b) Differentiate $\sin\frac{x}{2}$ with respect to x. — 1

c) $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$ — 2

d) Find the arc length, correct to 2 decimal places, given radius is 5.9 cm and angle subtended is $23^{\circ}12'$. — 2

e)



An ornamental arch window 2 metres wide and 2 metres high is to be made in the shape of an arc of a cosine curve, as illustrated an axes above. — 2

If the arch is made in the shape of the curve

$$y=2\cos\frac{\pi}{2}x$$

find the area of the window (your answer may be left in terms of π)

f) Find the volume of the solid formed if the curve $y=\sec x$ is rotated about the x-axis from x = 0 to x = $\frac{\pi}{4}$. — 3

Question Seven (12 marks) Marks

a) Find primitives (i.e. indefinite integrals) of:

i) $x^{\frac{1}{3}}$ 1

ii) $\frac{1}{\sqrt[3]{x}}$ 1

iii) $\frac{1}{\sqrt{7x-1}}$ 2

b) Sketch the curve $y = x^3$. Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -1$ and $x = 3$. 4

c) The area under the curve $y = 2x - x^2$ between $x = 0$ and $x = 2$ is rotated about the x-axis through one complete revolution. Find the volume of the solid so formed. 4

Question Eight (12 marks)

a) Simplify $\log_6 4 + \log_6 63 - \log_6 7$ 2

b) Solve $2^x = 7$ (to 2 decimal places) 2

c) Find the equation of the tangent to $y = e^{3x}$ at the point (0,1) 2

d) Find $\int_0^1 (e^{6x} - 1)dx$ 2

e) Consider the function y = ln(x-1) for x > 1.

i) Sketch the function, showing its essential features. 2

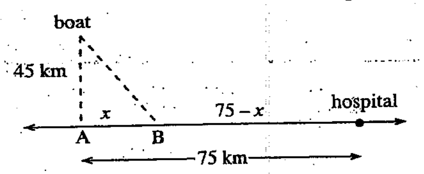
ii) Use Simpson's rule with three function values to find an approximation to (to 2 decimal places) 2

$$\int_2^4 \ln(x-1)dx.$$

Question Nine (12 marks) Marks

a) Sketch the curve $y = 4 + 3x - x^3$, showing any turning points or points of inflexion. 8

b) Imagine that you are the captain of a ship and one of your passengers has been injured and is bleeding internally. Your ship is 45 kilometres from the closest point on the coast. A hospital is a further 75 km down the coast along a straight road from this point. You can contact the hospital to send an ambulance to meet you at any point along the road, The boat travels at 40 km/h and the ambulance averages 70 km/h. Initial conditions are represented by the diagram below.



i) Show that the total time taken is represented by:- 1

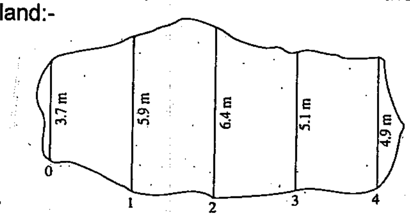
$$T = \frac{\sqrt{45^2 + x^2}}{40} + \frac{75 - x}{70}.$$

ii) You want to get the patient to the hospital as quickly as you can. Determine the point along the road, to 2 decimal places, that the ambulance should meet the boat to minimise travelling time to the hospital. 3

Question Ten (12 marks) Marks

a) A tree 3 metres tall grows an 1 metre in the first year and then $\frac{1}{3}$ of the previous year's additional height each year thereafter. **2**
What will its ultimate height be?

b) When surveyors need to find the area of an irregular piece of land, they measure regular strips and use an approximation method such as the trapezoidal rule. Consider the following piece of land:- **3**



The table below gives the measurements:-

X m	0	1	2	3	4
Y m	3.7	5.9	6.4	5.1	4.9

Use the trapezoidal rule to find its area, correct to 2 decimal places.

c) Some banks offer a "honeymoon" period on their loans. This usually takes the form of a lower interest rate for the first year. Suppose that a couple borrowed $170 000 for their first house, to be paid back monthly over 15 years. They work out that they can afford to pay $1650 per month to the bank. Loan payments are made monthly. The standard rate of interest is 8.4% pa, but the bank also offers a special rate of 6% pa for one year to people buying their first home.

i) Let M be the amount of the monthly payment needed to pay off the loan. Show that the A_{12} the amount owing after twelve months is:-

$$A_{12} = 170000(1.005)^{12} - M(1 + 1.005 + ... + 1.005^{11})$$ **3**

ii) Use this value, A_{12}, as the principal of the loan at the standard rate for the next 14 years. Calculate the value of the monthly payment that is needed to pay the loan off. Can the couple afford to agree to the contract? **4**

SOLUTIONS MATHS TRIAL 2003

Question 1

a) $2x - (x-2) = 2x - x + 2$ ✓
$= x + 2$ ✓

b) i) $-3 \leq 1 - x < 4$
$-4 \leq -x < 3$ ✓
$4 \geq x > -3$
$-3 < x \leq 4$ ✓

ii) $|x+1| = 4$
$x + 1 = 4$ or $x + 1 = -4$
$x = 3$ ✓ $x = -5$ ✓

c) $\frac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{2\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ ✓
$= \frac{6+3\sqrt{6}+2}{3-2}$
$= 8 + 3\sqrt{6}$ ✓

d) i) $f(-2) + f(-1) + f(0)$
$= 0 + (-1) + 0$ ✓
$= -1$ ✓

ii) $a^2 \geq 0$ ✓
$f(x) = x$ for $x \geq 0$
$f(a^2) = a^2$ ✓

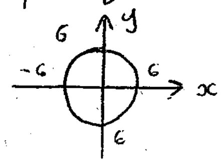
Question 2

a) $\sin 1.7 = 0.9916648 = 0.992$ (3 dp) ✓

b) i) $\cot 65° \times \sin 65°$
$= \frac{\cos 65°}{\sin 65°} \times \sin 65°$ ✓
$= \cos 65° = 0.423$ (3 dp) ✓

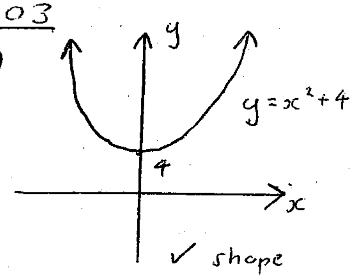
ii) $\sin^2 20° + \sin^2 70°$
$= \cos^2 70° + \sin^2 70°$ ✓
$= 1$ ✓

c) i)



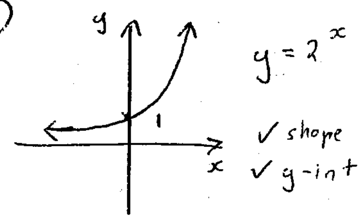
$x^2 + y^2 = 36$
✓ shape
✓ co-ords

ii)



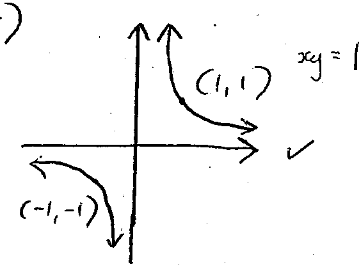
✓ shape
✓ vertex

iii)



✓ shape
✓ y-int

iv)



✓

Question 3

a) i) AB = AC (given)
∠ABC = ∠ACB
(angles opp equal sides) ✓
∠ABC = ∠ADE
(corres ∠s BC ∥ DE)
∠ACB = ∠AED
(corres ∠s BC ∥ DE) ✓
∴ ∠ADE = ∠AED
AD = AE
(sides opp equal angles) ✓
∴ △ADE is isosceles triangle

ii) AB = AC (given)
AD = AE (proved above)
AB − AD = AC − AE
∴ DB = EC ✓

iii) In △s DBC and ECB
BC is common ✓
∠DBC = ∠ECB (isosceles △) ✓
DB = EC (proved above) ✓
∴ △DBC ≡ △ECB (SAS) ✓

b) $\cos\theta = \frac{1}{2}$ $0° \leq \theta \leq 360°$
$\theta = 60°, 300°$ ✓✓
$\sin 60° = \frac{\sqrt{3}}{2}$ ✓ $\sin 300° = -\sin 60°$
$= -\frac{\sqrt{3}}{2}$ ✓

Question 4

a) i) $\frac{d}{dx}\left(\frac{2x-1}{3x+2}\right) = \frac{2(3x+2)-3(2x-1)}{(3x+2)^2}$ ✓
$= \frac{7}{(3x+2)^2}$ ✓

ii) $\frac{d}{dx}\left(x\sqrt{1+x}\right) = x\,\frac{1}{2}(1+x)^{-\frac{1}{2}} + (1+x)^{\frac{1}{2}}$ ✓
$= \frac{x}{2\sqrt{1+x}} + \sqrt{1+x}$ ✓
$= \frac{x}{2\sqrt{1+x}} + \frac{2(1+x)}{2\sqrt{1+x}}$
$= \frac{3x+2}{2\sqrt{1+x}}$

b) $t_n = a + (n-1)d$
$t_1 = a = 4$
$t_3 = 4 + 2d$ ✓
$t_5 = 4 + 4d$ ✓
now $t_5 = 4t_3$
$4 + 4d = 16 + 8d$ ✓
$-4d = 12$
$d = -3$ ✓

c) i) $t_n = ar^{n-1}$
$\frac{t_7}{t_6} = 3$ $\frac{ar^6}{ar^5} = 3$
$r = 3$ ✓

ii) $t_2 = ar = 6$
$a = 2$ ✓

iii) $S_n = \frac{a(r^n - 1)}{r-1}$
$= \frac{2(3^{12}-1)}{3-1}$ ✓
$= 3^{12} - 1$
$= 531\,440$ ✓

Question 5

a) ✓

	Cube 2: 0	1	2	3	4	5
Cube 1: 0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	6	8	10
3	0	3	6	9	12	15
4	0	4	8	12	16	20
5	0	5	10	15	20	25

i) α) $P(0) = \frac{11}{36}$
β) $P(\geq 16)$
$= \frac{4}{36} = \frac{1}{9}$ ✓

ii) The following combinations give scores of at least 41:
(16, 25) (25, 16)
(20, 25) (25, 20)
(20, 25) (25, 20)
(25, 25) ✓
$P(> 41) = \frac{7}{36 \times 36}$ ✓
$= \frac{7}{1296}$

b) i) P (wins first prize) $= \frac{10}{100} = \frac{1}{10}$ ✓

ii) P (wins both) $= \frac{10}{100} \times \frac{9}{99} = \frac{1}{110}$ ✓

iii) P (wins neither) $= \frac{90}{100} \times \frac{89}{99} = \frac{89}{110}$ ✓

iv) P (at least one prize) $= 1 - P(LL)$

$= 1 - \frac{89}{110}$

$= \frac{21}{110}$ ✓

c) $x + \frac{1}{x} = 3$

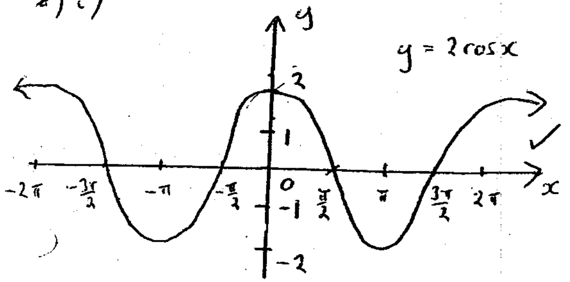
$x^2 - 3x + 1 = 0$ ✓

$x = \frac{3 \pm \sqrt{(-3)^2 - 4}}{2}$ ✓

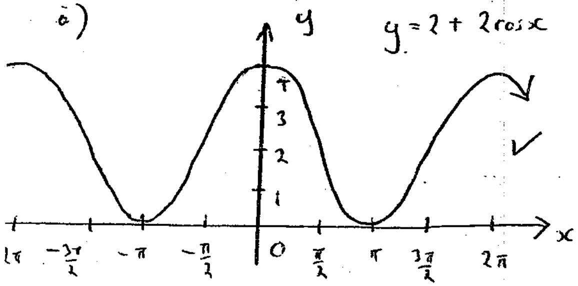
$= \frac{3 \pm \sqrt{5}}{2}$ ✓

Question 6

a) i)



ii)



b) $\frac{d}{dx}\left(\sin \frac{x}{2}\right) = \frac{1}{2} \cos \frac{x}{2}$ ✓

c) $\int_0^{\frac{\pi}{3}} \sin 2x \, dx$

$= -\frac{1}{2} \left[\cos 2x\right]$

$= -\frac{1}{2} \left(\cos \frac{2\pi}{3} - \cos 0\right)$

$= -\frac{1}{2} \left(-\frac{1}{2} - 1\right)$

$= \frac{3}{4}$ ✓

d) $l = 5.9 \times 23°12' \times \frac{\pi}{180}$

$= 2.39$ cm (to 2 dp)

e) $A = 2\int_0^1 3 \cos \frac{\pi}{2} x \, dx$

$= \frac{8}{\pi} \left[\sin \frac{\pi}{2} x\right]_0^1$ ✓

$= \frac{8}{\pi} \left(\sin \frac{\pi}{2} - \sin 0\right)$

$= \frac{8}{\pi}$ metres^2 ✓

f) $V = \pi \int_0^{\frac{\pi}{4}} \sec^2 x \, dx$ ✓

$= \pi \left[\tan x\right]_0^{\frac{\pi}{4}}$ ✓

$= \pi (1 - 0)$

$= \pi$ units^3 ✓

Question 9

a) $y = 4 + 3x - x^3$ ✓

$\frac{dy}{dx} = 3 - 3x^2$ ✓ $\frac{d^2y}{dx^2} = -6x$

For stationary points $\frac{dy}{dx} = 0$

$3(1 - x^2) = 0$

$x = \pm 1$ ✓ ✓

∴ Stationary pts at (1, 6) (−1, 2)

at (1, 6) $\frac{d^2y}{dx^2} < 0$ ∴ (1, 6) max pt ✓

(−1, 2) $\frac{d^2y}{dx^2} > 0$ ∴ (−1, 2) min pt ✓

For pts of inflexion

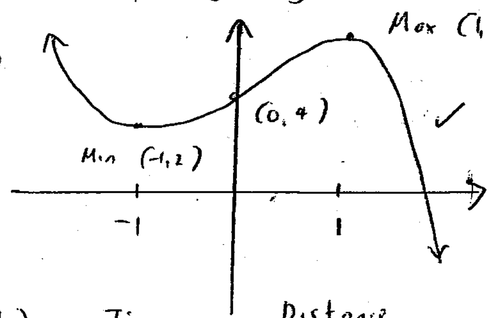
$\frac{d^2y}{dx^2} = 0$ $\quad -6x = 0$

$x = 0 \quad y = 4$ ✓

when x a little less than 0 $\frac{d^2y}{dx^2} > 0$

x a little more than 0 $\frac{d^2y}{dx^2} < 0$ ✓

Thus pt of inflexion



b) Time = $\frac{\text{Distance}}{\text{Speed}}$

Total Time = Time$_{Ship}$ + Time$_{Ambulance}$

$T = \frac{\sqrt{45^2 + x^2}}{40} + \frac{75 - x}{70}$ ✓

$T' = \frac{\frac{1}{2} \times 2x \, (45^2 + x^2)^{-\frac{1}{2}}}{40} -$

$= \frac{x}{40\sqrt{45^2 + x^2}} - \frac{1}{70}$

For $T' = 0$

$\frac{x}{40\sqrt{45^2 + x^2}} = \frac{1}{70}$

$\frac{7}{4} x = \sqrt{45^2 + x^2}$

$\frac{49}{16} x^2 = 45^2 + x^2$

$x^2 = 45^2 \times \frac{16}{33}$

$x = 31.33$ m

when x is a little less than 31.33 $T' < 0$

when x is a little less than 31.33 $T' > 0$ ✓

∴ min at $x = 31.33$

∴ ship should proceed to a point 31.33 km from the closest point on coast

Question 10

a) $S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{3}{\frac{2}{3}}$ ✓

$= \frac{9}{2} = 4\frac{1}{2}m$ ✓

∴ ultimate height is $4\frac{1}{2}m$

b) $\int_a^b f(x)\,dx \doteq \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$ ✓

where $h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

$\int_0^4 f(x)\,dx \doteq \frac{1}{2}\left[3.7+4.9+2(5.9+6.4+5.1)\right]$ ✓

$= 21.70\,m^2$ (to 2dp) ✓

c) i) let M be monthly repayment

after 1 mth $A_1 = 170000(1.005) - M$ ✓

" 2 mths $A_2 = (170000(1.005)-M)1.005 - M$

$= 170000(1.005)^2 - M(1+1.005)$ ✓

" 3 mths $A_3 = 170000(1.005)^3 - M(1+1.005+1.005^2)$

12 mths $A_{12} = 170000(1.005)^{12} - M(1+1.005+\ldots+1.005^{11})$

ii) After 15 years

$A_{180} = A_{12}(1.007)^{168} - M(1+1.007+\ldots+1.007^{167}) = 0$ ✓

$170000(1.005^{12}) - M(1+1.005+\ldots+1.005^{11}) \times 1.007^{168}$

$= M(1+1.007+\ldots 1.007^{167})$

$\left[170000(1.005^{12}) - M\frac{(1.005^{12}-1)}{1.005-1}\right] \times 1.007^{168} = \frac{M(1.007^{168}-1)}{1.007-1}$

$170000(1.005^{12}) \times (1.007)^{168} - M\frac{(1.005^{12}-1)}{0.005} \times 1.007^{168}$

$= M\frac{(1.007^{168}-1)}{0.007}$

$M\left[\frac{1.005^{12}-1}{0.005} \times 1.007^{168} + \frac{1.007^{168}-1}{0.007}\right]$

$= 170000(1.005)^{12} \times (1.007)^{168}$

$M = \dfrac{170000(1.005)^{12} \times (1.007)^{168}}{\frac{1.005^{12}-1}{0.005} \times (1.007)^{168} + \frac{1.007^{168}-1}{0.007}}$ ✓

$\doteq \dfrac{170000 \times 1.0617 \times 3.2281}{\left(\frac{0.0617}{0.005} \times 3.2281\right) + \frac{2.2281}{0.007}}$

$= \$1626.86$ ✓

Yes the couple can afford the loan. ✓